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New Results in Four and Five Loop QED calculations *

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We report on two recent multiloop results in QED: (i) the four-loop corrections to the conversion relations between the QED charge renormalized in the on-shell and $\overline{\text{MS}}$ schemes; (ii) analytical evaluation of a class of asymptotic contributions to the muon anomaly at five-loops.

1. Introduction

The study of the anomalous magnetic moment of the muon a_μ is a long-standing challenge for both theory and experiment. It has been measured with impressive accuracy at the level of 0.5 parts per million [1]: $a_\mu^{\text{exp}} = 116592080(63)\cdot 10^{-11}$. From theory side the anomalous magnetic moment has been studied in great detail through the computation of higher order corrections (see, e.g. reviews [2–4]). These higher order corrections to a_μ^{theo} are basically classified into three classes: pure QED, electroweak and hadronic contribution.

A discussion of the electroweak and hadronic corrections can be found in Ref. [5] and references therein. Within this work we consider higher order corrections to the pure QED part. Starting from two-loop, diagrams with internal fermion-loops can arise, where the fermion-type of the internal loop can in general be different from the external muon. The one-loop [6] and two-loop [7–13] contributions have been computed more than 50 years ago. The three-loop order has been computed numerically and analytically [14–20]

The complete calculation of hundreds of diagrams contributing to the muon anomaly at four loop order is only possible by numerical integration, which was performed in a remarkable long-term effort by Kinoshita and his collaborators [21,22]. Even some numerically important five-loop diagrams have been computed by now [23].

Accurate numerical calculation of highly divergent multiloop Feynman amplitudes is highly nontrivial task. The analytical calculation (of even a very particular class of diagrams) should be quite useful as an independent check *in both directions!* (see e.g. instructive examples in Refs. [24–27,21,28,29]).

In particular starting from two-loop there arise logarithmic contributions of the type $\log(M_\mu/M_e)$, where M_μ is the mass of the muon and M_e the mass of the electron, respectively. In view of the large mass ratio $M_\mu/M_e \sim 200$ one

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can expect, that these logarithms play a dominant role. The logarithmically enhanced contributions arise from the insertion of the electron vacuum polarization (eVP) into the first order muon vertex diagram, but they can also appear through light-by-light (LBL) scattering diagrams. Examples for both diagram types are shown in Fig. 1.

Figure 1. Example diagrams leading to dominant logarithmic contributions from electron vacuum polarization (VP) insertions and light-by-light (LBL) scattering diagrams.

In what follows we will consider only eVP contributions to a_μ which are produced from diagrams like Fig 1 (eVP) but with the photon propagator receiving all possible QED perturbative corrections made of electron loops and photon exchanges only. The general structure of eVP was throughly studied long ago in the pioneering publications of B. Lautrup and E. De Rafael and R. Barbieri and E. Remiddi [30,31].

There it was found, that the asymptotic part of the muon anomaly a_μ^{as}, which contains these logarithmic contributions originating form the electron vacuum polarization function insertions and the mass independent term, can be obtained with the help of the electron vacuum polarization function in the asymptotic limit $M_e \to 0$. The corresponding master formula reads

$$a_\mu^{\mathrm{eVP}} = a_\mu^{\mathrm{as}} + \mathcal{O}\left(\frac{M_e}{M_\mu}\right) = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x) \left[d_R^{\mathrm{as}}\left(\frac{x^2}{1-x}\frac{M_\mu^2}{M_e^2},\alpha\right) - 1\right] + \mathcal{O}\left(\frac{M_e}{M_\mu}\right). \quad (1)$$

Here

$$d_R^{\mathrm{as}}(-q^2/M^2,\alpha) = \frac{1}{1+\Pi^{\mathrm{as}}(-q^2/M^2,\alpha)}, \quad (2)$$

$$\Pi(-q^2/M^2,\alpha) \underset{-q^2\to\infty}{=\!=} \Pi^{\mathrm{as}}(-q^2/M^2,\alpha) + \mathcal{O}\left(\frac{M^2}{-q^2}\right). \quad (3)$$

In the above formulas Π is the photon polarization operator (in QED incorporating *exactly one* fermion field), M is the on-shell fermion mass, and the fine structure constant α is defined in the classical OS-scheme by the condition:

$$\Pi(-q^2 = 0, M^2, \alpha) \equiv 0. \quad (4)$$

Once Π^{as} is computed to some order in α the master formula (1) immediately delivers the corresponding eVP contribution to a_μ.

At present this technique has been applied in order to find the complete eVP contributions to a_μ up to three-loop and four-loop order [30,31,25,27].

In our talk we extend these results by one order. As a by-product we will also derive the four-loop corrections to the conversion relations between the QED charge renormalized in the on-shell and $\overline{\mathrm{MS}}$ schemes.

2. Photon polarization operator in $\overline{\mathrm{MS}}$-scheme

Thus, the main problem is to compute the asymptotic photon polarization operator ($Q^2 \equiv -q^2$)

$$\Pi^{\mathrm{as}}(Q/M,\alpha) = \sum_i \Pi^{\mathrm{as},(i)}(Q/M)\left(\frac{\alpha}{\pi}\right)^i, \quad (5)$$

with every $\Pi^{\mathrm{as},(i)}(Q/M)$ being, in fact[3], a polynomial of order not higher than i in $\ell_{QM} = \log(\frac{Q^2}{M^2})$.

It is very useful to consider first the photon polarization operator in $\overline{\mathrm{MS}}$-scheme written as $\overline{\Pi}(-q^2, m^2, \mu, \overline{\alpha})$ where $\overline{\alpha} \equiv \alpha^{\overline{\mathrm{MS}}}(\mu)$ and $m \equiv m^{\overline{\mathrm{MS}}}(\mu)$ are the running coupling constant and the fermion mass in the $\overline{\mathrm{MS}}$-scheme, while μ stands for the $\overline{\mathrm{MS}}$ renormalization point. Indeed,

[3]This is a consequence of the Weinberg theorem [32].

the $\overline{\rm MS}$ polarization function $\overline{\Pi}$ has a smooth massless limit:

$$\overline{\Pi}(Q^2, m^2 = 0, \overline{\alpha}) = \sum_i \overline{\Pi}^{(i)}(\mu^2/Q^2) \left(\frac{\overline{\alpha}(\mu)}{\pi}\right)^i . \quad (6)$$

A use of the fundamental concept of the *invariant charge* [33,34] directly leads to the connection between $\overline{\Pi}$ and $\Pi^{\rm OS}$

$$\frac{\alpha^{\rm OS}}{1 + \Pi^{\rm OS}(Q^2, M^2, \alpha^{\rm OS})} = \frac{\overline{\alpha}}{1 + \overline{\Pi}(Q^2, m^2, \overline{\alpha})}, \quad (7)$$

Eq. (7) allows one (see e.g. [30]) to relate $\alpha^{\overline{\rm MS}}$ and $\alpha^{\rm OS}$ through the $\overline{\rm MS}$ polarization operator at zero momentum transfer

$$\overline{\Pi}(Q^2 = 0, m^2, \overline{\alpha}) = \sum_i \overline{\Pi}^{(i)}(\mu^2/m^2) \left(\frac{\overline{\alpha}(\mu)}{\pi}\right)^i . \quad (8)$$

Thus, to compute the Π^{as} at four loops we need to know the $\overline{\rm MS}$ polarization function $\overline{\Pi}$ in the massless and momentum-less limits as well as the relation between on-shell and $\overline{\rm MS}$ masses of a fermion in QED at three loops. The latter, fortunately, is available from [35,36].

2.1. Massless limit of $\overline{\Pi}$

We have computed $\overline{\Pi}^{(4)}(\mu^2/Q^2)$ using the parallel version of `FORM` [37–39]. The contributing diagrams were first generated with the package `QGRAF` [40]. The reduction to master integrals was performed with the help of an auxiliary integral representation [41]. The result reads ($\ell_{\mu Q} = \log(\frac{\mu^2}{Q^2})$):

$$\begin{aligned}\overline{\Pi}^{(4)}(\mu^2/Q^2) &= \frac{1075825}{373248} - \frac{13}{8640}\pi^4 + \frac{13051}{2592}\zeta_3 \\ &- \frac{5}{3}\zeta_3^2 + \frac{45}{32}\zeta_5 - \frac{35}{4}\zeta_7 \\ &+ l_{\mu Q}\left[\frac{9403}{10368} + \frac{23}{108}\zeta_3 - \frac{5}{3}\zeta_5\right] \\ &+ l_{\mu Q}^2\left[\frac{19}{144} - \frac{1}{9}\zeta_3\right] + l_{\mu Q}^3\frac{1}{108}. \end{aligned} \quad (9)$$

2.2. Momentum-less limit of $\overline{\Pi}$

The limit $q^2 \to 0$ leads to the evaluation of massive tadpole diagrams. Their computation has been performed with `FORM` [37,42,43] based programs. All appearing tadpole diagrams have been reduced to master integrals with the help of Laporta's algorithm [20,44]. The arising polynomials in the space-time dimension $d = 4-2\,\varepsilon$ have been simplified with the program `FERMAT` [45]. The remaining master integrals are known analytically to sufficient high order in ε and have been taken from Refs. [46–54]. The following result for $\overline{\Pi}^{(4)}$ at $Q = 0$ was found

$$\begin{aligned}\overline{\Pi}^{(4)}(Q = 0) &= -\frac{24254383}{9331200} + \frac{69437}{86400}\pi^4 \\ &- \frac{1780741}{43200}\zeta_3 + \frac{10087}{480}\zeta_5 - \frac{106}{675}\pi^4\ln 2 \\ &+ \frac{2227}{720}\pi^2\ln^2 2 - \frac{32}{135}\pi^2\ln^3 2 - \frac{2227}{720}\ln^4 2 \\ &+ \frac{32}{225}\ln^5 2 - \frac{2227}{30}a_4 - \frac{256}{15}a_5 \\ &+ \left[-\frac{9383}{10368} - \frac{29}{48}\zeta_3\right]\ell_{\mu m} - \frac{25}{216}\ell_{\mu m}^2 - \frac{1}{108}\ell_{\mu m}^3. \end{aligned} \quad (10)$$

Here $\ell_{\mu m} = \log(\frac{\mu^2}{m^2})$, $\zeta_n = \zeta(n)$ is Riemann's zeta function and $a_n = {\rm Li}_n(1/2) = \sum_{i=1}^{\infty} 1/(2^i i^n)$.

3. Conversion formulas for α

Let us define the conversion factor $C_{\overline{\alpha}\alpha}$, which converts the fine structure constant $\overline{\alpha}$ in the $\overline{\rm MS}$-scheme into α in OS-scheme: $\overline{\alpha} = C_{\overline{\alpha}\alpha}\,\alpha$:

$$C_{\overline{\alpha}\alpha} = 1 + \sum_{i\ge 1} C_{\overline{\alpha}\alpha}^{(i)}\left(\frac{\alpha}{\pi}\right)^i . \quad (11)$$

A use of eq. (10) directly leads to our result for $C_{\alpha\overline{\alpha}}^{(4)}$ (for brevity we skip the lower order expressions for $C_{\alpha\overline{\alpha}}^{(i)}$ with $i = 1, 2, 3$ which can be found in [55])

$$\begin{aligned}C_{\overline{\alpha}\alpha}^{(4)}(\mu^2/M^2) &= \frac{14327767}{9331200} + \frac{8791}{3240}\pi^2 + \frac{204631}{259200}\pi^4 \\ &-\frac{175949}{4800}\zeta_3 + \frac{1}{24}\pi^2\zeta_3 + \frac{9887}{480}\zeta_5 \\ &-\frac{595}{108}\pi^2\ln 2 - \frac{106}{675}\pi^4\ln 2 + \frac{6121}{2160}\pi^2\ln^2 2 \\ &-\frac{32}{135}\pi^2\ln^3 2 - \frac{6121}{2160}\ln^4 2 + \frac{32}{225}\ln^5 2 \\ &-\frac{6121}{90}a_4 - \frac{256}{15}a_5 \\ &+ \ell_{\mu M}\left[-\frac{383}{31104} + \frac{23}{108}\pi^2 - \frac{41}{144}\zeta_3\right.\end{aligned}$$

$$
- \frac{2}{9}\,\pi^2 \ln 2 \Big]
$$

$$
+ \quad \frac{43}{144}\,\ell^2_{\mu M} + \frac{13}{108}\,\ell^3_{\mu M} + \frac{1}{81}\,\ell^4_{\mu M}. \qquad (12)
$$

Note that in the process of deriving (12) one needs also $\overline{\Pi}^{(i)}(\mu^2/m^2)$ and $\overline{\Pi}^{(i)}(\mu^2/Q^2)$ for $i = 1,2,3$. We have not written the corresponding results explicitly to save space; the reader could find them e.g. in Ref. [27].

4. Asymptotic photon polarization operator in on-shell scheme

The combined use of eqs. (9,11,12) immediately leads us to the the following expression for the four-loop contribution to the asymptotic photon polarization[4]

$$
\begin{aligned}
\Pi^{\mathrm{as},(4)}(Q/M) = {} & \frac{5132143}{1555200} - \frac{8791}{3240}\,\pi^2 - \frac{205021}{259200}\,\pi^4 \\
& + \frac{5281673}{129600}\,\zeta_3 - \frac{1}{24}\,\pi^2\,\zeta_3 - \frac{5}{3}\,\zeta_3^2 - \frac{2303}{120}\,\zeta_5 \\
& - \frac{35}{4}\,\zeta_7 + \frac{595}{108}\,\pi^2 \ln 2 + \frac{106}{675}\,\pi^4 \ln 2 \\
& - \frac{6121}{2160}\,\pi^2 \ln^2 2 + \frac{32}{135}\,\pi^2 \ln^3 2 + \frac{6121}{2160} \ln^4 2 \\
& - \frac{32}{225} \ln^5 2 + \frac{6121}{90}\,a_4 + \frac{256}{15}\,a_5 \\
& + \ell_{QM} \left[-\frac{11833}{10368} - \frac{23}{108}\,\zeta_3 + \frac{5}{3}\,\zeta_5 \right] \\
& + \ell^2_{QM} \left[\frac{19}{144} - \frac{1}{9}\,\zeta_3 \right] - \frac{1}{108}\,\ell^3_{QM}. \qquad (13)
\end{aligned}
$$

Note that in the process of deriving (13) one needs also $\overline{\Pi}^{(i)}(\mu^2/m^2)$ and $\overline{\Pi}^{(i)}(\mu^2/Q^2)$ for $i = 1,2,3$. We have not write the corresponding results explicitly to save space; the reader could find them e.g. in Ref. [27].

[4]All terms proportional to ℓ in (13) could be derived in easy way from renormalization group arguments as explained in [30]. In addition, the process of constructing d_R^{as} one should invert the power series for $(1+\Pi^{\mathrm{as}})$, which also produces *remarkably easy-to-compute* factorizable fourth order contributions to d_R^{as} like $\alpha_s^4\,\Pi^{as,(2)}\,\Pi^{as,(2)}$ and so on (see e.g. [56] and references therein). However, to include power suppressed terms of order $(M/Q)^n$ to Π^{OS} is much less trivial even for the factorizable contributions [57,58].

5. eVP contribution to the muon anomaly at fifth order

Combining eqs. (1,13) we arrive to our main result: the complete eVP contribution to the muon anomaly at order α^5 (below we write down only the new five-loop result; we have confirmed all previously known eVP contributions to a_μ at three and four loops)

$$
a_\mu^{\mathrm{as}} = \sum_{i \ge 2} a_\mu^{\mathrm{as},(i)} \left(\frac{\alpha}{\pi}\right)^i, \qquad (14)
$$

$$
\begin{aligned}
a_\mu^{\mathrm{as},(5)} = {} & -\frac{296496193}{41990400} + \frac{45709}{58320}\,\pi^2 + \frac{212701}{518400}\,\pi^4 \\
& - \frac{4488523}{259200}\,\zeta_3 + \frac{35}{144}\,\pi^2\,\zeta_3 + \frac{4}{3}\,\zeta_3^2 + \frac{10909}{720}\,\zeta_5 \\
& + \frac{35}{8}\,\zeta_7 - \frac{55}{24}\,\pi^2 \ln 2 - \frac{53}{675}\,\pi^4 \ln 2 \\
& + \frac{6121}{4320}\,\pi^2 \ln^2 2 - \frac{16}{135}\,\pi^2 \ln^3 2 - \frac{6121}{4320} \ln^4 2 \\
& + \frac{16}{225} \ln^5 2 - \frac{6121}{180}\,a_4 - \frac{128}{15}\,a_5 \\
& + \ell_{\mu e} \left[\frac{1416095}{279936} + \frac{41}{972}\,\pi^2 - \frac{1855}{432}\,\zeta_3 \right. \\
& \qquad \left. - \frac{10}{3}\,\zeta_5 - \frac{2}{9}\,\pi^2 \ln 2 \right] \\
& + \ell^2_{\mu e} \left[-\frac{1507}{1944} + \frac{8}{81}\,\pi^2 + \frac{4}{3}\,\zeta_3 \right] \\
& - \frac{83}{243}\,\ell^3_{\mu e} + \frac{8}{81}\,\ell^4_{\mu e}, \qquad (15)
\end{aligned}
$$

where $\ell_{\mu e} = \log(M_\mu/M_e)$. Finally, numerically, the result reads:

$$
a_\mu^{\mathrm{as},(5)} = 62.2667 = 63.481_{\mathrm{NS}} - 1.21429_{\mathrm{SI}}. \qquad (16)
$$

Eq. (16) also displays the decomposition of the full result into two pieces, corresponding to *non-singlet* and *singlet* contributions to the photon polarization operator (see, e.g. [59]).

6. Conclusion and Acknowledgment

We hope that our main result (16) for the complete fifth order eVP contribution to the muon anomaly could be of some use for testing the complicated numerical simulations like it happened a few times in the past.

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Note added. Very recently, two months after the conference, the result for the five-loop eVP contribution to the muon anomaly from singlet diagrams have been published by T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio and N. Watanabe [60]. Their result reads:

$$A_2^{(10)}(m_\mu/m_e)(\text{Set } I(j)_{(e,e)} : \text{combined}) = -1.24726(12). \tag{17}$$

One observes a good agreement for the singlet case between the numerical result (17) and our prediction (16). The remaining 2.6 % difference presumably comes from power suppressed corrections to the asymptotic result.

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