Heavy-Flavor Contribution to Bhabha Scattering

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We evaluate the last missing piece of the two-loop QED corrections to the high-energy electron-positron scattering cross section originated from the vacuum polarization by heavy fermions. The calculation is performed within a new approach applicable to a wide class of perturbative problems with mass hierarchy. The result is crucial for the high-precision physics program at existing and future e^+e^- colliders.

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Electron-positron scattering at high energy has been originally studied by J.H. Bhabha almost a century ago [1] and nowadays plays an ultimate role in particle phenomenology. It provides a very efficient tool for luminosity determination at electron-positron colliders and thus mediates the process of extracting physical information from the raw experimental data [2]. The accuracy of theoretical predictions for the cross section of *Bhabha scattering* directly affects our quest for "new physics" at operating and planned e^+e^- colliders. For example, the future International Linear Collider (ILC) requires the accuracy of 0.1 permille for the small angle scattering to provide the most accurate tests of the Standard Model of electroweak interactions [3]. At the same time the low-energy experiments [4, 5] require about one permille accuracy of the cross section at large scattering angles to reduce the hadronic uncertainty in the theoretical prediction for the muon anomalous magnetic moment, which is a renowned probe of the physics beyond the Standard Model [6]. Thus, insufficient theoretical accuracy may jeopardize the whole high-precision physics program at e^+e^- colliders. To match the experimental needs, the two-loop QED corrections must be included into the theoretical analysis.

The calculation of the radiative corrections to the Bhabha cross section is among the classical problems of perturbative quantum field theory with a long history and remarkable efforts have been devoted to the study of the two-loop QED contributions. The corrections due to the vacuum polarization by virtual electron-positron pairs have been evaluated in [7] including the full dependence on the electron mass m_e. Pure photonic corrections have been obtained in Ref. [8] (see also [9]) in the leading order of the small electron mass expansion through the *infrared matching* to the massless case [10]. This result is sufficient for all the phenomenological applications and has been recently confirmed within a slightly different framework [11]. For the vacuum polarization by virtual heavy-flavor fermion-antifermion pairs the result is available only in the limit when the heavy fermion mass m_f is much less than the center-of-mass energy $\sqrt{s}$ [11, 12]. This condition, however, does not hold for τ-lepton, c and b-quark in the practically interesting energy range of about a few GeV as well as for the top quark at the typical ILC energies 500 GeV $\lesssim \sqrt{s} \lesssim$ 1000 GeV. The result for an arbitrary value of m_f^2/s constitutes the last missing part of the QED two-loop corrections. This two-loop scattering problem with two different masses is a real challenge for modern methods of perturbative quantum field theory.

In this Letter we present the heavy-flavor contribution retaining the full dependence on m_f^2/s. We first outline a new approach based on the general theory of infrared divergences, which makes the whole calculation possible. Then we present the complete numerical result for the correction in two phenomenologically most important cases and give the analytical expression in two limits $m_f^2 \ll s$ and $m_f^2 \gg s$. For completeness, we include the mixed QED-QCD two-loop corrections in the numerical analysis of the heavy-quark contribution.

The perturbative expansion for the Bhabha cross section in the fine structure constant α is defined as follows $\mathrm{d}\sigma = \sum_{n=0}^{\infty} (\alpha/\pi)^n \delta^{(n)} \mathrm{d}\sigma^{(0)}$, where $\delta^{(n)}$ is the correction factor, with $\delta^{(0)} = 1$. In the phenomenologically interesting kinematical region all the Mandelstam invariants are large relative to the electron mass $s,\ t,\ u \gg m_e^2$. Therefore all the terms suppressed by m_e can be neglected. The leading order differential cross section in the small electron mass approximation takes the form

$$\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2 + \mathcal{O}(m_e^2/s)\,, \qquad (1)$$

where $x = (1-\cos\theta)/2$ and θ is the scattering angle. Virtual corrections are infrared divergent. These *soft* divergencies are canceled in the physical inclusive cross section when one adds the photonic bremsstrahlung [13]. We regulate the soft divergencies by dimensional regu-

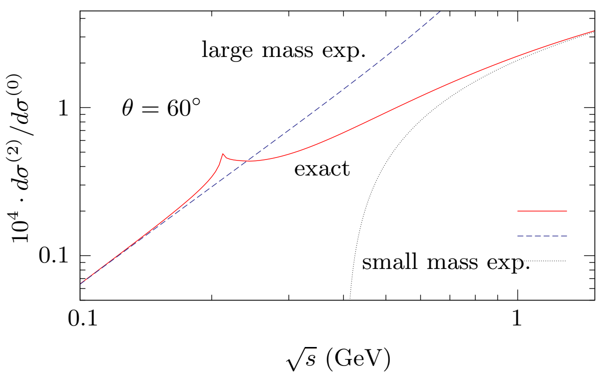

FIG. 1: Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60°$ due to a closed loop of muon ($m_\mu = 113$ MeV)

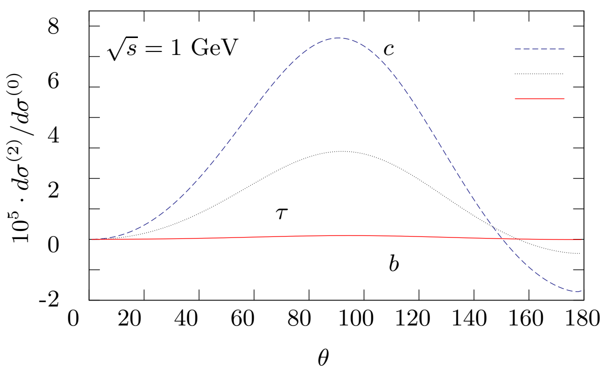

FIG. 2: Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed loop of τ-lepton ($m_\tau = 1.7$ GeV), c-quark ($m_c = 1.25$ GeV) and b-quark ($m_b = 4.7$ GeV).

larization with $d = 4 - 2\varepsilon$ space-time dimensions. The standard approach to deal with the bremsstrahlung is to split it into a soft part, which accounts for the emission of the photons with the energy below a given cutoff $\varepsilon_{cut} \ll m_e$, and a hard part corresponding to the emission of the photons with the energy larger than ε_{cut}. The infrared finite hard part is then computed numerically using Monte-Carlo methods with physical cuts dictated by the experimental setup. At the same time, the soft part is computed analytically and combined with the virtual corrections ensuring the cancellation of the soft divergencies. Thus we consider the finite sum of two terms $\delta^{(2)} = \delta^{(2)}_{vv} + \delta^{(1)}_s \delta^{(1)}$ corresponding to the two-loop virtual correction [20] and the one-loop correction to the single soft photon emission, which factorizes into the product of the first-order contributions [14]. In the small electron mass limit the two-loop correction can be decomposed according to the asymptotic dependence on m_e

$$\delta^{(2)} = \delta^{(2)}_1 \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_0 + \mathcal{O}(m_e^2/s)\,. \qquad (2)$$

The logarithmic term in Eq. (2) is a remnant of the *collinear* divergence which is regulated by the electron mass. Now we come to the crucial point of our approach. The collinear divergencies and, hence, the singular dependence of the corrections on m_e have a peculiar structure which was extensively studied in the context of QCD. In particular, the collinear divergencies factorize and can be absorbed in the external field renormalization [15]. This property is crucial for our analysis because it allows us to perform the most difficult part of the calculation with a strictly massless electron. Indeed, due to the factorization, the singular dependence on m_e is the same for the Bhabha amplitude and (the square of) the vector form factor [8]. This attributes the total logarithmic correction ot the amplitude to the one-particle reducible diagram given by the product of the one-loop contributions. In calculating the cross section one must take into account also the contributions coming from the interference of the one-loop corrections to the amplitude and the corresponding soft emission; both contributions have a factorized form. Thus it is straightforward to obtain the coefficient of the logarithmic term in Eq. (2) which reads $\delta^{(2)}_1 = [4\ln(\varepsilon_{cut}/\varepsilon) + 3]\,\delta^{(1)}$, where $\varepsilon = \sqrt{s}/2$. At the same time, the sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass limit and can be computed with $m_e = 0$. Let us emphasize that this property in general holds only for the sum of the diagrams. The individual diagrams computed in a covariant gauge do exhibit collinear divergencies for $m_e = 0$. This however does not pose any additional problem since we work in dimensional regularization. In this case the collinear divergencies show up as the extra poles in ε which are not related to the soft emission and disappear in the sum of the one-particle irreducible diagrams. Thus all the "true" two-loop diagrams contribute only to the non logarithmic term in Eq. (2) which can be written as follows

$$\begin{aligned}\delta^{(2)}_0 &= -4\left[1 + \ln\left(\frac{1-x}{x}\right)\right]\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right)\delta^{(1)} \\ &\quad + \left(\frac{x}{1-x+x^2}\right)^2 Q_f^2 N_c f(\rho, x)\,, \end{aligned} \qquad (3)$$

where the first term is determined by the soft emission, $f(\rho, x)$ is a function of two dimensionless variables: $\rho = m_f^2/s$ and $x = -t/s$, and we explicitly factored out the heavy-flavor electric charge Q_f and the number of colors N_c, which is 1 for leptons and 3 for quarks. The two-loop problem with massless electron falls it the same complexity class as the one considered in Ref. [7]. The most difficult part of the calculation is given by the two-loop box diagrams. We use the Laporta Algorithm [16] to identify the set of so-called master integrals which are subsequently evaluated by means of the differential equation method [17]. Finally, the result for the function $f(\rho, x)$ can be expressed in closed analytic form in terms of mathematical functions known as one and two-dimensional generalized harmonic polylogarithms of maximum weight three. In this Letter we focus on the numerical impact of the correction in the phenomenologically important cases and postpone the details of the analysis and the complete analytical result for a future publication. However, the analytical structure of the result becomes much simpler in the small-mass limit $m_f^2 \ll s$ and the large-mass limit $m_f^2 \gg s$ where it is given by the series $f(\rho,x)=\sum_{n=0}^{\infty}\rho^n f_n(\rho,x)$ and $f(\rho,x)=\sum_{n=0}^{\infty}\rho^{-n}\bar{f}_n(\rho,x)$, respectively. The series are asymptotical and the coefficients $f_n(\rho,x)$ and $\bar{f}_n(\rho,x)$ are in general logarithmic functions of ρ. With the exact result at hand we can compute any term of these expansions. The leading order coefficient $f_0(\rho,x)$ has been computed in [11, 12] and we confirm this result. On the other hand the leading order coefficient $\bar{f}_0(\rho,x)$ vanishes due to on-shell renormalization of the fine structure constant. Thus we present the results for the first unknown terms of the expansions which read

$$\begin{aligned}
f_1(\rho,x) = \frac{1}{x^3}\Bigg[&(4x^5-6x^4+8x^3-8x^2+6x-4)\left[\ln^2(\rho)\right.\\
&\left.+4\mathrm{Li}_2(x)+12\right]+\ln(\rho)\left[(2x^5-7x^4+10x^3-10x^2+7x-2)\right.\\
&-(4x^5-6x^4+10x^3-14x^2+12x-8)\ln(x)\\
&\left.+(4x^5-6x^4+8x^3-8x^2+6x-4)\ln(1-x)\right]\\
&-(40x^5-54x^4+50x^3-17x^2-12x+8)\zeta(2)\\
&+(12x^5-17x^4+18x^3-18x^2+17x-12)\ln(1-x)\\
&+\frac{1}{2}(8x^5-14x^4+15x^3-15x^2+14x-8)\ln^2(1-x)\\
&-(12x^5-21x^4+26x^3-26x^2+21x-14)\ln(x)\\
&+(8x^5-10x^4+8x^3-x^2-6x+4)\ln(x)\ln(1-x)\\
&-\frac{1}{2}(8x^5-7x^4-x^3+12x^2-15x+8)\ln^2(x)\\
&+\left(Q_f^2-1\right)\left[(6-9x+12x^2-12x^3+9x^4-6x^5)\ln(\rho)\right.\\
&\left.-\left(6-9x+9x^2-3x^3\right)\ln(x)\right]\Bigg]. \qquad (4)
\end{aligned}$$

$$\begin{aligned}
\bar{f}_1(\rho,x) = &\frac{5x^3-22x^2+22x-5}{30x}\ln(\rho)+\frac{955x^3-3926x^2}{1350x}\\
&+\frac{3926x-955}{1350x}-\frac{12x^3-19x^2+14x-6}{10x}\zeta(2)\\
&+\frac{3x^3+x^2-x-3}{30x}\ln(1-x)+\frac{2x^3-5x^2+5x-2}{20x}\ln^2(1-x)\\
&-\frac{20x^3-78x^2+93x-58}{90x}\ln(x)-\frac{6x^2-x-4}{60}\ln^2(x)\\
&+\frac{12x^3-19x^2+14x-6}{30x}\ln(x)\ln(1-x)\\
&+\frac{4x^3-8x^2+8x-4}{5x}\mathrm{Li}_2(x)\\
&-\left(Q_f^2-1\right)\frac{41x^3-82x^2+82x-41}{54x}. \qquad (5)
\end{aligned}$$

In Fig. 1 the second-order muon contribution $(\alpha/\pi)^2\delta^{(2)}$ is plotted as function of $\sqrt{s}$ in the threshold region around $2m_\mu \approx 216$ MeV at the fixed scattering angle $\theta = 60^\circ$. All the terms involving a power of the logarithm $\ln(\varepsilon_{cut}/\varepsilon)$ are excluded from the numerical estimates because the corresponding contribution critically depends on the event selection algorithm and cannot be unambiguously estimated without imposing specific cuts on the photon bremsstrahlung. We present the exact result along with the first nonvanishing terms of the small and large mass expansions. The asymptotic results fail to approximate the exact one with 10% accuracy in the interval $1.7 \lesssim \sqrt{s}/m_f \lesssim 10$. In particular they are completely useless for the description of the top quark contribution at the typical energies of the ILC. Both expansions break down near the threshold $\sqrt{s}\sim 2m_f$.

Let us discuss the numerical impact of the corrections on the luminosity determination at present and future electron-positron colliders. As the first application, we consider the Bhabha scattering at $\sqrt{s} = 1$ GeV. This is the center-of-mass energy of the KLOE experiment at DAΦNE, which plays crucial role in determination of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment. The second-order contributions of τ-lepton, c, and b-quark to the differential cross section are plotted as functions of the scattering angle in Fig. 2. Note that the energy under consideration is sufficiently below the quarkonium threshold and the heavy-quarks can be treated perturbatively. Thus we included into the numerical estimates the mixed $\mathcal{O}(\alpha\alpha_s)$ corrections due to the virtual gluon exchange inside the heavy quark loop. The contributions of the τ-lepton, c, and b-quark are suppressed with respect to the muon at least by one order of magnitude (*c.f.* Fig. 1). Thus the total heavy-flavor contribution is dominated by the muon loop and reaches about 1/2 permille at large scattering angles. Let us now discuss the top quark contribution to Bhabha scattering at high energies characteristic to the ILC. The large electroweak decay width of the top quark serves as an infrared regulator and suppresses the hadronization effects. Thus our perturbative result is applicable also at the energy about and above the top-antitop threshold. In Fig. 3 the contribution of top quark is plotted as function of the scattering angle at $\sqrt{s} = 500$ GeV. At large angles it reaches about 1/4 permille in magnitude. Note that the corrections are of the same magnitude as the nonlogarithmic pure photonic terms [8] if θ is not close to 0° or 180° and vanish for strictly forward scattering.

To this moment we have not mentioned the contribution of light quarks. In principle, it requires a special treatment because of hadronization effects (see *e.g.* [18]). However, this contribution can be estimated by naïve use of the perturbative result with the constituent quark masses. Such estimates are normally in rough agreement with the result of the rigorous analysis based on the dispersion approach. Adopting $m_d = m_u = 130$ MeV dictated by the hadronic threshold, we find the contribution of d and u-quark to be below one permille for $\sqrt{s} = 1$ GeV. It is not clear whether a more refined calculation is necessary at the present level of experimental

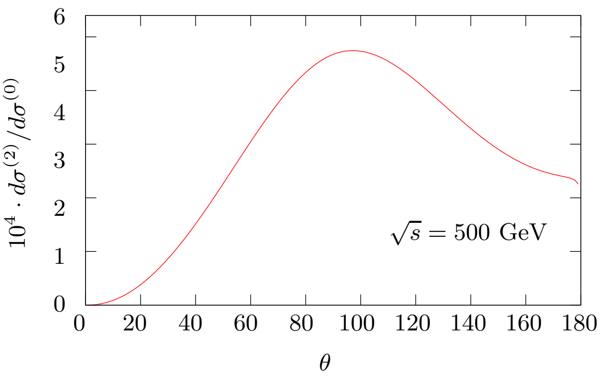

FIG. 3: Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 500$ GeV due to a closed loop of t-quark ($m_t = 170.9$ GeV).

accuracy. Thus, we have full control over the two-loop corrections to Bhabha scattering at low energies where the QED contributions are dominant. The accuracy of the luminosity determination is now restricted only by the precision of the Monte-Carlo event generators for the hard photon and electron-positron pair emission [19]. To achieve such an accuracy for the large angle scattering at the ILC one has to perform more careful analysis of the hadronic contributions and take into account also yet unknown two-loop electroweak corrections.

To conclude, we have derived the two-loop radiative corrections to Bhabha scattering due to vacuum polarization by the virtual heavy-flavor fermion-antifermion pairs which completes the QED analysis in two loops. This solves one of the current front-line problems in perturbative quantum field theory. The result is crucial for the high-precision physics at electron-positron colliders. It removes the last piece of pure theoretical uncertainty in luminosity determination at low-energy accelerators and gives the proper account for the top quark effects at the ILC. We believe that our approach will be very useful for the high-order perturbative analysis in a wide class of the processes with mass hierarchy.

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[1] J. H. Bhabha, Proc.Roy.Soc. **154A**, 195 (1935).

[2] S. Jadach *et al.* in G. Altarelli, T. Sjöstrand and F. Zwirner (eds.), *Physics at LEP2*, CERN-96-01.

[3] J. A. Aguilar-Saavedra *et al.* [ECFA/DESY LC Physics Working Group], arXiv:hep-ph/0106315.

[4] A. Aloisio *et al.* [KLOE Collaboration], Phys. Lett. B **606**, 12 (2005).

[5] S. Eidelman [CMD-2 and SND Collaborations], PoS **HEP2005**, 114 (2006).

[6] A. Czarnecki and W. J. Marciano, Phys. Rev. D **64**, 013014 (2001).

[7] R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, and J. J. van der Bij, Nucl. Phys. B **701** 121 (2004); *ibid.* **B716**, 280 (2005).

[8] A. A. Penin, Phys. Rev. Lett. **95**, 010408 (2005); Nucl. Phys. **B734**, 185 (2006).

[9] E. W. N. Glover, J. B. Tausk, and J. J. Van der Bij, Phys. Lett. B **516** 33 (2001).

[10] Z. Bern, L. Dixon, and A. Ghinculov, Phys. Rev. D **63**, 053007 (2001).

[11] T. Becher and K. Melnikov, JHEP **0706**, 084 (2007).

[12] S. Actis, M. Czakon, J. Gluza, and T. Riemann, Nucl. Phys. **B786**, 26 (2007).

[13] T. Kinoshita, J. Math. Phys. **3**, 650 (1962); T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).

[14] D. R. Yennie, S. C. Frautschi, and H. Suura, Annals Phys. **13**, 379 (1961).

[15] J. Frenkel and J. C. Taylor, Nucl. Phys. B **116**, 185 (1976).

[16] S. Laporta and E. Remiddi, Phys. Lett. B **379** (1996) 283; S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [arXiv:hep-ph/0102033].

[17] A. V. Kotikov, Phys. Lett. B **254** (1991) 158; E. Remiddi, Nuovo Cim. A **110** (1997) 1435.

[18] B. A. Kniehl, M. Krawczyk, J. H. Kuhn, and R. G. Stuart, Phys. Lett. B **209**, 337 (1988).

[19] G. Balossini, C. M. Carloni Calame, G. Montagna, O. Nicrosini, and F. Piccinini, Nucl. Phys. B **758**, 227 (2006).

[20] We do not consider the trivial correction given by two heavy-fermion loop insertions.