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Precise Determinations of the Strong Coupling Constant

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Recent results for the cross section of electron-positron annihilation into hadrons and for the decay rates of the Z-boson and the τ-lepton into hadrons including corrections of order α_s^4 are reviewed. The consistency between two values of α_s measured at vastly different energies constitutes a striking test of asymptotic freedom. Combining the results from Z and τ decays we find $\alpha_s(M_Z) = 0.1198 \pm 0.0015$ as one of the most precise and presently only NNNLO result for the strong coupling constant.

The strong coupling constant α_s is one of the three fundamental gauge couplings constants of the Standard Model (SM) of particle physics. Its precise determination is one of the most important aims of particle physics. Experiments at different energies allow to test the predictions for its energy dependence based on the renormalization group equations, the comparison of the results obtained from different processes leads to critical tests of the theory and potentially to the discovery of physics beyond the Standard Model. Last not least, the convergence of the three gauge coupling constants related by SU(3)xSU(2)xU(1) to a common value, after evolving them to high energies, allows to draw conclusions about the possibility of embedding the SM in the framework of a Grand Unified Theory.

A characteristical example is shown in Fig.1, which demonstrates in particular the anticipated improvement in our information about coupling constant unification at the GUT scale, if the experimental uncertanty in α_s will be reduced from 0.0027 (corresponding to the value quoted in [1]) to 0.0009, a precision that might be within reach with 10^9 Z-events at a GIGA-Z [2]. In the latter case it may well be possible to deduce nontrivial constraints on threshold corrections and the structure of symmetry breaking at the GUT-scale.

One of the most precise and theoretically safe determination of α_s is based on measurements of the cross section for electron-positron annihilation into hadrons. These have been performed in the low-energy region between 2 GeV and 10 GeV

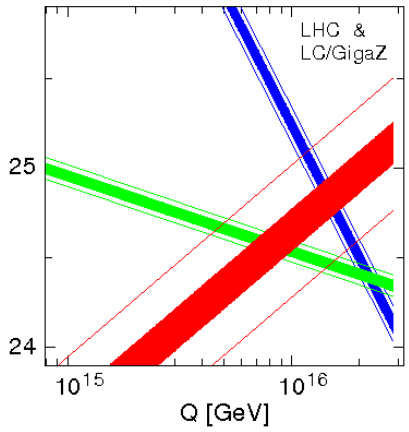

Figure 1. Renormalization group equation of the three coupling constants α_1, α_2 and α_3 [3]

and, in particular, at and around the Z resonance at 91.2 GeV. Conceptually closely related is the measurement of the semileptonic decay rate of the τ-lepton, leading to a determination of α_s at a scale below 2 GeV.

From the theoretical side, in the framework of perturbative QCD, these rates and cross sections are evaluated as inclusive rates into massless quarks and gluons [4,5]. (Power suppressed mass effects are well under control for e^+e^--annihilation, both at low energies and around the Z resonance, and for τ decays [6–9], and the same applies to mixed QCD and electroweak correc-

tions [10,11]).

The ratio $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is expressed through the absorptive part of the correlator of the electromagnetic current j_μ:

$$R(s) = 12\pi\,\Im\Pi(-s-i\epsilon), \quad (1)$$

$$3\,Q^2\,\Pi(Q^2) = i\int \mathrm{d}^4x e^{iq\cdot x}\langle 0|\mathrm{T}j_\mu(x)j^\mu(0)|0\rangle, \quad (2)$$

with $Q^2 = -q^2$. It is also convenient to introduce the Adler function as

$$D(Q^2) = -12\,\pi^2 Q^2\,\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) = \int_0^\infty \frac{Q^2\;R(s)ds}{(s+Q^2)^2},$$

We define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} d_i a_s^i(Q^2),\;\; R(s) = \sum_{i=0}^{\infty} r_i a_s^i(s), \quad (3)$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = Q^2$ or to $\mu^2 = s$ for the Euclidian and Minkowskian functions respectively. The results for generic values of μ can be easily recovered with standard RG techniques.

Note that the first three terms of the perturbative series for D and R coincide. Starting from r_3, terms proportional π^2 arise which can be predicted from those of lower order. It has been speculated that these "π^2-terms", also called "kinematical terms", might constitute a major part of the full higher order corrections (see, e.g. [12]); however, the validity of this hypothesis can only be established by the full calculation. Indeed, for the scalar correlator this assumption has been shown to fail completely [13].

For the vector correlator the terms of order a_s^2 and a_s^3 have been evaluated nearly thirty and about fifteen years ago [14,15], respectively. The a_s^4 corrections are conveniently classified according to their power of n_f, with n_f denoting the number of light quarks. The $a_s^4 n_f^3$ term is part of the "renormalon chain", the evaluation of the next term, of order $a_s^4 n_f^2$, was a test case for the techniques used extensively in this paper and, furthermore, led to useful insights into the structure of the perturbative series already [16].

The complete five-loop calculation has been performed in [17] using QGRAF [18] for the automatic generation of Feynman diagrams and the field theoretical methods described in more detail in [17,19–22]. Using a parallel version [23] of FORM [24], this direct and largely automatic procedure required enormous computing resources.

The so-called "non-singlet" diagrams, evaluated in [17], are sufficient for a complete description of τ-decays. For e^+e^- annihilation they correspond to the dominant terms proportional $\sum_i Q_i^2$. The singlet contributions proportional $(\sum_i Q_i)^2$ arise for the first time in $\mathcal{O}(\alpha_s^3)$ and are known to be small. Similar comments apply to the singlet contributions in Z decays.

The analytic result for the Adler function contains transcendental numbers up to ζ_7, with ζ_4 and ζ_6 being absent. It leads straightforwardly to R at order α_s^4. In numerical form

$$\begin{aligned} R &= 1 + a_s + (1.9857 - 0.1152\,n_f)\,a_s^2 \\ &+ (-6.63694 - 1.20013 n_f - 0.00518 n_f^2)\,a_s^3 \\ &+ (-156.61 + 18.77\,n_f - 0.7974\,n_f^2 \\ &+ 0.0215\,n_f^3)\,a_s^4. \end{aligned} \quad (4)$$

Using the Principle of Fastest Apparent Convergences [25] and the exact result for d_4, the coefficients d_5 and r_5 can be predicted (following [26,27]) for $n_f = 3, 4, 5$, namely

$$\begin{aligned} d_5^{\text{pred}}(n_f = 3,4,5) &= 275,\;\; 152,\;\; 89, \quad (5) \\ r_5^{\text{pred}}(n_f = 3,4,5) &= -505,\;\; -134,\;\; 168\,. \end{aligned}$$

These estimates may become of relevance for the International Linear Collider (ILC) running in the GIGA-Z mode with an anticipated precision of $\delta\alpha_s = 0.0005$ — 0.0007 [2], and already today for the analysis of τ-decays.

Let us now move to the analysis of present data for e^+e^- annihilation and τ decays. Measurements of $R(s)$ at lower energies, with their correspondingly larger values of α_s, are in principle more sensitive to $\alpha_s(M_Z)$ if the same relative precision could be obtained. At present, however, the systematic experimental error of 2% is a limiting element for a competitive measurement. Nevertheless, it is interesting, that the result of a recent analysis $\alpha_s(9^2\;\text{GeV}^2) = 0.160 \pm 0.034$

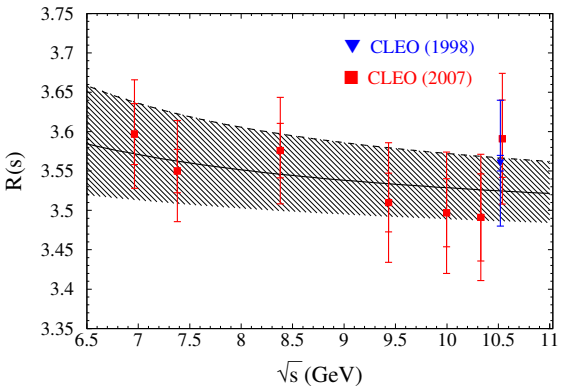

Figure 2. $R(s)$ as measured by CLEO [32, 33]. The full solid line and the hatched band correspond to the theory prediction based on $\alpha_s(M_Z^2) = 0.110^{-0.014}_{-0.017}$. The dashed line represents theory prediction where results from the CLEO analysis [33], $\alpha_s^{(5)}(M_Z) = 0.126$, has been used.

[28] is a clear 3σ signal for the influence of QCD corrections also in this energy range. Furthermore, as shown in [28], proper inclusion of quark mass effects and matching at flavour threshold is required to obtain the correct result (Fig. 2).

Evolving this value from 9 GeV to M_Z leads to $\alpha_s(M_Z^2) = 0.110^{+0.014}_{-0.017}$. It is instructive to combine this result with the one obtained in Ref. [29], $\alpha_s^{(4)}(5^2\text{GeV}^2) = 0.235^{+0.047}_{-0.047}$, which was based on earlier measurements by BES [30], MD-1 [31] and CLEO [32]. Adding the correlated and uncorrelated errors of the different experiments in quadrature, the final result $\alpha_s^{(4)}(9^2\text{GeV}^2) = 0.182^{+0.022}_{-0.025}$ represents the combined information on the strong coupling from measurements in the region below the bottom threshold. The shift in α_s from the inclusion of the α_s^4 term amounts to $\delta\alpha_s(9\text{GeV}) = 0.002$ and is thus irrelevant compared to the large experimental error.

In contrast to the situation below 10 GeV the extraction of α_s from Z-decays is affected by the α_s^4 terms. The analysis of the electroweak working group [34] is based on eq. (4) with $n_f = 5$, including term up to $\mathcal{O}(\alpha_s^3)$ and leads to

$$\alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026^{\text{exp}} \qquad (6)$$

Including the α_s^4 term leads to a shift $\delta\alpha_s(M_Z) = 0.0005$

$$\alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\text{exp}}, \qquad (7)$$

The theory error can now safely be neglected.

Higher orders are of particular relevance in the low-energy region, for example in τ decays. The correction from perturbative QCD to the ratio

$$\begin{aligned} R_{\tau,V+A} &= \frac{\Gamma(\tau\to\text{hadrons}_{S=0}+\nu_\tau)}{\Gamma(\tau\to l+\bar{\nu}_l+\nu_\tau)} \\ &= 3|V_{ud}|^2 S_{EW}(1+\delta_0+\delta'_{EW}+\delta_2+\delta_{NP}) \end{aligned} \qquad (8)$$

is given by

$$1+\delta_0 = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right) R(s). \qquad (9)$$

The perturbative quantity δ_0 can be evaluated in Fixed Order perturbation theory or with "Contour Improvement" as proposed in [35,36]

$$\delta_0^{FO} = a_s + 5.202\, a_s^2 + 26.366\, a_s^3 + 127.079\, a_s^4, \qquad (10)$$

$$\delta_0^{CI} = 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + 64.29\, a_s^4. \qquad (11)$$

From the experimental side $\delta_0^{\text{exp}} = 0.1998 \pm 0.0043_{\text{exp}}$ has been used [37]. Starting from δ_0^{exp} one easily obtains the new values of $\alpha_s(M_\tau)$. Its dependence on the choice of d_4 (previously estimate and the new exact result) are summarized in Table I.

As stated above the theory error for α_s from Z decays is small compared to the experimental uncertainties. The situation is more problematic for τ decays and to some extent the theory error remains to be matter of choice. As anticipated in [27] it decreases significantly, once α_s^4 terms are included. However, the difference between the two methods stabilizes (this was checked in [27]

Table 1
Results for $\alpha_s(M_\tau)$ for different values of d_4. The first line displays the α_s^3 results (with the α_s^4 terms set to zero in eqs. (10,11)). The second line uses the previously predicted value for d_4, the last one uses the exact result (4). The first error is the experimental one; the second (theoretical) uncertainty in the value of α_s corresponds to changing the renormalization scale μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$.

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

by adding an estimate for the α_s^5 term) and must be considered as irreducible uncertainty, with the final result

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm th}. \quad (12)$$

For the central value the mean value of FO and CI has been used, for the theory error half of the difference between two methods (0.01) plus (module of) the estimated correction from α_s^5 term (-0.005), the latter being based on $d_5 = 275$ (see eq. (5)). Applying four-loop running and matching [38–41] to (12) we arrive

$$\begin{aligned} a_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm th} \\ &\quad \pm 0.0003_{\rm evol}\ , \\ &= 0.1202 \pm 0.0019. \end{aligned} \quad (13)$$

Here the evolution error receives contributions from the uncertainties in the c-quark mass (0.00003, $m_c(m_c) = 1.286(13)$ GeV [42]) and the b-quark mass (0.00001, $m_b(m_b) = 4.164(25)$ GeV [42]), the matching scale (0.0001, μ varied between $0.7\, m_q(m_q)$ and $3.0\, m_q(m_q)$), the four-loop truncation in the matching expansion (0.0001) and the four-loop truncation in the RGE equation (0.0003). (For the last two errors the size of the shift due the highest known perturbative term was treated as systematic uncertainty.) The errors had been added in quadrature.

Summary: The exact result for the α_s^4 term in the Adler function allows to extract the strong coupling constant from Z and τ decays with high precision. Including the exact α_s^4 leads to small shifts of the central value and to a significant reduction of the theory uncertainty. Note that the shifts in $\alpha_s(M_Z)$ from Z- and τ-decays, are opposite in sign and move the values in the *proper* direction, decreasing, thus, the current slight mismatch between two independent determinations of α_s. The final results

$$\alpha_s(M_Z)|_Z = 0.1190 \pm 0.0026 \quad (14)$$

$$\alpha_s(M_Z)|_\tau = 0.1202 \pm 0.0019 \quad (15)$$

from these two observables, although based on measurements of vastly different energy scales, are in remarkable agreement. This constitutes a striking test of asymptotic freedom in QCD. The two values can be combined to

$$\alpha_s(M_Z) = 0.1198 \pm 0.0015.$$

This is one of the most precise and presently only N^3LO result.

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REFERENCES

1. W.-M. Yao et al, Journal of Physics G33, 1 (2006)
2. M. M. Winter (2001), `ECFA-DESY Linear Collider Note LC-PHSM-2001-016`.
3. B. C. Allanach, G. A. Blair, S. Kraml, H. U. Martyn, G. Polesello, W. Porod and P. M. Zerwas, arXiv:hep-ph/0403133.
4. K. G. Chetyrkin, J. H. Kühn, and A. Kwiatkowski, Phys. Rep. **277**, 189 (1996), `and references therein.`
5. M. Davier, A. Hocker, and Z. Zhang, Rev. Mod. Phys. **78**, 1043 (2006), `and references therein.`
6. K. G. Chetyrkin, J. H. Kühn, and M. Steinhauser, Phys. Lett. **B371**, 93 (1996),

hep-ph/9511430; Nucl. Phys. **B482**, 213 (1996), hep-ph/9606230.
7. K. G. Chetyrkin and J. H. Kühn, Phys. Lett. **B248**, 359 (1990); Nucl. Phys. **B432**, 337 (1994), hep-ph/9406299.
8. K. G. Chetyrkin, R. V. Harlander, and J. H. Kühn, Nucl. Phys. **B586**, 56 (2000), hep-ph/0005139.
9. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Nucl. Phys. Proc. Suppl. **135**, 243 (2004).
10. A. Czarnecki and J. H. Kühn, Phys. Rev. Lett. **77**, 3955 (1996), hep-ph/9608366.
11. R. Harlander, T. Seidensticker, and M. Steinhauser, Phys. Lett. **B426**, 125 (1998), hep-ph/9712228.
12. D. V. Shirkov (2000), hep-ph/0009106.
13. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **96**, 012003 (2006), hep-ph/0511063.
14. K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **B85**, 277 (1979).
15. S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. **B259**, 144 (1991).
16. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. **88**, 012001 (2002), hep-ph/0108197.
17. P. A. Baikov, K. G. Chetyrkin and J. H. Kuhn, arXiv:0801.1821 [hep-ph].
18. P. Nogueira, J. Comput. Phys. **105**, 279 (1993).
19. A. A. Vladimirov, Theor. Math. Phys. **43**, 417 (1980).
20. K. G. Chetyrkin and V. A. Smirnov, Phys. Lett. **B144**, 419 (1984).
21. K. G. Chetyrkin, Phys. Lett. **B391**, 402 (1997), hep-ph/9608480.
22. P. A. Baikov, Phys. Lett. **B385**, 404 (1996), hep-ph/9603267; Nucl. Instrum. Meth. **A389**, 347 (1997), hep-ph/9611449; Phys. Lett. **B634**, 325 (2006), hep-ph/0507053.
23. M. Tentyukov et al. (2004), cs.sc/0407066.
24. J. A. M. Vermaseren (2000), math-ph/0010025.
25. G. Grunberg, Phys. Rev. **D29**, 2315 (1984).
26. A. L. Kataev and V. V. Starshenko, Nucl. Phys. Proc. Suppl. **39BC**, 312 (1995), hep-ph/9408395.
27. P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. **D67**, 074026 (2003), hep-ph/0212299.
28. J. H. Kühn, M. Steinhauser, and T. Teubner, Phys. Rev. **D76**, 074003 (2007), arXiv:0707.2589 [hep-ph].
29. J. H. Kühn and M. Steinhauser, Nucl. Phys. B **619** (2001) 588 [Erratum-ibid. B **640** (2002) 415] [arXiv:hep-ph/0109084].
30. J. Z. Bai *et al.* [BES Collaboration], Phys. Rev. Lett. **88** (2002) 101802 [arXiv:hep-ex/0102003].
31. A. E. Blinov *et al.* [MD-1 Collaboration], Z. Phys. C **70** (1996) 31.
32. R. Ammar *et al.* [CLEO Collaboration], Phys. Rev. D **57** (1998) 1350 [arXiv:hep-ex/9707018].
33. D. Besson *et al.* [CLEO Collaboration], arXiv:0706.2813 [hep-ex].
34. J. Alcaraz et al. (LEP Collaboration and ALEPH Collaboration and DELPHI Collaboration and L3 Collaboration and OPAL Collaboration and LEP Electroweak Working Group) (2007), arXiv:0712.0929 [hep-ex].
35. A. A. Pivovarov, Z. Phys. **C53**, 461 (1992), hep-ph/0302003.
36. F. Le Diberder and A. Pich, Phys. Lett. **B286**, 147 (1992).
37. M. Davier, A. Hocker, and Z. Zhang, Nucl. Phys. Proc. Suppl. **169**, 22 (2007), hep-ph/0701170.
38. T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. **B400**, 379 (1997), hep-ph/9701390.
39. M. Czakon, Nucl. Phys. **B710**, 485 (2005), hep-ph/0411261.
40. K. G. Chetyrkin, J. H. Kühn, and C. Sturm (2005), hep-ph/0512060.
41. Y. Schroder and M. Steinhauser, JHEP **01**, 051 (2006), hep-ph/0512058.
42. J. H. Kühn, M. Steinhauser, and C. Sturm, Nucl. Phys. **B778**, 192 (2007), hep-ph/0702103.