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The Strong Coupling Constant at Low and High Energies

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Recent results for the cross section of electron-positron annihilation into hadrons and for the decay rates of the Z boson and the τ lepton into hadrons including corrections of order α_s^4 are reviewed. The consistency between two values of α_s measured at vastly different energies constitutes a striking test of asymptotic freedom and requires the proper matching conditions at charm and bottom thresholds. Combining the results from Z and τ decays leads to $\alpha_s(M_Z) = 0.1198 \pm 0.0015$ as one of the most precise and presently only NNNLO result for the strong coupling constant. We report a recent determination of α_s which is based on a lattice evaluation of the pseudoscalar correlator for the charm quarks combined with continuum perturbation theory.

The strong coupling constant α_s is one of the three fundamental gauge couplings constants of the Standard Model (SM) of particle physics. Its precise determination is one of the most important aims of particle physics. Experiments at different energies allow to test the predictions for its energy dependence based on the renormalization group equations, the comparison of the results obtained from different processes leads to critical tests of the theory and potentially to the discovery of physics beyond the Standard Model. Last not least, the convergence of the three gauge coupling constants related by SU(3)×SU(2)×U(1) to a common value, after evolving them to high energies, allows to draw conclusions about the possibility of embedding the SM in the framework of a Grand Unified Theory.

A characteristical example is shown in Fig.1, which demonstrates in particular the anticipated improvement in our information about coupling constant unification at the GUT scale, if the experimental uncertainty in α_s will be reduced from 0.0027 (corresponding to the value quoted by the "Electroweak Working Group" [1]) to 0.0009, a precision that might be within reach with 10^9 Z-events at a GIGA-Z [2]. In this latter case it may well be possible to deduce non-trivial constraints on threshold corrections and the structure of symmetry breaking at the GUT-scale.

One of the most precise and theoretically safe determination of α_s is based on measurements of the cross section for electron-positron annihilation into hadrons. These have been performed in the low-energy region between 2 GeV and 10 GeV and, in particular, at and around the Z resonance at 91.2 GeV. Conceptually closely related is the measurement of the semileptonic decay rate of the τ-lepton, leading to a determination of α_s at a scale below 2 GeV.

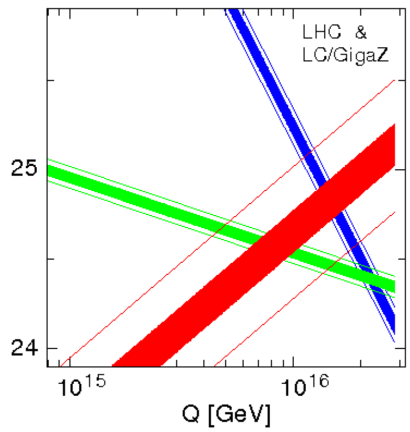

Figure 1. Renormalization group equation of the three coupling constants α_1, α_2 and α_3 [3]

From the theoretical side, in the framework of perturbative QCD, these rates and cross sections are evaluated as inclusive rates into massless quarks and gluons [4,5]. (Power suppressed mass effects are well under control for e^+e^--annihilation, both at low energies and around the Z resonance, and for τ decays [6–9], and the same applies to mixed QCD and electroweak corrections [10,11]).

The ratio $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is expressed through the absorptive part of the correlator of the electromagnetic current j_μ:

$$R(s) = 12\pi\, \Im\Pi(-s - i\epsilon), \quad (1)$$

$$3\,Q^2\,\Pi(Q^2) = i\int \mathrm{d}^4x e^{iq\cdot x}\langle 0|\mathrm{T}j_\mu(x)j^\mu(0)|0\rangle, \quad (2)$$

with $Q^2 = -q^2$. It is also convenient to introduce the Adler function as

$$D(Q^2) = -12\,\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) = \int_0^\infty \frac{Q^2\;R(s)ds}{(s+Q^2)^2},$$

Conversely, in the perturbative expansion $R(s)$ is given by

$$R(s) = D(s) - \pi^2\beta_0^2\{\frac{d_1}{3}a_s^3 + (d_2 + \frac{5}{6\,\beta_0}d_1\,\beta_1)\,a_s^4\} + \ldots\,. \quad (3)$$

Here we define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} d_i a_s^i(Q^2),\;\; R(s) = \sum_{i=0}^{\infty} r_i a_s^i(s), \quad (4)$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = Q^2$ or to $\mu^2 = s$ for the Euclidian and Minkowskian functions respectively. The results for generic values of μ can be easily recovered with standard RG techniques.

Note that the first three terms of the perturbative series for D and R coincide. Starting from r_3, terms proportional π^2 arise which can be predicted from those of lower order. It has been speculated that these "π^2-terms", also called "kinematical terms", might constitute a major part of the full higher order corrections (see, e.g. [12]); however, the validity of this hypothesis can only be established by the full calculation. Indeed, for the scalar correlator this assumption has been shown to fail completely [13].

For the vector correlator the terms of order a_s^2 and a_s^3 have been evaluated nearly thirty and about fifteen years ago [14–16], respectively. The a_s^4 corrections are conveniently classified according to their power of n_f, with n_f denoting the number of light quarks. The $a_s^4 n_f^3$ term is part of the "renormalon chain", the evaluation of the next term, of order $a_s^4 n_f^2$, was a test case for the techniques used extensively in this paper and, furthermore, led to useful insights into the structure of the perturbative series already [17].

The complete five-loop calculation has been performed in [18]. It required the evaluation of about twenty thousand diagrams generated automatically via QGRAF [19]. Using "infrared rearrangement" [20], the R^* operation [21] and the prescriptions formulated in [22] to algorithmically resolve the necessary combinatorics, it was possible to express the absorptive part of the five-loop diagrams in terms of four-loop massless propagator integrals. Using then a representation for Feynman integrals proposed in [23,24], these integrals were reduced to a sum of 28 master integrals, with coefficients which are rational functions of the space-time dimension D. These coefficients were, in a first step, evaluated in the large-D limit, and, after evaluating sufficiently many terms of the $1/D$ expansion, fully reconstructed [25]. This direct and largely automatic procedure required enormous computing resources and was performed on a 32+8-node SGI parallel computer with SMP architecture and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using a parallel version [26] of FORM [27].

The so-called "non-singlet" diagrams, where one and the same closed quark line is connected to the external currents, and which were evaluated in [18], are sufficient for a complete description of τ-decays. For e^+e^- annihilation, they correspond to the dominant terms proportional $\sum_i Q_i^2$. The singlet contributions proportional $(\sum_i Q_i)^2$ arise for the first time in $\mathcal{O}(\alpha_s^3)$ and are known to be small. Similar comments apply to the singlet contributions in Z decays.

The analytic result for the Adler function contains transcendental numbers up to ζ_7, with ζ_4 and ζ_6 being absent. It leads straightforwardly to R at order α_s^4. In numerical form

$$\begin{aligned} R &= 1 + a_s + (1.9857 - 0.1152\, n_f)\, a_s^2 \\ &+ (-6.63694 - 1.20013 n_f - 0.00518 n_f^2)\, a_s^3 \\ &+ (-156.61 + 18.77\, n_f - 0.7974\, n_f^2 \\ &+ 0.0215\, n_f^3)\, a_s^4. \end{aligned} \quad (5)$$

It is also instructive to explicitly display the genuine five-loop contributions to d_4 (underlined in (6,7) below) and the "kinematical" terms originating from the analytic continuation:

$$\begin{aligned} r_3 &= \underline{18.2} - 24.9 + (\underline{-4.22} + 3.02)\, n_f \quad (6) \\ &+ (\underline{-0.086} + 0.091)\, n_f^2, \\ r_4 &= \underline{135.8} - 292.4 + (\underline{-34.4} + 53.2)\, n_f \quad (7) \\ &+ (\underline{1.88} - 2.67)\, n_f^2 + (\underline{-0.010} + 0.032)\, n_f^3. \end{aligned}$$

Since it will presumably take a long time until the next term of the perturbative series will be evaluated, it is of interest to investigate the predictive power of various optimization schemes empirically. Using the principles of "Fastest Apparent Convergence" (FAC) [28] or of "Minimal Sensitivity" (PMS) [29], which happen to coincide in this order, the central values of the predictions [30,31]

$$d_4^{\rm pred}(n_f = 3,4,5) = 27 \pm 16, 8 \pm 18, -8 \pm 44$$

differ significantly from the exact result

$$d_4^{\rm exact}(n_f = 3,4,5) = 49.08, \quad 27.39, \quad 9.21. \quad (8)$$

However, within the error estimates [31] predicted and exact values are in agreement. The picture changes, once these estimates are used to predict the coefficient r_4. Although sizable cancellations between "dynamical" and "kinematical" terms are observed for the individual n_f coefficients in (7) the predictions for the final results are significantly closer (in relative sense) to the results of the exact calculation:

$$\begin{aligned} r_4^{\rm pred}(n_f = 3,4,5) &= -129 \pm 16, \; -112 \pm 30, \\ & \quad -97 \pm 44, \\ r_4^{\rm exact}(n_f = 3,4,5) &= -106.88, \quad -92.898, \\ & \quad -79.98. \end{aligned} \quad (9)$$

(This is in striking contrast to the case of the scalar correlator, where the predictions for the dynamical terms work well, but, as a consequence of the strong cancellations between dynamical and kinematical terms fail completely in the Minkowskian region.) Using FAC and the exact result for d_4, the coefficients d_5 and r_5 can be predicted (following [31,32]) for $n_f = 3,4,5$, namely

$$\begin{aligned} d_5^{\rm pred}(n_f = 3,4,5) &= 275, \quad 152, \quad 89, \\ r_5^{\rm pred}(n_f = 3,4,5) &= -505, \quad -134, \quad 168 \\ &. \end{aligned} \quad (10) \; (11)$$

These terms may become of relevance for the International Linear Collider (ILC) running in the GIGA-Z mode with an anticipated precision of $\delta\alpha_s = 0.0005$ — 0.0007 [2], and already today for the analysis of τ-decays.

Let us now move to the analysis of present data for e^+e^- annihilation and τ decays. Measurements of $R(s)$ at lower energies, with their correspondingly larger values of α_s, are in principle more sensitive to $\alpha_s(M_Z)$ if the same relative

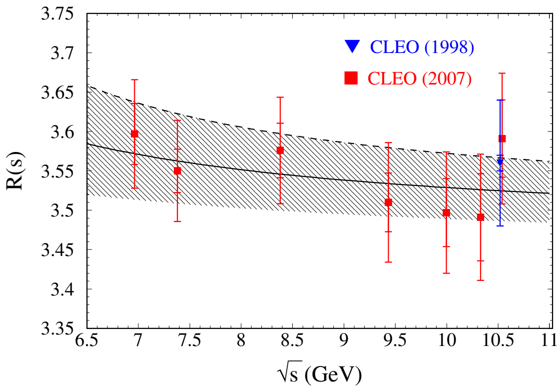

Figure 2. $R(s)$ as measured by CLEO [35, 36]. The full solid line and the hatched band correspond to the theory prediction based on $\alpha_s(M_Z^2) = 0.110^{-0.014}_{-0.017}$. The dashed line represents theory prediction where results from the CLEO analysis [36], $\alpha_s^{(5)}(M_Z) = 0.126$, has been used.

precision could be obtained. At present, however, the systematic experimental error of 2% is a limiting element for a competitive measurement. Nevertheless, it is interesting, that the result of a recent analysis $\alpha_s(9^2\ \mathrm{GeV}^2) = 0.160 \pm 0.034$ [33] is a clear 3σ signal for the influence of QCD corrections also in this energy range. Furthermore, as shown in [33], proper inclusion of quark mass effects and matching at flavour threshold is required to obtain the correct result (Fig. 2). A convenient FORTRAN program, called RHAD, which predicts $R_{\mathrm{had}}(s)$ and where all quark mass effects and higher order QCD corrections are implemented, is described in [34]. Ignoring quark mass effects and proper matching conditions [35], leads to a shift of α_s by one standard deviation, as shown in Fig. 2 (dashed line).

Evolving the result from 9 GeV to M_Z leads to $\alpha_s(M_Z^2) = 0.110^{+0.014}_{-0.017}$. It is instructive to combine this value with the one obtained in Ref. [37], $\alpha_s^{(4)}(5^2\mathrm{GeV}^2) = 0.235^{+0.047}_{-0.047}$, which was based on earlier measurements by BES [38], MD-1 [39] and CLEO [35] in the region between 3 and 10 GeV. Adding the correlated and uncorrelated errors of the different experiments in quadrature, the final result

$$\alpha_s(9^2\mathrm{GeV}^2)^{NNLO} = 0.182^{+0.022}_{-0.025} \qquad (12)$$

represents the combined information on the strong coupling from measurements in the region below the bottom threshold. The shift in α_s from the inclusion of the α_s^4 term amounts to $\delta\alpha_s(9^2\mathrm{GeV}^2) = 0.002$ and is thus irrelevant compared to the large experimental error.

In contrast to the situation below 10 GeV the extraction of α_s from Z decays is affected by the α_s^4 terms. The analysis of the electroweak working group [1] is based on eq. (5) with $n_f = 5$, including terms up to $\mathcal{O}(\alpha_s^3)$ and leads to

$$\alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026^{\mathrm{exp}}. \qquad (13)$$

Including the α_s^4 term leads to a shift $\delta\alpha_s(M_Z) = 0.0005$

$$\alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\mathrm{exp}}. \qquad (14)$$

The theory error may either be conservatively based on the shift produced by the last term (0.0005) or on the scale variation with $\mu/M_Z = \frac{1}{3} \div 3$, leading to ± 0.0002 and can now be neglected in both cases.

Higher orders are of particular relevance in the low-energy region, for example in τ decays. The correction from perturbative QCD to the ratio

$$\begin{aligned} R_{\tau,V+A} &= \frac{\Gamma(\tau\to \mathrm{hadrons}_{S=0}+\nu_\tau)}{\Gamma(\tau\to l+\bar\nu_l+\nu_\tau)} \\ &= 3|V_{ud}|^2 S_{EW}(1+\delta_0+\delta'_{EW}+\delta_2+\delta_{NP}) \end{aligned} \qquad (15)$$

is given by

$$1+\delta_0 = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right) R(s). \qquad (16)$$

The perturbative quantity δ_0 can be evaluated in Fixed Order perturbation theory or with "Contour Improvement" as proposed in [40,41]

$$\delta_0^{FO} = a_s + 5.202\, a_s^2 + 26.366\, a_s^3 + 127.079\, a_s^4, \qquad (17)$$

$$\delta_0^{CI} = 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + 64.29\, a_s^4. \qquad (18)$$

From the experimental side $\delta_0^{\mathrm{exp}} = 0.1998 \pm 0.0043_{\mathrm{exp}}$ has been used [42]. More recent evaluations of the experimental input correspond to slightly larger values $\delta_0^{\mathrm{exp}} = 0.2066 \pm 0.0070$ [43] and $\delta_0^{\mathrm{exp}} = 0.2042 \pm 0.0050$ [44], but are still consistent with this previous value. Starting from δ_0^{exp} and using eq. (17) and eq. (18) respectively, one easily obtains $\alpha_s(M_\tau)$. Its dependence on the choice of d_4 (previous estimate and the new exact result) is summarized in Table I.

As stated above the theory error for α_s from Z decays is small compared to the experimental uncertainties. The situation is more problematic for τ decays and to some extent the theory error remains to be matter of choice. As anticipated in [31] it decreases significantly, once α_s^4 terms are included. However, the difference between the two methods stabilizes (this was checked in [31] by adding an estimate for the α_s^5 term) and must be considered as irreducible uncertainty, with the final result

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\mathrm{exp}} \pm 0.015_{\mathrm{th}}. \qquad (19)$$

Table 1
Results for $\alpha_s(M_\tau)$ for different values of d_4, extracted from eq. (17). The first line displays the α_s^3 results (with the α_s^4 terms set to zero in eqs. (17,18)). The second line uses the previously predicted value for d_4, the last one uses the exact result (5). The first error is the experimental one; the second (theoretical) uncertainty in the value of α_s corresponds to changing the renormalization scale μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$.

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

For the central value the mean value of FO and CI has been used, for the theory error half of the difference between two methods (0.01) plus (module of) the estimated correction from α_s^5 term (-0.005), the latter being based on $d_5 = 275$ (see eq. (10)).

Recently it has been argued [43], that predictions based on FO are unstable and should therefore be abandoned. Using CI only and the new value for d_4, and assuming small higher order terms in the expansion of the Adler function, leads to a relatively large value $\alpha_s(m_\tau) = 0.344 \pm 0.005_{\rm exp} \pm 0.007_{\rm th}$ and an optimistic theory error. This line of reasoning is in striking contrast with a new study performed in [44]. In this paper the Adler function series is described by a model which incorporates the exactly known coefficients up to d_4 and theory constraints on the large-order behaviour and it is argued, that FO smoothly approaches the correct asymptotic result. The result $\alpha_s(m_\tau) = 0.320 \pm 0.003_{\rm exp}{}^{+0.011}_{-0.006}$ is significantly lower than the previous one. As expected the two results, taken at face value, are in agreement with the result for FO and CI respectively, as listed in Table 1. On the other hand, they are mutually exclusive. Individually, after evolving α_s to M_Z the associated error corresponds to about or less than 10^{-4} and the respective result would completely dominate all determinations of α_s. For the time being it seems prudent to stay with eq. (19) which takes the difference between FO and CI into account.

Applying four-loop running and matching [45–48] as implemented in the updated version of the MATHEMATICA-routine RunDec [49] to (19) we arrive at

$$
\begin{aligned}
a_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm th} \\
&\quad \pm 0.0003_{\rm evol} , \\
&= 0.1202 \pm 0.0019. \qquad (20)
\end{aligned}
$$

Here the evolution error receives contributions from the uncertainties in the c-quark mass (0.00003, $m_c(m_c) = 1.286(13)$ GeV [50]) and the b-quark mass (0.00001, $m_b(m_b) = 4.164(25)$ GeV [50]), the matching scale (0.0001, μ varied between $0.7\, m_q(m_q)$ and $3.0\, m_q(m_q)$), the four-loop truncation in the matching expansion (0.0001) and the four-loop truncation in the RGE equation (0.0003). (For the last two errors the size of the shift due the highest known perturbative term was treated as systematic uncertainty.) The errors had been added in quadrature.

A recent, completely different determination of α_s is based on the combination of lattice simulation and continuum pertubation theory [51]. Derivatives of the correlator of two charm-quark currents $m_c\overline{\psi}\gamma_5\psi$ were evaluated at vanishing momentum q^2. Relating lattice and perturbative results for the same quantity then allowed for a determination of m_c and α_s. Specifically, higher moments, corresponding to higher derivatives with respect to q^2, were used for an accurate determination of m_c. From the second derivative, being dimensionless, a value $\alpha_s(M_Z) = 0.113(4)$ was obtained. Although this method cannot yet compete with the determinations discussed above, the combination of lattice and continuum methods may well lead to precission determinations of α_s.

As shown in Fig. 3, it is instructive to compare the results of α_s measurements at m_τ, at intermediate energies around 9 GeV and at M_Z directly with the predictions based on four-loop running and matching, starting from a value $\alpha_s(M_Z) = 0.1198 \pm 0.0015$. Theory prediction and the three experimental results are in remarkable agreement.

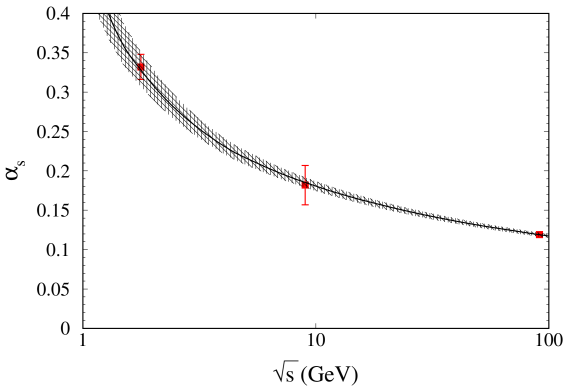

Figure 3. Comparison of the experimental results eqs. (12), (14) and (19) with the four-loop evaluation and a starting value $\alpha_s(M_Z) = 0.1198 \pm 0.0015$

Summary: The exact result for the α_s^4 term in the Adler function allows to extract the strong coupling constant from Z and τ decays with high precision. Including the exact α_s^4 leads to small shifts of the central value and to a significant reduction of the theory uncertainty. Note that the shifts in $\alpha_s(M_Z)$ from Z and τ decays, are opposite in sign and move the values in the *proper* direction, decreasing, thus, the current slight mismatch between two independent determinations of α_s. The final results

$$\alpha_s(M_Z)|_Z = 0.1190 \pm 0.0026\,, \qquad (21)$$

$$\alpha_s(M_Z)|_\tau = 0.1202 \pm 0.0019\,, \qquad (22)$$

from these two observables, although based on measurements of vastly different energy scales, are in remarkable agreement. This constitutes a striking test of asymptotic freedom in QCD. The two values can be combined to

$$\alpha_s(M_Z) = 0.1198 \pm 0.0015.$$

This is one of the most precise and presently only N^3LO result.

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