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The second physical moment of the heavy quark vector correlator at $\mathcal{O}(\alpha_s^3)$

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Abstract

The second moment of the heavy quark vector correlator at $\mathcal{O}(\alpha_s^3)$ is presented. The implications of this result on recent determinations of the charm and bottom quark mass are discussed.

Key words: Perturbative calculations, Quantum Chromodynamics, Dispersion Relations, Charm Quarks, Bottom Quarks
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1 Introduction

Correlators of quark currents are of prime interest for several phenomenological applications. Their low-energy expansions, in particular, allow for the precise determination of charm and bottom quark masses via QCD sum rules [1–5]. For this reason, heavy quark correlators have been frequently investigated in the framework of perturbation theory.

Up to $\mathcal{O}(\alpha_s^2)$, analytic expansions to great depth are known for the low energy region. The three-loop QCD corrections to the correlator of two vector currents were first calculated in [6]. In [7] up to seven terms in the low energy expansion were obtained. This calculation also included further currents, namely the scalar, pseudo-scalar, and axial-vector current. Recently the calculation at three-loops has been extended to moments up to $n = 30$ for all four currents [8,9].

The moments of the vector correlator can then be used to extract the value of the masses of the charm and bottom quark from e^+e^- data in the threshold region using the R-ratio, since they are related via a dispersion relation. A brief outline of this method is given in Section 2, which was first applied at three loops in [3].

At three loops a significant, sometimes dominant part of the error arises from the theoretical uncertainty due to higher orders, often estimated by the renormalization scale dependence. Therefore the calculation had to be taken to the four-loop level [10,11] to reach a precision comparable to or below the experimental data. The contributions from double-fermionic loop insertions of heavy and/or light quarks are known explicitely up to 30 terms in the low energy expansion [12]. The contributions due to light quark loop insertions of $\mathcal{O}(\alpha_s^n n_l^{n-1})$ are known to all orders in α_s [13]. Recently the lower moments were also calculated for the remaining three currents in [14].

In [4] the first moment of the vector correlator was used to extract the masses of the charm and bottom quarks. Since all but constant terms are known from renormalization group arguments, the analysis was done for up to the fourth moment, employing a conservative error estimate for the missing constant terms.

In this paper we present the calculation of the second moment of the vector correlator and discuss its impact on the determination of the charm and bottom quark masses.

The outline of this paper is as follows: In Section 2 we set the framework and notations used throughout the paper. In Section 3 we explain the details of the calculation, present the result for the second physical moment and discuss its impact on the the quark mass determination. A brief summary and conclusions are given in Section 4.

2 Notation

The correlator $\Pi^{\mu\nu}(q)$ of two vector currents is defined as

$$\Pi^{\mu\nu}(q) = i \int dx\, e^{iqx} \langle 0|Tj^{\mu}(x)j^{\nu}(0)|0\rangle \,, \tag{1}$$

with the current $j^{\mu}(x) = \bar{\Psi}(x)\gamma^{\mu}\Psi(x)$ being composed of the heavy quark fields $\Psi(x)$. The function $\Pi^{\mu\nu}(q)$ is conveniently written in the form

$$\Pi^{\mu\nu}(q) = (-q^2 g^{\mu\nu} + q^{\mu}q^{\nu})\Pi(q^2) \,. \tag{2}$$

It can be related to the ratio $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ with the help of the dispersion relation

$$\Pi(q^2) = \frac{1}{12\pi^2}\int_0^\infty ds \frac{R(s)}{s\,(s-q^2)}\,, \tag{3}$$

where the normalization $\Pi(0) = 0$ has been adopted.

To extract the quark masses the experimental data on the right hand side of (3) has to be compared with the theoretical evaluation of $\Pi(q^2)$ on the left hand side. This is best be done by comparing the corresponding Taylor series in q^2. The n-th derivatives with respect to q^2 at $q^2 = 0$ define the experimental moments

$$\mathcal{M}_n^{exp} = \int ds \frac{R(s)}{s^{n+1}}\,, \tag{4}$$

which can be compared with the theoretical moments

$$\mathcal{M}_n^{th} = Q_q^2 \frac{9}{4}\left(\frac{1}{4\bar{m}_q^2}\right)^n \bar{C}_n\,. \tag{5}$$

The latter are related to the Taylor coefficients $\bar{C}_n$ of the vacuum polarization function

$$\bar{\Pi}(q^2) = \frac{3Q_q^2}{16\pi^2}\sum_{n\geq 0}\bar{C}_n \bar{z}^n \tag{6}$$

with $\bar{z} = q^2/(4\bar{m}^2)$. Symbols carrying a bar indicate that the renormalization has been performed in the $\overline{\text{MS}}$ scheme. The coefficients $\bar{C}_n$ can be expanded in a power series in $\frac{\alpha_s}{\pi}$

$$\bar{C}_n = \bar{C}_n^{(0)} + \frac{\alpha_s}{\pi}\bar{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \bar{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \bar{C}_n^{(3)} + \cdots\,.$$

The four-loop contribution $\bar{C}_n^{(3)}$ can be decomposed according to the number of quark loops and colour structures as follows:

$$\begin{aligned}\bar{C}_n^{(3)} =& C_F T_F^2 n_l^2 \bar{C}_{ll,n}^{(3)} + C_F T_F^2 n_h^2 \bar{C}_{hh,n}^{(3)} + C_F T_F^2 n_l n_h \bar{C}_{lh,n}^{(3)} \\ &+ C_F T_F n_l \left(C_A \bar{C}_{lNA,n}^{(3)} + C_F \bar{C}_{lA,n}^{(3)}\right) \\ &+ C_F T_F n_h \left(C_A \bar{C}_{hNA,n}^{(3)} + C_F \bar{C}_{hA,n}^{(3)}\right) \\ &+ \bar{C}_{n_f^0,n}^{(3)} + \frac{n_h}{N_C} d^{abc} d^{abc} \bar{C}_{S,n}^{(3)}\,.\end{aligned} \tag{7}$$

$\bar{C}_{n_f^0,n}^{(3)}$ contains the purely bosonic contributions, where we set the number of colours $N_C = 3$ for simplicity, while $\bar{C}_{S,n}^{(3)}$ denotes the contribution from singlet diagrams. $C_F = \frac{N_C^2-1}{2N_C}$ and $C_A = N_C$ are the Casimir operators of the fundamental and adjoint representation of the $SU(N_C)$ group, respectively. $T_F = \frac{1}{2}$ is the index of the fundamental representation. d^{abc} is the symmetric structure constant. n_l and $n_h = 1$ denote the number of light and heavy quarks, respectively.

3 Calculation and Results

The diagrams have been generated using `QGRAF` [15]. Expanding them in q^2 results in four-loop tadpole integrals. Using `EXP` [16] they are mapped to six topologies with the maximum of nine lines. The reduction to the 13 master integrals is then performed with the help of `CRUSHER` [17] using Integration-By-Parts identities and the Laporta algorithm [18,19]. A supplementary technique to perform the reduction to master integrals is based on the idea that self energy subgraphs of the integral can be reduced independently in order to effectively reduce the number of loops of the diagram. This can be useful because these integrals have up to two more propagator powers than integrals without an internal self energy and are therefore more cumbersome for traditional Laporta algorithm. In combination with Groebner Bases and the `Mathematica` package `FIRE` [20–22] it is also possible to calculate integrals without internal self energies [23]. In total the reduction of 1.8 million integrals was needed in order to perform the calculation, which is done using `FORM` [24] in combination with the MATAD [25] setup. The necessary master integrals have been calculated in [26–32]. We confirm the results for the zeroth and first moment given in [8,10,11].

Inserting the master integrals and performing the renormalization of the strong coupling constant and the mass in the $\overline{\mathrm{MS}}$ scheme leads to the following result for the second moment at $\mu^2 = m^2$ as defined in Eq. (7):

$$
\begin{aligned}
\bar{C}^{(3)}_{n_f^0,2} = &+ \frac{649850742588811347}{353072079360000} - \frac{2900811008}{3648645} a_5 \\
&- \frac{1662518706713}{21016195200}\left(24 a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) + \frac{362601376}{54729675}\log^5 2 \\
&- \frac{725202752}{10945935}\zeta_2 \log^3 2 - \frac{1684950406}{3648645}\zeta_4 \log 2 \\
&+ \frac{112680551036302633}{47076277248000}\zeta_3 - \frac{26401638588211}{28021593600}\zeta_4 - \frac{164928917}{270270}\zeta_5\,, \\
\bar{C}^{(3)}_{S,2} = &+ \frac{5881974201847}{8369115955200} + \frac{97011619}{696729600}\left(24 a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
&+ \frac{796232393699}{371960709120}\zeta_3 - \frac{745372259}{185794560}\zeta_4\,, \\
\bar{C}^{(3)}_{hNA,2} = &- \frac{20427854209619}{5649153269760} - \frac{31595849}{11612160}\left(24 a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
&- \frac{29638030087837}{697426329600}\zeta_3 + \frac{968787977}{15482880}\zeta_4 + \frac{362}{63}\zeta_5\,,
\end{aligned}
$$

$$
\begin{aligned}
\bar{C}^{(3)}_{lNA,2} =& -\frac{22559166733}{16796160000} - \frac{520999}{4354560}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
& - \frac{309132631}{12902400}\zeta_3 + \frac{167529079}{5806080}\zeta_4 \,, \\
\bar{C}^{(3)}_{hA,2} =& -\frac{37320009196157}{271593907200} - \frac{130387543}{2177280}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
& - \frac{5811074101069}{6706022400}\zeta_3 + \frac{2218910663}{1451520}\zeta_4 \,, \\
\bar{C}^{(3)}_{lA,2} =& +\frac{357543003871}{11757312000} + \frac{520999}{2177280}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
& - \frac{36896356307}{174182400}\zeta_3 + \frac{598455689}{2903040}\zeta_4 \,, \\
\bar{C}^{(3)}_{lh,2} =& +\frac{95040709}{62705664} - \frac{2029}{41472}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \\
& - \frac{12159109}{4644864}\zeta_3 + \frac{99421}{55296}\zeta_4 \,, \\
\bar{C}^{(3)}_{hh,2} =& +\frac{1842464707}{646652160} - \frac{2744471}{1064448}\zeta_3 \,, \\
\bar{C}^{(3)}_{ll,2} =& +\frac{15441973}{19136250} - \frac{32}{45}\zeta_3 \,,
\end{aligned}
$$

where Riemann's zeta function ζ_n and the polylogarithm $\mathrm{Li}_n(1/2)$ are defined by

$$
\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \text{and} \quad a_n = \mathrm{Li}_n(1/2) = \sum_{k=1}^{\infty} \frac{1}{2^k k^n} \,. \tag{8}
$$

For completeness we also give the results for the singlet contribution to the zeroth and first moment:

$$
\bar{C}^{(3)}_{S,0} = \frac{2411}{20160} - \frac{6779}{4480}\zeta_3 + \frac{2189}{768}\zeta_4 - \frac{5}{48}\zeta_5 - \frac{73}{576}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \,, \tag{9}
$$

$$
\bar{C}^{(3)}_{S,1} = \frac{664837}{2566080} - \frac{2017831}{855360}\zeta_3 + \frac{175}{48}\zeta_4 - \frac{739}{4320}\left(24a_4 + \log^4 2 - 6\zeta_2 \log^2 2\right) \,. \tag{10}
$$

Numerically one finds at $\mu^2 = m^2$

$$
\bar{C}^{(3)}_2|_{n_l=3} = -\,3.49373 + 0.155877 \,, \tag{11}
$$

$$
\bar{C}^{(3)}_2|_{n_l=4} = -\,2.64381 + 0.155877 \,, \tag{12}
$$

which lies well in the range estimated in [4], $-6.0 \leq \bar{C}^{(3)}_2|_{n_l=3} \leq 7.0$ and $-8.0 \leq \bar{C}^{(3)}_2|_{n_l=4} \leq 9.5$. The second term in each of these equations corresponds to the singlet contribution.

Extracting the charm and bottom quark mass from the second moment using the input data given in [4] with the new value of $C^{(3)}_2$ leads to a shift of $-3\,\mathrm{MeV}$

for m_c and $-2\,\mathrm{MeV}$ for m_b and yields

$$m_c(3\,\mathrm{GeV}) = 0.976(16)\,\mathrm{GeV}\,, \tag{13}$$
$$m_b(10\,\mathrm{GeV}) = 3.607(19)\,\mathrm{GeV}\,. \tag{14}$$

This can be converted to the values at m_c and m_b, respectively:

$$m_c(m_c) = 1.277(16)\,\mathrm{GeV}\,, \tag{15}$$
$$m_b(m_b) = 4.162(19)\,\mathrm{GeV}\,. \tag{16}$$

The values of the quark masses extracted using the first and the second moment are in very good agreement. For the bottom quark the final result given in [4], $m_b(m_b) = 4.164(25)\,\mathrm{GeV}$, was obtained using the second moment and a conservative error estimate was used for the missing contributions to the second moment. While the overall shift in the bottom quark mass induced by the full results for the second moment is only 2 MeV the error gets reduced by 6 MeV, which makes this the most precise determination of the bottom quark mass.

4 Summary and Conclusion

We have presented the second physical moment in the low energy expansion of the heavy quark vector correlator at four-loop order, including the singlet contribution. Although this contribution only causes a rather small shift in the quark masses obtained from the second moment the error is reduced significantly. The values remain in good agreement with those extracted using the first moment.

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