Three-loop results in HQET

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Recent results and methods of three-loop calculations in HQET are reviewed.

1. Off-shell HQET propagator diagrams

Two-loop HQET propagator diagrams were reduced [1] to two master integrals, using integration by parts [2] identities.

There are 10 generic topologies of three-loop HQET propagator diagrams (Fig. 1). They can be reduced [3], using integration by parts relations, to 8 master integrals (Fig. 2). The algorithm has been constructed by hand, and implemented as a REDUCE package Grinder [3]. It is analogous to the massless package Mincer [4].

The first 5 master integrals can be easily expressed via Γ functions. The next two ones are two-loop integrals with a single d-dependent index. The last one is truly three-loop.

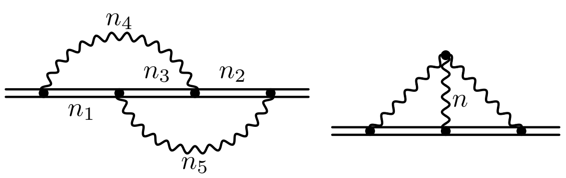


Figure 3. Two-loop propagator diagrams

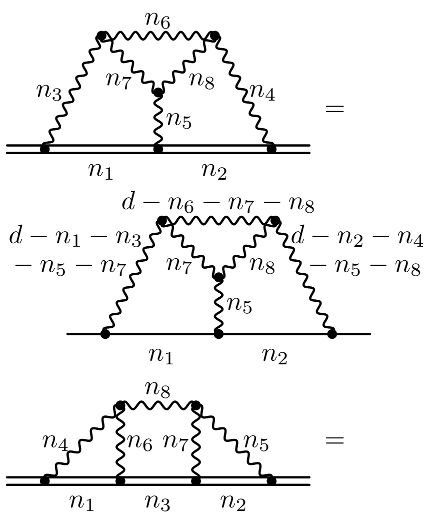
The first two-loop diagram (Fig. 3) in coordinate space is just a double integral, and can be easily calculated for arbitrary $n_{1\ldots5}$ [3]:

$$J(n_1,n_2,n_3,n_4,n_5) = \Gamma(n_1+n_2+n_3+2(n_4+n_5-d)) \frac{\Gamma(n_1+n_3+2n_4-d)\Gamma(d/2-n_4)\Gamma(d/2-n_5)}{\Gamma(n_1+n_2+n_3+2n_4-d)\Gamma(n_4)\Gamma(n_5)\Gamma(n_1+n_3)} {}_3F_2\left(\begin{array}{c} n_1, d-2n_5, n_1+n_3+2n_4-d \\ n_1+n_3, n_1+n_2+n_3+2n_4-d \end{array} \middle| 1 \right). \quad (1)$$

The second diagram in Fig. 3 for arbitrary n has been calculated [5] using Gegenbauer polynomials in x-space [6]:

$$I(1,1,1,1,n) = \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-n-1\right)}{\Gamma(d-2)} \left[2\frac{\Gamma(2n-d+3)\Gamma(2n-2d+6)}{(n-d+3)\Gamma(3n-2d+6)} {}_3F_2\left(\begin{array}{c} n-d+3, n-d+3, 2n-2d+6 \\ n-d+4, 3n-2d+6 \end{array} \middle| 1 \right) - \Gamma(d-n-2)\Gamma^2(n-d+3) \right]. \quad (2)$$

Similarly to the two-loop case [7], some HQET propagator integrals are related to on-shell massive propagator integrals by inversion of Euclidean integration momenta:



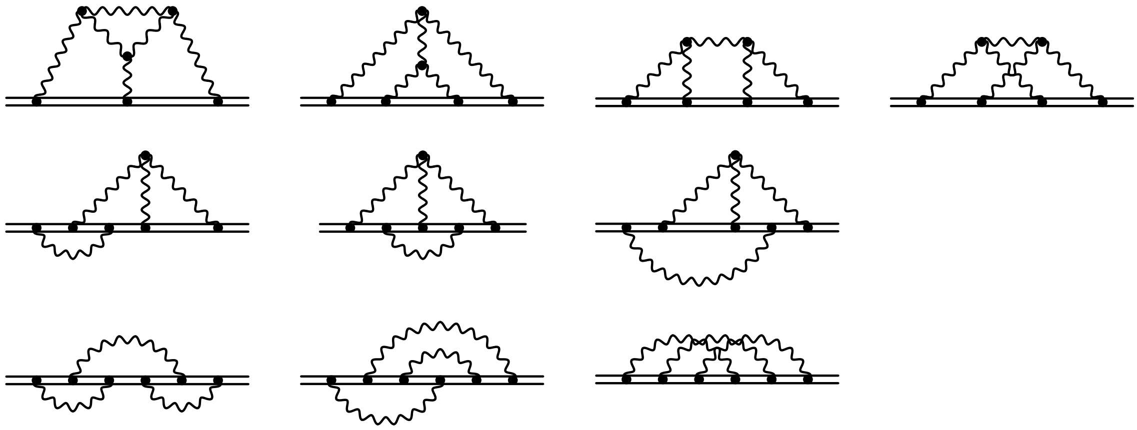


Figure 1. Topologies of three-loop HQET propagator diagrams

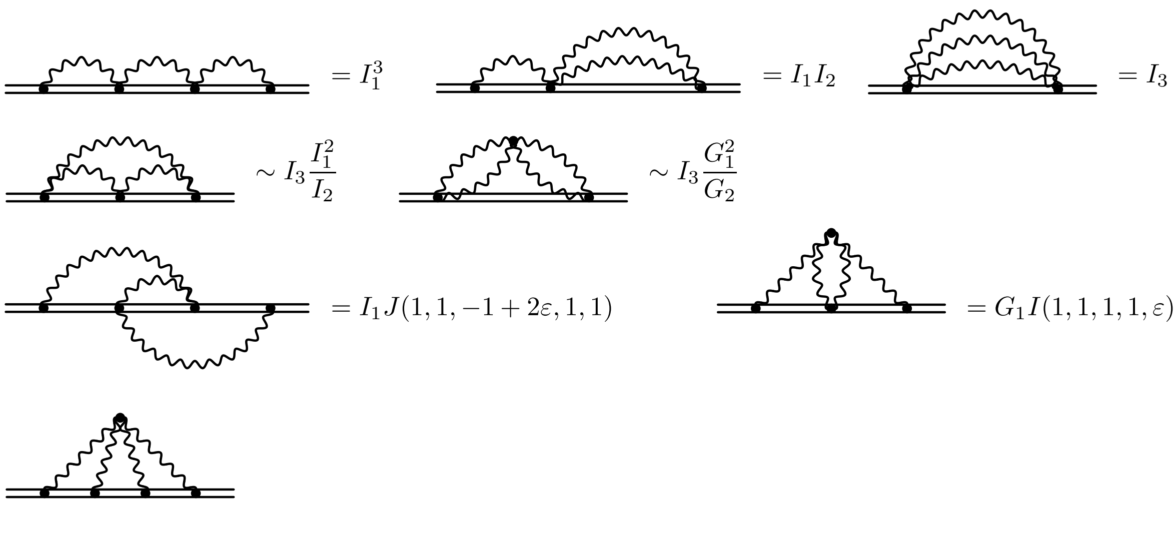
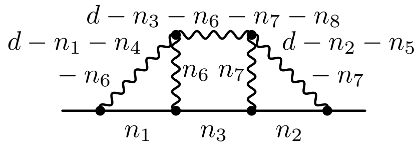
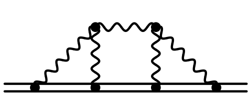
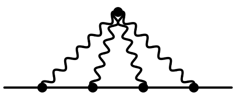


Figure 2. Master integrals for three-loop HQET propagator diagrams



(3)

In particular, at $d = 4$ we have

 = 

$= -5\zeta_5 + 12\zeta_2\zeta_3\,.$ (4)

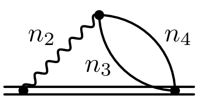
Reducing the left-hand side to the master integrals, one can express [8] the last master integral in Fig. 2 via the right-hand side of (4), which is an on-shell master integral known from [9]. Note that $\mathcal{O}(\varepsilon)$ corrections to (4) are not known.

Using this technique, the HQET heavy-quark propagator has been calculated up to three loops [10], and the heavy-quark field anomalous dimension (obtained earlier by a completely different method [11]) has been confirmed. The anomalous dimension of the HQET heavy–light quark current has been calculated [10]. The correlator of two heavy–light currents has been found, up to three loops, including light-quark mass corrections of order m and m^2 [12]. The quark-condensate contribution to this correlator has been also calculated up to three loops [12]. Its ultraviolet divergence yields the difference of twice the anomalous dimension of the heavy-quark current and the that of the quark condensate, thus providing a completely independent confirmation of the result obtained in [10]. The gluon-condensate contribution has been calculated up to two loops [12] (at one loop it vanishes).

2. On-shell HQET propagator diagrams with mass

2.1. Reduction

The two-loop on-shell HQET propagator integral with a massive loop can be expressed via Γ functions for arbitrary indices [13]:



=

$$\frac{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_3\right)\Gamma\left(\frac{n_1-d}{2}+n_2+n_4\right)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)\Gamma(n_1+2n_2+n_3+n_4-d)}
\frac{\Gamma\left(\frac{n_1}{2}+n_2+n_3+n_4-d\right)\Gamma\left(\frac{d-n_1}{2}-n_2\right)}{\Gamma\left(\frac{d-n_1}{2}\right)}\,. \quad (5)$$

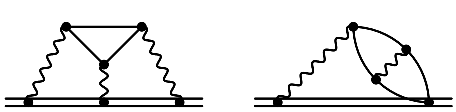
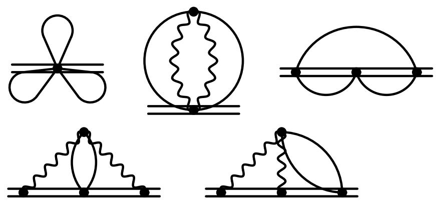


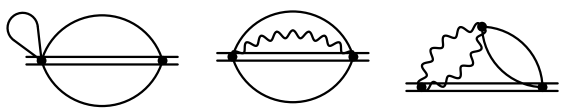
Figure 4. Topologies of on-shell HQET propagator diagrams with mass

There are two generic topologies of three-loop on-shell HQET propagator diagrams with a massive loop (Fig. 4). Algorithms of their reduction to master integrals, using integration by parts identities, have been constructed [13] by Gröbner bases technique [14]. All scalar integrals can be divided into two classes which are not mixed by recurrence relations: apparently even and apparently odd ones. They are integrals which would be even or odd with respect to $v \to -v$ if there were no $i0$ in denominators.

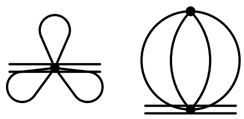
All apparently even integrals of the first topology reduce to



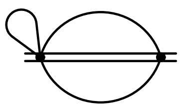
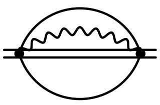
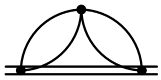
while apparently odd ones to



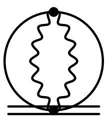
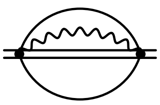
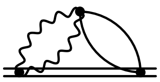
All apparently even integrals of the second topology reduce to



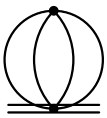
while apparently odd ones to

The master integrals

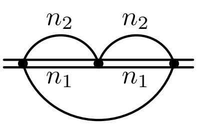
can be easily expressed via Γ functions. The master integral



has been investigated in detail [15,16].

2.2. A master integral

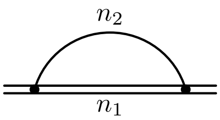
Now we shall discuss the integrals

$$I_{n_1 n_2 n_3} =$$ 

(6)

(I_{111} is one of the master integrals). Several approaches have been tried [13,17]. The best result was obtained [17] using a method similar to [15].

First we consider the one-loop subdiagram

$$I_{n_1 n_2}(p_0) =$$ 

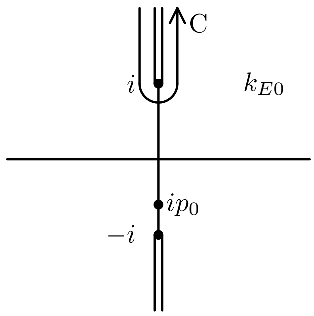
$= \quad (7)$

$$\frac{1}{i\pi^{d/2}} \int \frac{dk_0\, d^{d-1}\vec{k}}{[-2(k_0+p_0)-i0]^{n_1}[1-k^2-i0]^{n_2}} \,.$$

After the Wick rotation, we integrate in $d^{d-1}\vec{k}$:

$$I_{n_1 n_2}(p_0) = \frac{\Gamma(n_2-(d-1)/2)}{\pi^{1/2}\Gamma(n_2)} \times \int_{-\infty}^{+\infty} dk_{E0} \frac{(k_{E0}^2+1)^{(d-1)/2-n_2}}{(-2p_0-2ik_{E0})^{n_1}} \,.$$

If $p_0<0$, we can deform the integration contour



$$I_{n_1 n_2}(p_0) = 2\frac{\Gamma(n_2-(d-1)/2)}{\pi^{1/2}\Gamma(n_2)} \times \cos\left[\pi\left(\frac{d}{2}-n_2\right)\right] \int_1^\infty dk \frac{(k^2-1)^{(d-1)/2-n_2}}{(2k-2p_0)^{n_1}} \,.$$

This integral is

$$I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))} \times {}_2F_1\left(\left. \begin{array}{c} n_1, n_1+2n_2-4+2\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \right| \frac{1}{2}(1+p_0) \right) \,,$$

or, after using a ${}_2F_1$ identity,

$$I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1+n_2-2+\varepsilon)\Gamma(n_1+2n_2-4+2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1+n_2-2+\varepsilon))} \times {}_2F_1\left(\left. \begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1+n_2-2+\varepsilon \\ n_1+n_2-\frac{3}{2}+\varepsilon \end{array} \right| 1-p_0^2 \right) \,. \quad (8)$$

This result was obtained [17] using the HQET Feynman parametrization.

Now we can integrate in $d^{d-1}\vec{p}$ in the three-loop diagram:

$$I_{n_1 n_2 n_3} = \frac{\Gamma(n_3-3/2+\varepsilon)}{\pi^{1/2}\Gamma(n_3)} \times \int_{-\infty}^{+\infty} I_{n_1 n_2}^2(ip_{E0})(1+p_{E0}^2)^{3/2-n_3-\varepsilon} dp_{E0} \,. \quad (9)$$

The square of ${}_2F_1$ in (8) can be expressed via an ${}_3F_2$ using the Clausen identity. We analytically continue this ${}_3F_2$ from $1+p_{E0}^2>1$ to $z=1/(1+p_{E0}^2)<1$ and integrate (9) term by term. The result contains, in general, three ${}_4F_3$ of unit argument.

The convergent integral I_{122} is related to the master integral I_{111} by

$$I_{122} = -\frac{(d-3)^2(d-4)(3d-8)(3d-10)}{8(3d-11)(3d-13)} I_{111}$$

For this integral, we obtain [17]

$$\begin{aligned}\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = &-\frac{1}{2\varepsilon^2}\Bigg[\frac{1}{1+2\varepsilon} \\ &{}_4F_3\left(\begin{matrix}1, \frac{1}{2}-\varepsilon, 1+\varepsilon, -2\varepsilon \\ \frac{3}{2}+\varepsilon, 1-\varepsilon, 1-2\varepsilon\end{matrix}\,\middle|\, 1\right) \\ &- \frac{2}{1+4\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma^3(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+4\varepsilon)} \\ &{}_3F_2\left(\begin{matrix}\frac{1}{2}, 1+2\varepsilon, -\varepsilon \\ \frac{3}{2}+2\varepsilon, 1-\varepsilon\end{matrix}\,\middle|\, 1\right) \\ &+ \frac{1}{1+6\varepsilon} \\ &\frac{\Gamma^2(1-\varepsilon)\Gamma^4(1+2\varepsilon)\Gamma(1-2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+4\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+6\varepsilon)}\Bigg].\end{aligned} \tag{10}$$

Expansion of this result up to ε^7 agrees with [17]

$$\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = \frac{\pi^2}{3}\frac{\Gamma^3(1+2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^6(1+\varepsilon)\Gamma(2+6\varepsilon)}. \tag{11}$$

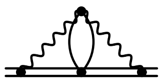
This equality has also been checked by high precision numerical calculations at some finite ε values. During the workshop, David Broadhurst has rewritten this conjectured hypergeometric identity in a nice form

$$\begin{aligned}b(\varepsilon) &= \frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}, \\ g_n(\varepsilon) &= \frac{b^n(\varepsilon)}{b(n\varepsilon)(1+2n\varepsilon)}, \\ g_1(\varepsilon)&{}_4F_3\left(\begin{matrix}1, \frac{1}{2}-\varepsilon, 1+\varepsilon, -2\varepsilon \\ \frac{3}{2}+\varepsilon, 1-\varepsilon, 1-2\varepsilon\end{matrix}\,\middle|\, 1\right) \\ &- 2g_2(\varepsilon){}_3F_2\left(\begin{matrix}\frac{1}{2}, 1+2\varepsilon, -\varepsilon \\ \frac{3}{2}+2\varepsilon, 1-\varepsilon\end{matrix}\,\middle|\, 1\right) \\ &+ g_3(\varepsilon) = 0.\end{aligned} \tag{12}$$

We have no analytical proof.

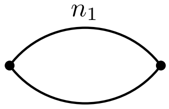
2.3. Other master integrals

Now we turn to other master integrals [13]. By applying Mellin–Barnes techniques to the α-representation and using the Barnes lemmas, we succeeded in calculating one integral exactly:

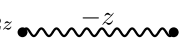


$$= \frac{\Gamma(1/2-\varepsilon)\Gamma(-\varepsilon)\Gamma^2(2\varepsilon)\Gamma(1+\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(3/2-\varepsilon)\Gamma(4\varepsilon)} \left[\psi(1/2-\varepsilon)+\psi(1-\varepsilon)-2\log 2+2\gamma_{\mathrm{E}}\right]. \tag{13}$$

In other master integral calculations, we use the Mellin–Barnes of the massive two-point function:

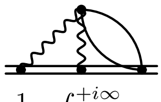


$$\begin{aligned}= &\frac{1}{i\pi^{d/2}}\int\frac{d^dk}{\left[m^2-k^2-i0\right]^{n_1}\left[m^2-(k+p)^2-i0\right]^{n_2}} \\ &= \frac{m^{d-2(n_1+n_2)}}{\Gamma(n_1)\Gamma(n_2)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\times \\ &\frac{\Gamma(n_1+z)\Gamma(n_2+z)\Gamma(n_1+n_2-d/2+z)}{\Gamma(n_1+n_2+2z)}\times \\ &m^{-2z}\end{aligned}$$



(14)

Applying this representation and eliminating one of the lines in the triangle with unit indices using integration by parts, we obtain a single Mellin–Barnes integral:



$$\begin{aligned}&= \frac{\Gamma^2(2\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(4\varepsilon)}\times \\ &\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(-z)\Gamma(-2\varepsilon-z)\Gamma(-\varepsilon-z)\times \\ &\frac{\Gamma(1+z)\Gamma(1/2+\varepsilon+z)\Gamma(1+\varepsilon+z)}{\Gamma(1-2\varepsilon-z)\Gamma(3/2+\varepsilon+z)} = \\ &-\Gamma^3(1+\varepsilon)\Bigg[\frac{\pi^2}{9\varepsilon^2}-\frac{6\zeta_3-5\pi^2}{9\varepsilon} \\ &+\frac{11}{270}\pi^4-\frac{10}{3}\zeta_3+\frac{19}{9}\pi^2\end{aligned}$$

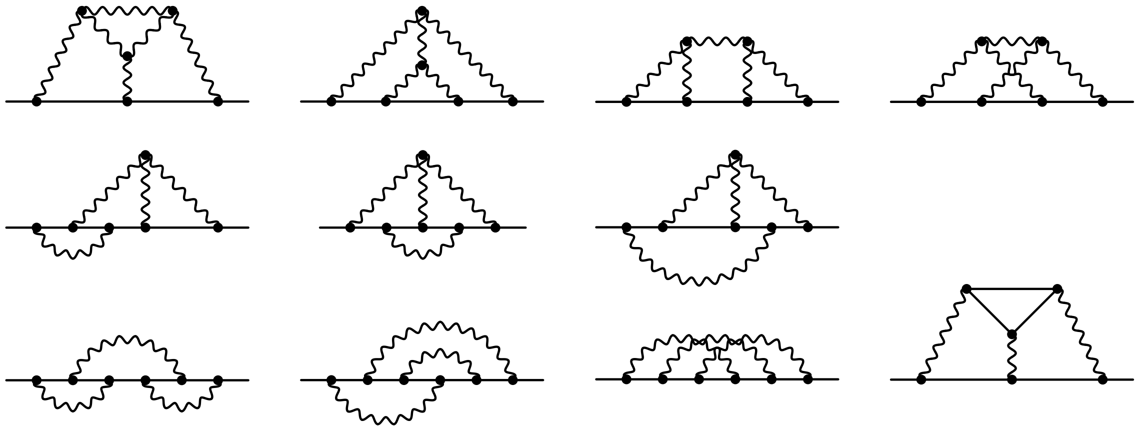
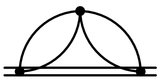


Figure 5. Topologies of three-loop massive on-shell propagator diagrams

$$+\left(-\frac{8}{3}\zeta_5+\frac{8}{9}\pi^2\zeta_3+\frac{11}{54}\pi^4-\frac{38}{3}\zeta_3+\frac{65}{9}\pi^2\right)\varepsilon$$
$$+\cdots\Bigg] . \qquad (15)$$

Similarly, in this case, we apply (14) once and use (5):



$$= \frac{\pi^{3/2}}{4^\varepsilon\Gamma(3/2-\varepsilon)}\times$$
$$\frac{1}{2\pi i}\int\limits_{-i\infty}^{+i\infty} dz \frac{\Gamma(1+z)\Gamma(3/2-\varepsilon+z)\Gamma(\varepsilon+z)}{\Gamma(3/2+z)\Gamma(\varepsilon-z)}\times$$
$$\Gamma(-z)\Gamma(-1/2+\varepsilon-z)\Gamma(-3/2+2\varepsilon-z)$$
$$= -\frac{32}{3}\pi^2+\cdots \qquad (16)$$

Feynman integrals considered here were used [13] for calculating the matching coefficients for the HQET heavy-quark field and the heavy–light quark current between the b-quark HQET with dynamic c-quark loops and without such loops (the later theory is the low-energy approximation for the former one at scales below m_c). Another recent application — the effect of $m_c \neq 0$ on $b \to c$ plus lepton pair at three loops [18]. The method of regions was used; the purely soft region (loop momenta $\sim m_c$) gives integrals of this type. Two extra terms of ε expansion of the master integral of Sect. 2.2 were required for this calculation which were not obtained in [13]. This was the initial motivation for [17].

3. On-shell massive QCD propagator diagrams

These diagrams are used for calculation of QCD/HQET matching coefficients [19]. There are two generic topologies of two-loop on-shell propagator diagrams. They were reduced [20,15, 21] to three master integrals, using integration by parts.

There are 11 generic topologies of three-loop HQET propagator diagrams (Fig. 5) (10 of them are the same as in HQET, Fig. 1, plus the diagram with a closed heavy-quark loop which does not exist in HQET). They can be reduced [11] to 19 master integrals (Fig. 6) using integration by parts. The reduction algorithm was first implemented as a FORM package SHELL3 [11]; in some papers [22], reduction is performed using the Laporta method. The master integrals are known from [9,11] (see also [22]).

As an example, let's consider chromomagnetic interaction of the heavy quark in HQET. The two-loop anomalous dimension of the chromo-

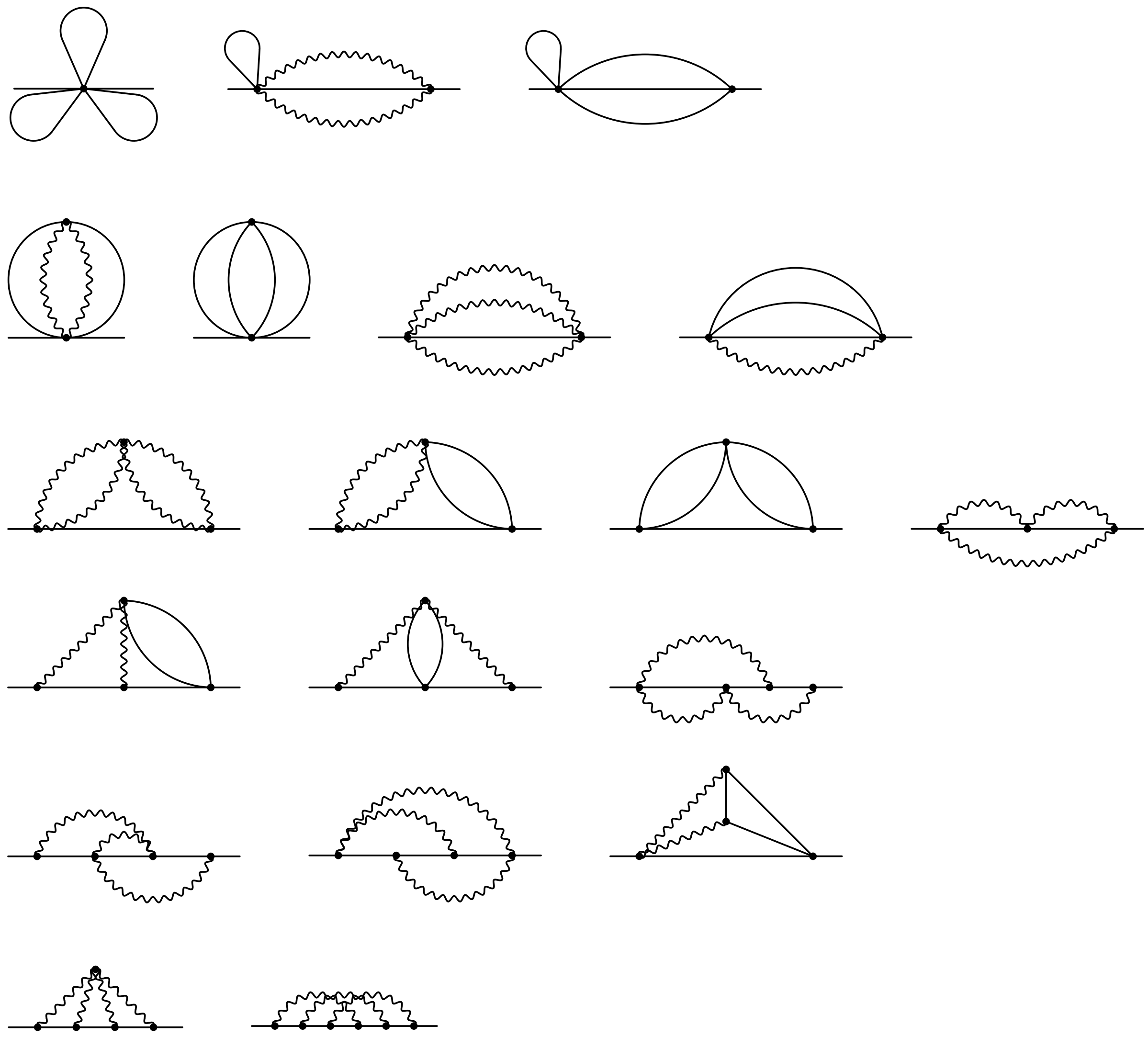


Figure 6. Master integrals for three-loop massive on-shell propagator diagrams

magnetic interaction operator has been calculated in [23], using off-shell HQET diagrams (Sect. 1) and R^* operation to get rid of infrared divergences. It was confirmed [24], a week later, using a completely different approach — QCD/HQET matching. In the same paper, also the coefficient of the chromomagnetic interaction operator in the HQET Lagrangian has been calculated up to two loops. Recently, both the anomalous dimension and the interaction coefficient have been calculated at three loops [25], also using the QCD/HQET matching. These results contain two colour factors which don't reduce to C_F and C_A (due to gluonic "light-by-light" diagrams; in the β-function and basic anomalous dimensions of QCD, these colour factors first appear at four loops.

I am grateful to S. Bekavac, D.J. Broadhurst, K.G. Chetyrkin, A. Czarnecki, A.I. Davydychev, T. Huber, D. Maître, P. Marquard, J.H. Piclum, D. Seidel, A.V. Smirnov, V.A. Smirnov, M. Steinhauser for collaboration on multi-loop projects in HQET; to the organizers of Loops and Legs 2008; and to D.J. Broadhurst for discussions of the master integral of Sect. 2.2.

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