Evaluating the three-loop static quark potential

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This is a status report of the evaluation of the three-loop corrections to the static QCD potential of a heavy quark and an antiquark. The families of Feynman integrals that appear in the evaluation are described. To reduce any integral of the families to master integrals we solve integration-by-parts relations by the algorithm called `FIRE`. To evaluate the corresponding master integrals we apply the Mellin–Barnes technique. First results are presented: the coefficients of n_l^3 and n_l^2, where n_l is the number of light quarks.

1. Introduction

The QCD potential between a static quark and its antiquark can be cast in the form

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s}{\vec{q}^2}\left[1+\frac{\alpha_s}{4\pi}a_1+\left(\frac{\alpha_s}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s}{4\pi}\right)^3\left(a_3+8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^2}\right)+\cdots\right], \quad (1)$$

where the renormalization scale of α_s is set to $\vec{q}^2$. The one-loop contribution a_1 is known since almost 30 years [1,2] and also the two-loop term has already been computed end of the nineties [3–5]. Furthermore logarithmic contributions are known at three- and four-loop level [6,7]. Explicit results are nicely summarized in the recent review [8]. The non-logarithmic third-order term, a_3, is still unknown.

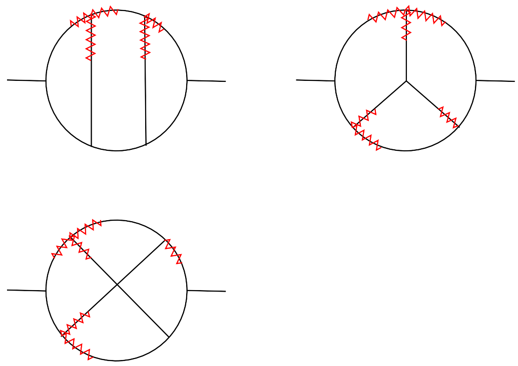

Figure 1. Families of Feynman integrals needed for the calculation of a_3. The solid lines correspond to relativistic massless propagators and the zigzag lines represents static propagators.

2. Reduction to master integrals

Any Feynman integral that contributes to a_3 can be mapped to one of the three graphs shown in Fig. 1 where solid lines stand for usual massless propagators of the form $1/(-p^2-i0)^{a_i}$ and wavy lines stand for linear propagators $1/(-v\cdot p-s_i i0)^{a_i}$, with $v\cdot p=p_0$, $s_i=\pm1$ and integer indices a_i. In case the latter type of propagators is absent the integrals reduce to usual massless two-point functions which can be treated with the help of `MINCER` [9]. Note, however, that the presence of the static lines significantly increases the complexity of the problem.

In general the integrals involve up to fifteen propagators (including an irreducible numerator). In order to simplify the reduction problem we ap-

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ply in a first step partial fraction identities to arrive at various subfamilies of integrals with at most three linear propagators. Thus any resulting integral is labeled by twelve indices one of which corresponds to an irreducible numerator and three indices correspond to linear propagators.

Let us in the following describe the Feynman integrals that are generated by the n_l contribution. Altogether we have to consider about 70 000 integrals (allowing for a general QCD gauge parameter ξ). As far as the "ladder" and "non-planar" diagrams are concerned we have to deal with the type of integrals shown in Fig. 2 where the linear propagators appear in the following form: If the loop momenta, k, l and r, in Fig. 2 are chosen as the momenta of the three upper lines, then the first diagram appears in two ways: either with the product

$$(-v\cdot k - i0)^{-a_9}(-v\cdot l - i0)^{-a_{10}}(-v\cdot r - i0)^{-a_{11}},$$

or with the product

$$(-v\cdot k + i0)^{-a_9}(-v\cdot l - i0)^{-a_{10}}(-v\cdot r - i0)^{-a_{11}}.$$

The second diagram appears with similar propagators where the momenta $\{k,l,r\}$ (in the first variant with $-i0$ in three places) are replaced by $\{k,k-l,r\}$. The third diagram in Fig. 2 corresponds to $\{k,l,l-r\}$ and the fourth one to $\{k,l,r\}$.

The integrals which correspond to the "Mercedes" graph are shown in Fig. 3. If we choose the loop momenta k, l and r as the momenta of the three lower lines these five diagrams appear with linear propagators of the form $(-v\cdot k - i0)$ with momenta $\{k,k-r,l\}$, $\{r,k-l,r-l\}$, $\{k,r,k-l\}$, $\{k,r,l\}$, $\{k,r-l,l\}$, respectively.

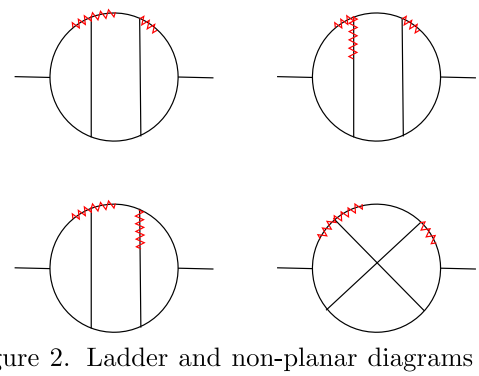

Figure 2. Ladder and non-planar diagrams contributing to the n_l part of a_3.

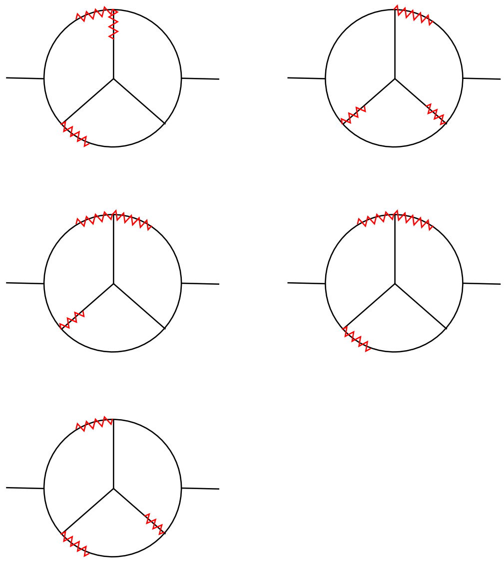

Figure 3. Mercedes diagrams contributing to the n_l part of a_3.

The next step is a reduction of all the nine types of these Feynman integrals to master integrals by solving integration-by-parts relations [10]. To do this we apply the algorithm called `FIRE` (Feynman Integral REduction) [11–16] which is based on an extension of the classical Buchberger algorithm to construct Gröbner bases (see, e.g., Ref. [17]).

Similarly to other approaches, we work in a given *sector*, i.e. a domain of integer indices a_i where some indices are positive and the rest of the indices are non-positive. The aim is to express any integral from the sector in terms of master integrals of this sector *and* integrals from lower sectors, where at least one more index is non-positive. It turns out that in the higher sectors (with a small number of non-positive indices) the corresponding s-basis [12–15] (a kind of a Gröbner basis) can be constructed easily (and, in most cases, even automatically).

In the opposite situation where a lot of non-positive indices occur, s-bases are constructed not so easily. Usually there is the possibility to explicitly perform an integration over some loop momentum for general value of ε with results in terms of gamma functions. A straightforward way to do this leads to multiple summations and turns out to be impractical. An advanced strategy within `FIRE` is to use s-bases for some regions of indices corresponding to a subintegral over such a loop momentum in order to reduce these indices to their boundary values. Then it is sufficient to use explicit integration formulae only for the boundary values. Integrals which are obtained from initial integrals by an explicit integration over a loop momentum in terms of gamma function usually involve a propagator with an analytic regularization by the shift ε or 2ε. After this integration we obtain a two-loop reduction problem with seven indices which is then solved by `FIRE`.

Finally, after using Gröbner bases in higher sectors and an explicit integration in lower sectors, it is still necessary to solve the reduction problem in a relatively small number of intermediate sectors. In these cases we turn to Laporta's algorithm [18,19] implemented as part of `FIRE`.

To reduce by `FIRE` all the integrals contributing to the n_l part of a_3 it took around ten days on a 2.3 GHZ Opteron computer with 8 GB operative memory. The most complicated integral in this reduction is $F(1,\ldots,1,1,-4,1,0)$ corresponding the nonplanar graph in Fig. 2 where the index of the irreducible numerator is zero and one of the static propagators is raised to the power -4. All other indices are equal to 1.

3. Evaluating master integrals

After using `FIRE` we know the master integrals. The number of the master integrals appearing in the n_l part of a_3 is around one hundred. For their evaluation we used the Mellin–Barnes (MB) technique which is based on a replacement of a sum of terms raised to some power by their products in some powers, at the cost of introducing additional integrations over contours of a complex plane. Then one takes explicitly all other integrations (over loop momenta and/or over alpha/Feynman parameters) and is left with a multiple MB integral.

For planar diagrams, experience shows that a minimal number of MB integrations is achieved if one introduces them loop by loop, i.e. one derives a MB representation for a one-loop subintegral, inserts it into a higher two-loop integral, etc.

Consider, for example, the dimensionally regularized Feynman integral corresponding to the first graph in Fig. 2 with the linear propagator $(-v\cdot k+i0)^{-a_9}$. Let us denote it by $F(a_1,\ldots,a_{11})$. A straightforward implementation of the loop-by-loop strategy leads to a six-fold MB representation which reads

$$
\begin{aligned}
F(a_1,\ldots,a_{11}) &= \frac{\left(i\pi^{d/2}\right)^3(-1)^{a_9}2^{a_{10,11}-1}}{(v^2)^{a_{9,10,11}/2}} \\
&\times\frac{(-q^2)^{6-a_{1,\ldots,8}-a_{9,10,11}/2-3\varepsilon}}{\prod_{i=3,4,7,8,11}\Gamma(a_i)\Gamma(4-a_{3,4,8,11}-2\varepsilon)} \\
&\times\frac{1}{(2\pi i)^6}\int_{-i\infty}^{+i\infty}\prod_{j=1}^{5}\left(\Gamma(-z_j)\mathrm{d}z_j\right)\frac{\Gamma(-2z_6)}{\Gamma(a_{10}-2z_6)} \\
&\times\frac{\Gamma(1/2-z_1)\Gamma(1/2-a_9/2+z_1)}{\Gamma(1/2+a_9/2+z_1)\Gamma(1/2-a_9/2-z_1)} \\
&\times\frac{\Gamma(4-a_{2,3,4,5,8}-a_{10,11}/2-2\varepsilon+z_1-z_{2,3})}{\Gamma(8-a_{1,\ldots,9}-a_{10,11}/2-4\varepsilon+z_1-z_{2,3})} \\
&\times\frac{\Gamma(-6+a_{1,\ldots,8}+a_{9,10,11}/2+3\varepsilon+z_{2,3})}{\Gamma(a_{1,2,3,4,5,7,8}+a_{10,11}/2+2\varepsilon-4+z_{1,2,3,4})} \\
&\times\frac{\Gamma(6-a_{1,2,3,4,5,7,8}-a_{9,10,11}/2-3\varepsilon-z_{2,3,4})}{\Gamma(a_6-z_4)\Gamma(a_5-z_5)} \\
&\times\frac{\Gamma(a_{2,3,4,5,7,8}+a_{10,11}/2+2\varepsilon-4+z_{1,2,3,4})}{\Gamma(6-a_{2,3,4,5,7,8,10}-a_{11}/2-3\varepsilon-z_2+z_6)} \\
&\times\frac{\Gamma(2-a_6-a_9/2-\varepsilon+z_{1,4})\Gamma(a_5+z_{3,4}-z_5)}{\Gamma(a_{2,3,4,8}+a_{11}/2+\varepsilon-2+z_{2,5,6})} \\
&\times\Gamma(a_4+z_{2,5})\Gamma(a_{10}/2+z_1-z_6)\Gamma(a_{11}/2+z_6) \\
&\times\Gamma(2-a_{4,8}-a_{11}/2-\varepsilon-z_{5,6}) \\
&\times\Gamma(2-a_{3,4}-a_{11}/2-\varepsilon-z_2+z_6) \\
&\times\Gamma(2-a_{5,7}-a_{10}/2-\varepsilon-z_{1,4}+z_{5,6}) \\
&\times\Gamma(a_{3,4,8}+a_{11}/2+\varepsilon-2+z_{2,5,6})\,, \qquad (2)
\end{aligned}
$$

where $a_{3,4,8,11}=a_3+a_4+a_8+a_{11}$, $z_{1,2,3}=z_1+z_2+z_3$, etc. By definition, any integration contour over z_i should go to the right (left) of poles of gamma functions with $+z$-dependence ($-z$-dependence).

There are two strategies for resolving the singularities in ε in MB integrals suggested in Refs. [20, 21] (see also Chapter 4 of [22,23]). The second one was formulated algorithmically [24,25], and the corresponding public code `MB.m` [25] has become by now a standard way to evaluate MB integrals in an expansion in ε. Using `MB.m` and evaluating the resulting finite MB integrals by corollaries of Barnes lemmas we have obtained, for example, the following result for one our master integrals:

$$F(1,\ldots,1,0,1) = -\frac{(i\pi^{d/2})^3}{(-q^2)^{3+3\varepsilon}v^2}\left[-\frac{64\pi^4}{135\varepsilon} - \frac{128\pi^4}{135} - \frac{32\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\varepsilon)\right]. \qquad (3)$$

In contrast to a similar Feynman integral with the linear propagator $1/(-v\cdot k - i0)$ (see Ref. [26]) one needs, at a first step of the resolution of the singularities in ε, an auxiliary analytic regularization. In fact, the result in Eq. (3) and the corresponding one for the case of $1/(-v\cdot k - i0)^{a_9}$ (see Ref. [26]) differ by a term which is nothing but a residue taken when performing the initial analytical continuation to the auxiliary parameter of analytic regularization.

The above integral is an example where products of the type $1/(x+i0)^a 1/(x-i0)^b$ appear, with $x = -v\cdot k, -v\cdot l$, etc. If a and b are positive integers, these products are ill-defined. For example, they do appear in diagrams resulting from iterations of the Coulomb potential. However, for integrals in Eq. (2), and many other integrals that we encounter, in these products at least one of the exponents is regularized by an amount proportional to ε and/or z (i.e. a variable of the MB integration). As a consequence they are well-defined, in the sense of analytic continuation. More explicitly, we have

$$(x+i0)^a(x-i0)^b = \frac{\sin(\pi a)e^{-i\pi b}}{\sin(\pi(a+b))}(x+i0)^{a+b} + \frac{\sin(\pi b)e^{i\pi a}}{\sin(\pi(a+b))}(x-i0)^{a+b}, \qquad (4)$$

for general complex numbers a and b.

All the master integrals that we encounter are expressed in terms of $\zeta(i)$, i=2,3,4,5,6, powers of $\ln 2$, $\mathrm{Li}_j(1/2), j = 4,5,6$, and the constant s_6 (see, e.g., [27]) up to transcendentality level six. With the assumption about the presence of these numbers in the results, one can apply the PSLQ algorithm [28] in situations where explicit integration over MB parameters by corollaries of Barnes lemmas is no longer possible. If this is just a onefold MB integral then it is possible to obtain a numerical result with more than 300 digits which are sufficient for the PSLQ algorithm. For twofold MB integrals, one can hope to obtain an accuracy of 50 digits depending on the complexity of the integrand. This accuracy can also be sufficient for a successful application of the PSLQ algorithm, at least when a given integral possesses homogeneous transcendentality. Unfortunately, sufficient criteria about homogeneous transcendentality are not known at the moment so that this property can be seen only experimentally, by considering lower terms of the expansion in ε.

4. Preliminary results and perspectives

The three-loop correction to the static quark potential can conveniently be parameterized in the form

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}, \qquad (5)$$

where n_l denotes the number of massless quarks. Using the techniques described above we evaluated the coefficients $a_3^{(3)}$ and $a_3^{(2)}$ which read

$$\begin{aligned} a_3^{(3)} &= -\left(\frac{20}{9}\right)^3 T_F^3, \\ a_2^{(2)} &= \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 \\ &\quad + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2. \end{aligned} \qquad (6)$$

The n_l^3 contribution together with the $C_A T_F^2$ part has already been presented in Ref. [26].

The coefficient $a_3^{(1)}$ is also reachable. Only four constants contributing to the higher-order expansion in ε of some master integrals are not known analytically at the moment.

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