TTP 08-22
SFB/CPP-08-36

Multi-Loop Calculations, Quark Masses and the Strong Coupling Constant

J. H. Kühn [a]

[a]Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany

The impact of multi-loop calculations on precise determinations of charm- and bottom-quark masses and the strong coupling constant is discussed. Recent N^3LO calculations, combined with new precision data, lead to a significant reduction of the errors. The results of these analyses, $m_c(3\text{ GeV}) = 0.986(13)$ GeV, $m_b(10\text{ GeV}) = 3.609(19)$ GeV and $\alpha_s(M_Z) = 0.1198(15)$ constitute the most precise and presently only N^3LO determinations of these fundamental quantities.

The strong coupling constant and the quark masses are the fundamental input parameters of the theory of strong interaction. Quark masses are an essential input for the evaluation of weak decay rates of heavy mesons and for quarkonium spectroscopy. Decay rates and branching ratios of a light Higgs boson, suggested by electroweak precision measurements, depend critically on the masses of the charm and bottom quarks, m_c and m_b. Last not least, confronting the predictions for these masses with experiment is an important task for all variants of Grand Unified Theories.

Similarly the strong coupling constant α_s is one of the three fundamental gauge couplings constants of the Standard Model (SM) of particle physics. Its precise determination is one of the most important aims of particle physics. Experiments at different energies allow to test the predictions for its energy dependence based on the renormalization group equations, the comparison of the results obtained from different processes leads to critical tests of the theory and potentially to the discovery of physics beyond the Standard Model. Last not least, the convergence of the three gauge coupling constants related by SU(3)xSU(2)xU(1) to a common value, after evolving them to high energies, allows to draw conclusions about the possibility of embedding the SM in the framework of a Grand Unified Theory.

Precise theoretical calculations, combined with excellent measurements, are the decisive prerequisite for an optimal determination of these fundamental parameters. Advanced techniques of perturbative QCD have been developed during the last years, which have led to significantly improved determinations of quark masses and the strong coupling constant. These techniques and a few selected results will be presented in the following.

Quark masses

A detailed analysis of m_c and m_b based on the ITEP sum rules [1] has been performed several years ago [2] and lead to $m_c(m_c) = 1.304(27)$ GeV and $m_b(m_b) = 4.191(51)$ GeV. During the past years new and more precise data for $\sigma(e^+e^- \to \text{hadrons})$ have become available in the low energy region, in particular for the parameters of the charmonium and bottomonium resonances. Furthermore, the error in the strong coupling constant $\alpha_s(M_Z)$ has been reduced. Last not least, the vacuum polarization induced by massive quarks has been computed in four-loop approximation [3,4]; more precisely: its first derivative at $q^2 = 0$ has been evaluated, which corresponds to the lowest moment of the familiar R-ratio. Based on these developments a new determination of the quark masses has been performed in Ref. [5].

The extraction of m_Q from low moments of the cross section $\sigma(e^+e^- \to Q\bar{Q})$ exploits its sharp rise close to the threshold for open charm and bottom production and the importance of the contributions from the narrow quarkonium resonances.

By evaluating the moments

$$\mathcal{M}_n^{\rm exp} \quad \equiv \quad \int \frac{{\rm d}s}{s^{n+1}} R_Q(s)\,, \tag{1}$$

with low values of n, the long distance contributions are averaged out and $\mathcal{M}_n$ involves short distance physics only, with a characteristic scale of order $E_{\rm threshold} = 2m_Q$. Through dispersion relations the moments are directly related to derivatives of the vacuum polarization function at $q^2 = 0$,

$$\mathcal{M}_n^{\rm theor} \quad = \quad \frac{12\pi^2}{n!}\left(\frac{\rm d}{{\rm d}q^2}\right)^n \Pi_Q(q^2)\bigg|_{q^2=0}\,, \tag{2}$$

which can be evaluated in perturbative QCD.

In its domain of analyticity $\Pi_Q(q^2)$ can be cast into the form

$$\Pi_Q(q^2) \quad = \quad Q_Q^2 \frac{3}{16\pi^2} \sum_{n\geq 0} \bar{C}_n \left(\frac{q^2}{4m_Q^2}\right)^n\,, \tag{3}$$

where $m_Q = m_Q(\mu)$ is the $\overline{\rm MS}$ mass at the scale μ. The perturbative series for the coefficients $\bar{C}_n$ in order α_s^2 was evaluated in Ref. [6], the four-loop contributions to $\bar{C}_0$ and $\bar{C}_1$ in Refs. [3,4]. More recently even $\bar{C}_2$ has been evaluated [7]. Combining Eqs. (1), (2) and (3), the charm quark mass can be obtained:

$$m_c(\mu) \quad = \quad \frac{1}{2}\left(\frac{\bar{C}_n}{\mathcal{M}_n^{\rm exp}}\right)^{1/(2n)}\,. \tag{4}$$

The moment with $n = 1$ is least sensitive to non-perturbative contributions from condensates, to the Coulombic higher order effects, the variation of μ and the parametric α_s dependence [5]:

$$m_c(3~{\rm GeV}) \quad = \quad 0.986(13)~{\rm GeV}\,, \tag{5}$$

Transforming this to the scale-invariant mass $m_c(m_c)$ one finds $m_c(m_c) = 1.286(13)$ GeV. Using the three-loop relation [8,9] between pole- and $\overline{\rm MS}$-mass this leads to $M_c^{(3-{\rm loop})} = 1.666$ GeV.

Instead of measuring the moments $\mathcal{M}_n^{\rm exp}$ in e^+e^- annihilation they can also be determined in lattice simulations. This approach has recently been pioneered in [10] using the Highly Improved Staggered Quarks (HISQ) discretization of the quark action [11,12] to simulate the moments of the pseudoscalar correlator. The final result, $m_c(3~{\rm GeV}) = 0.984(16)$ GeV is in excellent agreement with the determinations based on e^+e^- data.

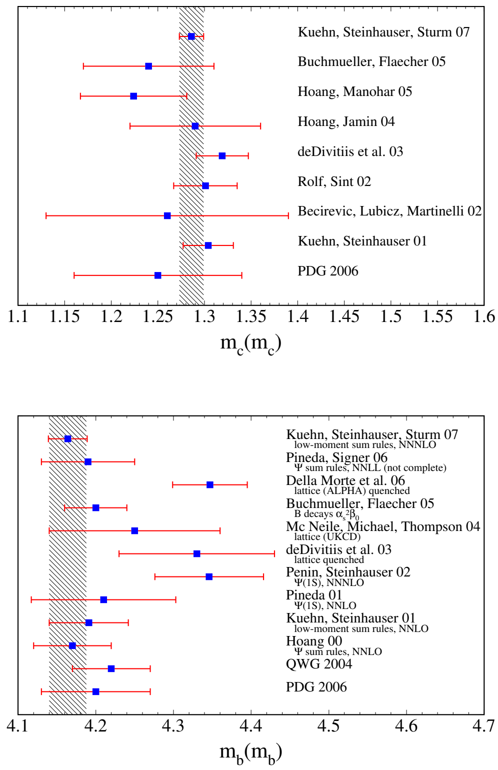

Figure 1. Comparison of recent determinations of $m_c(m_c)$ and $m_b(m_b)$.

The approach based on e^+e^- data is also applicable to the determination of m_b. The three results based on $n = 1, 2$ and 3 are of comparable precision. The relative size of the contributions from the threshold and the continuum region decreases for the moments $n = 2$ and 3. On the other hand, the theory uncertainty estimated from the variation of μ and the unknown four-loop contribution is still acceptable. Therefore the result from $n = 2$ is taken as the final answer

[5],

$$m_b(10 \text{ GeV}) = 3.609(25) \text{ GeV}, \tag{6}$$

corresponding to $m_b(m_b) = 4.164(25)$ GeV and a pole mass of $M_b^{(3-\text{loop})} = 4.800$ GeV. The recent evaluation [7] of $\bar{C}_2$ leads to a decrease of the central value by 2 MeV and a reduction of the error from 25 MeV to 19 MeV. A comparison of a few selected m_b- and m_c-determinations is shown in Fig. 1.

For various applications, either related to Z-boson decays or in connections to Grand Unified Theories (GUTs), the values of $m_b(\mu)$ at M_Z and $m_t(m_t) = 161.8 \pm 2.0$ GeV (as derived from $M_t = 171.4 \pm 2.1$ GeV) are of interest:

$$\begin{aligned} m_b(M_Z) &= 2.834 \pm 0.019 \pm 0.017 \text{ GeV} \\ m_b(161.8) &= 2.703 \pm 0.018 \pm 0.019 \text{ GeV}. \end{aligned} \tag{7}$$

The strong coupling constant

One of the most precise and theoretically safe determination of α_s is based on measurements of the cross section for electron-positron annihilation into hadrons. These have been performed in the low-energy region between 2 GeV and 10 GeV and, in particular, at and around the Z resonance at 91.2 GeV. Conceptually closely related is the measurement of the semileptonic decay rate of the τ-lepton, leading to a determination of α_s at a scale below 2 GeV.

From the theoretical side, in the framework of perturbative QCD, these rates and cross sections are evaluated as inclusive rates into massless quarks and gluons [13,14]. (Power suppressed mass effects are well under control for e^+e^--annihilation, both at low energies and around the Z resonance, and for τ decays [15–18], and the same applies to mixed QCD and electroweak corrections [19,20]).

The ratio $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is expressed through the absorptive part of the correlator of the electromagnetic current j_μ:

$$\begin{aligned} R(s) &= 12\pi\, \Im\Pi(-s - i\epsilon), \\ 3\,Q^2\,\Pi(Q^2) &= i \int \mathrm{d}^4x e^{iq\cdot x} \langle 0|\mathrm{T} j_\mu(x) j^\mu(0)|0\rangle\,, \end{aligned} \tag{8}$$

with $Q^2 = -q^2$. It is also convenient to introduce the Adler function as

$$D(Q^2) = -12\,\pi^2 Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) = \int \frac{Q^2\ R(s)ds}{(s+Q^2)^2}\,,$$

We define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} d_i a_s^i(Q^2), \quad R(s) = \sum_{i=0}^{\infty} r_i a_s^i(s), \tag{9}$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = Q^2$ or to $\mu^2 = s$ for the Euclidian and Minkowskian functions respectively.

Note that the first three terms of the perturbative series for D and R coincide. Starting from r_3, terms proportional π^2 arise which can be predicted from those of lower order. It has been speculated that these "π^2-terms", also called "kinematical terms", might constitute a major part of the full higher order corrections (see, e.g. [21]); however, the validity of this hypothesis can only be established by the full calculation. Indeed, for the scalar correlator this assumption has been shown to fail completely [22].

For the vector correlator the terms of order a_s^2 and a_s^3 have been evaluated nearly thirty and about fifteen years ago [23–26], respectively. The a_s^4 corrections are conveniently classified according to their power of n_f, with n_f denoting the number of light quarks. The $a_s^4 n_f^3$ term is part of the "renormalon chain" , the next term of order $a_s^4 n_f^2$ was evaluated in [27].

The complete five-loop calculation has been performed in [28] using QGRAF [29] for the automatic generation of Feynman diagrams and the field theoretical methods described in more detail in [28,30–33], using a parallel version [34] of FORM [35].

The so-called "non-singlet" diagrams, evaluated in [28], are sufficient for a complete description of τ-decays, for e^+e^- annihilation and Z decays they correspond to the dominant terms.

The analytic result for the Adler function contains transcendental numbers up to ζ_7, with ζ_4 and ζ_6 being absent. It leads straightforwardly to R at order α_s^4. In numerical form

$$\begin{aligned} R &= 1 + a_s + (1.9857 - 0.1152\,n_f)\,a_s^2 \\ &+ (-6.63694 - 1.20013 n_f - 0.00518 n_f^2)\,a_s^3 \end{aligned}$$

$$+ \quad (-156.61 + 18.77\, n_f - 0.7974\, n_f^2 \qquad (10)$$
$$+ \quad 0.0215\, n_f^3)\, a_s^4.$$

Using the Principle of Fastest Apparent Convergences [36] and the exact result for d_4, the coefficients d_5 and r_5 can be predicted (following [37,38])

$$\begin{aligned} d_5^{\rm pred}(n_f = 3,4,5) &= 275, \quad 152, \quad 89, \\ r_5^{\rm pred}(n_f = 3,4,5) &= -505, \quad -134, \quad 168\,. \end{aligned} \qquad (11)$$

These estimates may become of relevance for the International Linear Collider (ILC) running in the GIGA-Z mode with an anticipated precision of $\delta\alpha_s = 0.0005$ — 0.0007 [39], and already today for the analysis of τ-decays.

Let us now move to the analysis of present data for e^+e^- annihilation and τ decays. Measurements of $R(s)$ at lower energies, with their correspondingly larger values of α_s, are in principle more sensitive to $\alpha_s(M_Z)$ if the same relative precision could be obtained. At present, however, the systematic experimental error of 2% is a limiting element for a competitive measurement. Nevertheless, it is interesting, that the result of a recent analysis [40] $\alpha_s(9^2\ {\rm GeV}^2) = 0.160 \pm 0.034$ is a clear 3σ signal for the influence of QCD corrections also in this energy range. Furthermore, as shown in this paper, proper inclusion of quark mass effects and matching at flavour threshold is required to obtain the correct result.

Evolving this value from 9 GeV to M_Z leads to $\alpha_s(M_Z^2) = 0.110^{+0.014}_{-0.017}$. It is instructive to combine this result with the one obtained in Ref. [2], $\alpha_s^{(4)}(5^2{\rm GeV}^2) = 0.235^{+0.047}_{-0.047}$, which was based on earlier measurements in the region between 3 and 10GeV. The final result $\alpha_s^{(4)}(9^2{\rm GeV}^2) = 0.182^{+0.022}_{-0.025}$ represents the combined information on the strong coupling from these R measurements in the region below the bottom threshold. The shift in α_s from the inclusion of the α_s^4 term is small compared to the large experimental error.

In contrast to the situation below 10 GeV the extraction of α_s from Z-decays is affected by the α_s^4 terms. The $\mathcal{O}(\alpha_s^3)$ analysis of the electroweak working group [41] finds

$$\alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026^{\rm exp} \qquad (12)$$

Including the α_s^4 term [28] leads to a shift $\delta\alpha_s(M_Z) = 0.0005$

$$\alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\rm exp}, \qquad (13)$$

The theory error can now safely be neglected.

Higher orders are of particular relevance in the low-energy region, for example in τ decays. The correction from perturbative QCD to the ratio $\Gamma(\tau \to {\rm hadrons}_{S=0} + \nu_\tau)/\Gamma(\tau \to l + \bar\nu_l + \nu_\tau)$ is given by the factor

$$1 + \delta_0 = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2\left(1 + \frac{2s}{M_\tau^2}\right) R(s).$$

The perturbative quantity δ_0 can be evaluated in Fixed Order perturbation theory or with "Contour Improvement" as proposed in [42,43]

$$\delta_0^{FO} = a_s + 5.202\ a_s^2 + 26.366\ a_s^3 + 127.079\, a_s^4, \qquad (14)$$

$$\delta_0^{CI} = 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + 64.29\, a_s^4. \qquad (15)$$

From the experimental side $\delta_0^{\rm exp} = 0.1998 \pm 0.0043_{\rm exp}$ has been used [44]. Starting from $\delta_0^{\rm exp}$one easily obtains the new values of $\alpha_s(M_\tau)$. Its dependence on the choice of d_4 (previously estimate and the new exact result) is summarized in Table 1.

Table 1
Results for $\alpha_s(M_\tau)$ for different values of d_4. The first line displays the α_s^3 results (with the α_s^4 terms set to zero in eqs. (14,15)). The second line uses the previously predicted value for d_4, the last one uses the exact result (10). The first error is the experimental one; the second (theoretical) uncertainty in the value of α_s corresponds to changing the renormalization scale μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$.

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

As stated above the theory error for α_s from Z decays is small compared to the experimental uncertainties. The situation is more problematic for τ decays and to some extent the theory error remains to be matter of choice. As anticipated in [38] it decreases significantly, once α_s^4 terms are included. However, the difference between the two methods stabilizes (this was checked in [38] by adding an estimate for the α_s^5 term) and must be considered as irreducible uncertainty, with the final result [28]

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm th}. \quad (16)$$

For the central value the mean value of FO and CI has been used, for the theory error half of the difference between two methods (0.01) plus (module of) the estimated correction from α_s^5 term (-0.005), the latter being based on $d_5 = 275$ (see eq. (11)). Applying four-loop running and matching [45–48] one finds

$$\begin{aligned} a_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm th} \\ &\quad \pm 0.0003_{\rm evol}\ , \\ &= 0.1202 \pm 0.0019. \end{aligned} \quad (17)$$

Here the evolution error receives contributions from the uncertainties in the quark masses ($m_c(m_c) = 1.286(13)$ GeV and $m_b(m_b) = 4.164(25)$ GeV [5]), the matching scale, the four-loop truncation in the matching expansion and in the RGE equation.

Note that the shifts in $\alpha_s(M_Z)$ from Z- and τ-decays, are opposite in sign and move the values in the *proper* direction, decreasing, thus, the current slight mismatch between two independent determinations of α_s. The final results

$$\begin{aligned} \alpha_s(M_Z)|_Z &= 0.1190 \pm 0.0026 \\ \alpha_s(M_Z)|_\tau &= 0.1202 \pm 0.0019 \end{aligned}$$

from these two observables, are in remarkable agreement. This constitutes a striking test of asymptotic freedom in QCD. The two values can be combined to

$$\alpha_s(M_Z) = 0.1198 \pm 0.0015.$$

This is one of the most precise and presently only N^3LO result.

Acknowledgement
I would like to thank P. Baikov, K. Chetyrkin, M. Steinhauser and C. Sturm for a fruitful and pleasant collaberation. Work supported by SFB/TR-9 "Computational Particle Physics" .

REFERENCES

1. V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Phys. Rept. **41** (1978) 1.
2. J. H. Kühn and M. Steinhauser, Nucl. Phys. B **619** (2001) 588 [Erratum-ibid. B **640** (2002) 415] [arXiv:hep-ph/0109084].
3. K. G. Chetyrkin, J. H. Kühn and C. Sturm, Eur. Phys. J. C **48** (2006) 107 [arXiv:hep-ph/0604234].
4. R. Boughezal, M. Czakon and T. Schutzmeier, Phys. Rev. D **74** (2006) 074006 [arXiv:hep-ph/0605023].
5. J. H. Kühn, M. Steinhauser, and C. Sturm, Nucl. Phys. **B778**, 192 (2007), `hep-ph/0702103`.
6. K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, Phys. Lett. B **371** (1996) 93 [arXiv:hep-ph/9511430]; Nucl. Phys. B **505** (1997) 40 [arXiv:hep-ph/9705254].
7. A. Maier, P. Maierhöfer, P. Marquard, [arXiv:0806.3405[hep-ph]] and these proceedings
8. K. G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. **83** (1999) 4001 [arXiv:hep-ph/9907509]; Nucl. Phys. B **573** (2000) 617 [arXiv:hep-ph/9911434].
9. K. Melnikov and T. v. Ritbergen, Phys. Lett. B **482** (2000) 99 [arXiv:hep-ph/9912391].
10. I. Allison *et al.*, arXiv:0805.2999 [hep-lat].
11. E. Follana *et al.* [HPQCD Collaboration], Phys. Rev. D **75** (2007) 054502 [arXiv:hep-lat/0610092].
12. E. Follana, C. T. H. Davies, G. P. Lepage and J. Shigemitsu [HPQCD Collaboration], Phys. Rev. Lett. **100** (2008) 062002 [arXiv:0706.1726 [hep-lat]].
13. K. G. Chetyrkin, J. H. Kühn, and A. Kwiatkowski, Phys. Rep. **277**, 189 (1996), `and references therein.`
14. M. Davier, A. Hocker, and Z. Zhang, Rev. Mod. Phys. **78**, 1043 (2006), `and references therein.`

15. K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Lett. **B371**, 93 (1996), hep-ph/9511430; Nucl. Phys. **B482**, 213 (1996), hep-ph/9606230.
16. K. G. Chetyrkin and J. H. Kühn, Phys. Lett. **B248**, 359 (1990); Nucl. Phys. **B432**, 337 (1994), hep-ph/9406299.
17. K. G. Chetyrkin, R. V. Harlander, and J. H. Kühn, Nucl. Phys. **B586**, 56 (2000), hep-ph/0005139.
18. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. Proc. Suppl. **135**, 243 (2004).
19. A. Czarnecki and J. H. Kühn, Phys. Rev. Lett. **77**, 3955 (1996), hep-ph/9608366.
20. R. Harlander, T. Seidensticker, and M. Steinhauser, Phys. Lett. **B426**, 125 (1998), hep-ph/9712228.
21. D. V. Shirkov (2000), hep-ph/0009106.
22. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **96**, 012003 (2006), hep-ph/0511063.
23. K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **B85**, 277 (1979).
24. M. Dine and J. R. Sapirstein, Phys. Rev. Lett. **43** (1979) 668.
W. Celmaster and R. J. Gonsalves, Phys. Rev. Lett. **44** (1980) 560.
25. S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. **B259**, 144 (1991).
26. L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66** (1991) 560 [Erratum-ibid. **66** (1991) 2416].
K. G. Chetyrkin, Phys. Lett. B **391** (1997) 402 [arXiv:hep-ph/9608480].
27. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **88**, 012001 (2002), hep-ph/0108197.
28. P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, arXiv:0801.1821 [hep-ph].
29. P. Nogueira, J. Comput. Phys. **105**, 279 (1993).
30. A. A. Vladimirov, Theor. Math. Phys. **43**, 417 (1980).
31. K. G. Chetyrkin and V. A. Smirnov, Phys. Lett. **B144**, 419 (1984).
32. K. G. Chetyrkin, Phys. Lett. **B391**, 402 (1997), hep-ph/9608480.
33. P. A. Baikov, Phys. Lett. **B385**, 404 (1996), hep-ph/9603267;
34. M. Tentyukov et al. (2004), cs.sc/0407066.
35. J. A. M. Vermaseren (2000), math-ph/0010025.
36. G. Grunberg, Phys. Rev. **D29**, 2315 (1984).
37. A. L. Kataev and V. V. Starshenko, Nucl. Phys. Proc. Suppl. **39BC**, 312 (1995), hep-ph/9408395.
38. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. **D67**, 074026 (2003), hep-ph/0212299.
39. M. M. Winter (2001), ECFA-DESY Linear Collider Note LC-PHSM-2001-016.
40. J. H. Kühn, M. Steinhauser, and T. Teubner, Phys. Rev. **D76**, 074003 (2007), arXiv:0707.2589 [hep-ph].
41. J. Alcaraz et al. (LEP Collaboration and ALEPH Collaboration and DELPHI Collaboration and L3 Collaboration and OPAL Collaboration and LEP Electroweak Working Group) (2007), arXiv:0712.0929 [hep-ex].
42. A. A. Pivovarov, Z. Phys. **C53**, 461 (1992), hep-ph/0302003.
43. F. Le Diberder and A. Pich, Phys. Lett. **B286**, 147 (1992).
44. M. Davier, A. Hocker, and Z. Zhang, Nucl. Phys. Proc. Suppl. **169**, 22 (2007), hep-ph/0701170.
45. T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. **B400**, 379 (1997), hep-ph/9701390.
46. M. Czakon, Nucl. Phys. **B710**, 485 (2005), hep-ph/0411261.
47. K. G. Chetyrkin, J. H. Kühn, and C. Sturm (2005), hep-ph/0512060.
48. Y. Schroder and M. Steinhauser, JHEP **01**,051 (2006), hep-ph/0512058.