SFB/CPP-08-06
TTP08-03

Massless propagators: applications in QCD and QED

P. A. Baikov
Institute of Nuclear Physics, Moscow State University, Moscow 119899, Russia
E-mail: baikov@theory.sinp.msu.ru

K. G. Chetyrkin*
Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany
E-mail: chet@particle.uni-karlsruhe.de

J. H. Kühn
Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany
E-mail: jk@particle.uni-karlsruhe.de

We report on two recent results based on the evaluation of five-loop massless propagators in QCD and QED: (i) corrections of order α_s^4 to the absorptive part of the polarization function in QCD with $n_f = 3$; (ii) the five-loop contribution to the β function of quenched QED.

8th International Symposium on Radiative Corrections (RADCOR)
October 1-5 2007
Florence, Italy

*Speaker.

1. Introduction

Propagators — that is Feynman Integrals depending on only one external momenta — appeared in QFT from its very beginning, for instance, in the description of the vacuum polarization in QED and since then constitute an important set of Feynman integrals to deal with.

Massless propagators are an indispensable tool in Renormalization Group (RG) calculations within the framework of Dimensional Regularization [1] and the so-called Minimal Subtractions schemes [2].

Massless propagators appear in many important physical applications. The total cross-section of e^+e^- annihilation into hadrons, the Higgs decay rate into hadrons, the semihadronic decay rate of the τ lepton, the running of the fine structure coupling are all computable in the high energy limit in terms of massless propagators.

The strong coupling constant α_s is one of the three fundamental gauge couplings constants of the Standard Model of particle physics. Its precise determination is one of the most important aims of particle physics. One of the most precise and theoretically safe determination of α_s is based on measurements of the cross section for electron-positron annihilation into hadrons. These have been performed in the low-energy region between 2 GeV and 10 GeV and, in particular, at and around the Z resonance at 91.2 GeV. Conceptually closely related is the measurement of the semileptonic decay rate of the τ-lepton, leading to a determination of α_s at a scale below 2 GeV.

From the theoretical side, in the framework of perturbative QCD, these rates and cross sections are evaluated as inclusive rates into massless quarks and gluons [3, 4]. (Power suppressed mass effects are well under control for e^+e^--annihilation, both at low energies and around the Z resonance, and for τ decays [5, 6, 7, 8, 9, 10], and the same applies to mixed QCD and electroweak corrections [11, 12]).

The ratio $R(s) \equiv \sigma(e^+e^- \to \mathrm{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is expressed through the absorptive part of the correlator

$$\Pi_{\mu\nu}(q) = i\int \mathrm{d}x e^{iqx}\langle 0|T[\ j^{\mathrm{em}}_{\mu}(x) j^{\mathrm{em}}_{\nu}(0)\ ]|0\rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(-q^2)\,, \tag{1.1}$$

with the hadronic EM current $j^{\mathrm{em}}_{\mu} = \sum_f Q_f \overline{\Psi}_f \gamma_\mu \psi_f$, and Q_f being the EM charge of the quark f. The optical theorem relates the inclusive cross-section and thus the function $R(s)$ to the discontinuity of Π in the complex plane

$$R(s) = 12\,\pi\,\mathrm{Im}\,\Pi(-s - i\delta)\,. \tag{1.2}$$

The renormalization mode of the polarization operator $\Pi(Q^2)$ reads (see, e.g. [3])

$$1 + e^2\,\Pi(Q^2/\mu^2, \alpha_s) = Z_3^{\mathrm{ph}} + e^2\,\Pi_0(Q^2, \alpha_s^0),, \tag{1.3}$$

where $e = \sqrt{\alpha 4\pi}$ and α is the (renormalazied) fine structure constant. Eq. (1.3) can be naturally deduced from the connection between $\Pi(Q^2)$ and the photon propagator $D_{\mu\nu}(q)$

$$D_{\mu\nu}(q) = g_{\mu\nu}\frac{i}{q^2}\frac{1}{1 + e^2\,\Pi(q^2 = -Q^2)}.$$

It is also convenient to introduce the Adler function as

$$D(Q^2) = -12\pi^2 Q^2 \frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi(Q^2) = \int_0^\infty \frac{Q^2\, R(s) ds}{(s+Q^2)^2}. \tag{1.4}$$

We define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} d_i a_s^i(Q^2), \;\; R(s) = \sum_{i=0}^{\infty} r_i a_s^i(s), \tag{1.5}$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = Q^2$ or o $\mu^2 = s$ for the Euclidian and Minkowskian functions respectively. The results for generic values of μ can be easily recovered with standard RG techniques.

For the vector correlator the terms of order a_s^2 and a_s^3 have been evaluated nearly thirty and about fifteen years ago [13, 14], respectively. The a_s^4 corrections are conveniently classified according to their power of n_f, with n_f denoting the number of light quarks. The $a_s^4 n_f^3$ term is part of the "renormalon chain", the evaluation of the next term, of order $a_s^4 n_f^2$, was a test case for the techniques used extensively in this paper and, furthermore, led to useful insights into the structure of the perturbative series already [15]. The rest: two remaining most difficult terms of orders $a_s^4 n_f^2$ and $a_s^4 n_f^0$ of equivalent complexity[1] are not yet known. The results of their (partial) evaluation will be presented in the talk.

2. Calculation of d_4 at $n_f = 3$

The complete five-loop calculation requires the evaluation of 2671 Abelian quenched plus about seventeen thousand non-abelian and non-quenched diagrams (see Fig 1).

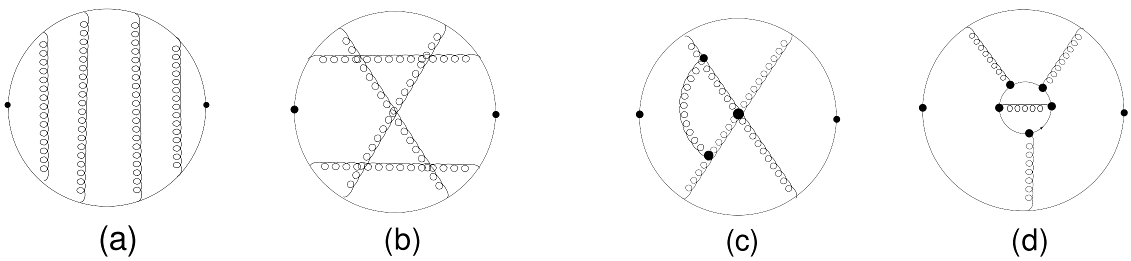

Figure 1: (a) (a),(b) Examples of five -loop quenched Abelian diagrams of order $a_s^4 n_f^0$. (c) A five -loop non-abelian diagram. (d) A five -loop non-quenched abelian diagram of order $a_s^4 n_f$

Using "infrared rearrangement" [16], the R^* operation [17] and the prescriptions formulated in [18] to algorithmically resolve the necessary combinatorics, it is possible to express the absorptive part of the five-loop diagrams in terms of four-loop massless propagator integrals ("p-integrals"). Using then a representation for Feynman integrals proposed in [19, 20], these p-integrals can be reduced to a sum of 28 master integrals, with coefficients which are rational functions of the space-time dimension D. These coefficients were, in a first step, evaluated in the large-D limit, and,

[1] In the sense that they both contain all possible topologies.

after evaluating sufficiently many terms of the $1/D$ expansion, fully reconstructed [21]. This direct and largely automatic procedure required enormous computing resources and was performed on a 32+8-node SGI parallel computer with SMP architecture and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using a parallel version [22] of FORM [23].

The construction of the large-D limit requires in general huge storage resources, which naturally constrains the structure of the input p-integrals: they should better not contain any extra parameters like color coefficients, n_f–the number of light quark flavours contributing to internal fermion loops–, and so on. As a result we are forced to use a "slice" approach: that is to set all color coefficients to their numerical values and to fix n_f to some integer. Thus, in order to compute the remaining two not-yet-known α_s^4 contributions to $R(s)$ one should compute two slices: $n_f = 0$ and $n_f = n_0$. Here n_0 stands for any non-zero integer which could be chosen at will. Since the evaluation of every single slice is a problem by itself, we decided to start from $n_0 = 3$ as the result has important physical applications for the analysis of the QCD corrections to hadronic τ-decays (see, e.g. [4]).

Our result reads (we suppress the trivial factor $3\sum_f Q_f^2$ throughout[2])

$$d_4(n_F = 3) = \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7 \tag{2.1}$$

$$\approx 49.0757. \tag{2.2}$$

The corresponding expression for $R(s)$ is:

$$1 + a_s + 1.6398\,a_s^2 + 6.3710\,a_s^3 - 106.8798\,a_s^4. \tag{2.3}$$

Since it will presumably take a long time until the next term of the perturbative series will be evaluated, it is of interest to investigate the predictive power of various optimization schemes empirically. Using the principles of "Fastest Apparent Convergence" (FAC) [24] or of "Minimal Sensitivity" (PMS) [25], which happen to coincide in this order, one gets [26, 27]

$$d_4^{\text{pred}}(n_f = 3) = 27 \pm 16,$$

with the central value of differing significantly from the exact result

$$d_4^{\text{exact}}(n_f = 3) = 49.08. \tag{2.4}$$

However, within the error estimates [27], predicted and exact values are in agreement. The picture changes, once these estimates are used to predict the coefficient r_4. The prediction for the final result is significantly closer (in a relative sense) to the result of the exact calculation:

$$r_4^{\text{pred}}(n_f = 3) = -129 \pm 16, \;\; r_4^{\text{exact}}(n_f = 3) = -106.88. \tag{2.5}$$

This is in striking contrast to the case of the scalar correlator, where the predictions for the dynamical terms work well, but, as a consequence of the strong cancellations between dynamical and kinematical terms fail completely in the Minkowskian region [28].

[2]In this paper we present the results for the so-called "non-singlet" diagrams, where one and the same closed quark line is connected to the external currents. These are sufficient for a complete description of τ-decays. For e^+e^- annihilation they correspond to the dominant terms proportional $\sum_f Q_f^2$. The singlet contributions proportional $(\sum_f Q_f)^2$ arise for the first time in $\mathcal{O}(\alpha_s^3)$. They are known to be small, and will be evaluated at a later point.

3. Colour structure of d_4

In general, it would be of great interest to know the α_s^4 contribution for a generic color group. The structure of the result for, say, Adler function, may be predicted from [29]:

$$d_4 = \sum_{i=1,12} d_{4,i}\, c_{4,i}, \tag{3.1}$$

with the coefficients $d_{4,i}$ being polynomials in $\ln(\frac{\mu^2}{Q^2})$ and the following colour factors read

$$\begin{aligned} c_4 = \{ & d_R C_F^4\,, d_R C_F^3 C_A\,, d_R C_F^2 C_A^2\,, d_R C_F C_A^3\,, d_R C_F^3 T_F n_f\,, d_R C_F^2 C_A T_F n_f\,, \\ & d_R C_F C_A^2 T_F n_f\,, d_R C_F^2 T_F^2 n_f^2\,, d_R C_F C_A T_F^2 n_f^2\,, d_R C_F T_F^3 n_f^3\,, d_F^{abcd} d_A^{abcd}\,, n_f d_F^{abcd} d_F^{abcd} \}. \end{aligned} \tag{3.2}$$

Here $[T^a T^a]_{ij} = C_F \delta_{ij}$ and $f^{acd} f^{bcd} = C_A \delta^{ab}$ are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra and $\mathrm{tr}(T^a T^b) = T_F \delta^{ab}$ is the trace normalization of the fundamental representation. d_R is the dimension of the fermion representation (i.e. the number of quark colours) and n_f is the number of quark flavors. The exact definitions for the higher order group invariants $d_F^{abcd} d_A^{abcd}$ and $d_F^{abcd} d_F^{abcd}$ are given in [29]. For QCD with SU(3):

$$C_F = 4/3\,, C_A = 3\,, T_F = 1/2\,, d_R = 3\,, d_F^{abcd} d_A^{abcd} = \frac{15}{2}\,, d_F^{abcd} d_F^{abcd} = \frac{5}{12}.$$

We observe that to get the full colour structure of d_4 we should compute as many as eight extra slices (as four slices are already available: $d_{4,10}$ [30], $d_{4,9}$ and $d_{4,8}$ [15], as well as $SU(3)$ with $n_f = 3$). Thus, complete evaluation of d_4 for generic colour group is a matter of future.

However, not all slices are equally difficult. For the case of the abelian U(1) group (QED!) the colour coefficients have especially simple form:

$$C_F = 1\,, C_A = 0\,, T_F = 1\,, d_R = 1\,, d_F^{abcd} d_A^{abcd} = 0\,, d_F^{abcd} d_F^{abcd} = 1.$$

In addition the corresponding diagrams are somewhat simpler due to the absence of the three- and four-gluon couplings as well as any vertexes with ghost lines. An even simpler slice corresponds to case of quenched QED (qQED):

$$C_F = 1\,, C_A = 0\,, T_F = 1\,, d_R = 1\,, n_f = 0\,, d_F^{abcd} d_A^{abcd} = 0\,, d_F^{abcd} d_F^{abcd} = 0.$$

Thus, in general the quenched QED contribution to d_n is equal to the coefficient in front of the $d_R\,(C_F)^n$ structure in the colour decomposition of d_n.

Up to order α^3 the Adler function for qQED reads [14]

$$D^{\mathrm{qQED}}(\alpha, Q) = 1 + 3A - \frac{3}{2}A^2 - \frac{69}{2}A^3, \tag{3.3}$$

where $A = A(\mu) = \frac{\alpha(\mu)}{4\pi}$. Note that $D^{\mathrm{qQED}}(\alpha, \mu/Q)$ does not depend on its second argument at all, which means that it is an essentially finite, scheme-independent quantity. It is instructive to compare eq. (3.3) to the β-function of qQED, also known to the same order from [31, 32, 33]:

$$\beta^{\mathrm{qQED}} = \frac{4}{3}A\left\{1 + 3A - \frac{3}{2}A^2 - \frac{69}{2}A^3\right\}. \tag{3.4}$$

One observes the validity of the relation

$$D^{\mathrm{qQED}}(\alpha, Q) = \frac{4}{3} A \, \beta^{\mathrm{qQED}} \tag{3.5}$$

which can be easily proven in all orders of perturbation theory starting from (1.3). Thus, the evaluation of the five-loop contribution to β^{qQED} provides with us another slice, namely with $d_{4,1}$.

On the other hand, such a calculation is of importance by itself. Indeed, the β- function of qQED possesses a number of remarkable and intriguing features.

- It is gauge and scheme independent in all orders.
- Its coefficients are simple rational numbers at three and four loops[3].
- If $\beta^{\mathrm{qQED}}(\alpha_0) = 0$ then $\alpha = \alpha_0$ leads to a self-consistent **finite** solution of (massless) QED [34, 35]

A detailed knot-theoretic explanation of the rationality property at three loop level was given in [36, 37]. At four loop the problem was thoroughly investigated with the help of a dimensionally background-field method in [37, 38]. Some hope that the rationality property is not accidental but also holds in higher orders has been expressed in [38]. Unfortunately, no clear unambiguous prediction as for the structure of higher (five loops and beyond) orders has ever been made.

We have computed $d_{4,1}$. Our five loop result for the β function reads:

$$\beta^{\mathrm{qQED}} = \frac{4}{3}A + 4A^2 - 2A^3 - 46A^4 + \left(\frac{4157}{6} \; + \; 128\,\zeta_3\right) A^5. \tag{3.6}$$

Thus our calculation of $d_{4,1}$ has solved the problem: the rationality ceases to exist starting from fifth loop.

Acknowledgments. We thank David Broadhurst and Dirk Kreimer for useful discussions. This work was supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 “Computational Particle Physics”, by INTAS (grant 03-51-4007) and by RFBR (grant 05-02-17645). The computer calculations were partially performed on the HP XC4000 super computer of the federal state Baden-Württemberg at the High Performance Computing Center Stuttgart (HLRS) under the grant “ParFORM”.

Note added. Recently we have finished the calculation of d_4 for a generic value of n_f. The reader is referred to [39] for details and for the discussion of phenomenological applications of the result.

References

[1] G. ’t Hooft and M.J.G. Veltman, Nucl. Phys. B44 (1972) 189.

[2] G. ’t Hooft, Nucl. Phys. B61 (1973) 455.

[3] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Rep. 277 (1996) 189, and references therein.

[4] M. Davier, A. Hocker and Z. Zhang, Rev. Mod. Phys. 78 (2006) 1043, and references therein.

[3] The rationality at one and two loops takes place for any β-function irrespectively of the (massless) theory.

[5] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, Phys. Lett. B371 (1996) 93, hep-ph/9511430.
[6] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, Nucl. Phys. B482 (1996) 213, hep-ph/9606230.
[7] K.G. Chetyrkin and J.H. Kühn, Phys. Lett. B248 (1990) 359.
[8] K.G. Chetyrkin and J.H. Kühn, Nucl. Phys. B432 (1994) 337, hep-ph/9406299.
[9] K.G. Chetyrkin, R.V. Harlander and J.H. Kühn, Nucl. Phys. B586 (2000) 56, hep-ph/0005139.
[10] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Nucl. Phys. Proc. Suppl. 135 (2004) 243.
[11] A. Czarnecki and J.H. Kühn, Phys. Rev. Lett. 77 (1996) 3955, hep-ph/9608366.
[12] R. Harlander, T. Seidensticker and M. Steinhauser, Phys. Lett. B426 (1998) 125, hep-ph/9712228.
[13] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, Phys. Lett. B85 (1979) 277.
[14] S.G. Gorishnii, A.L. Kataev and S.A. Larin, Phys. Lett. B259 (1991) 144.
[15] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 88 (2002) 012001, hep-ph/0108197.
[16] A.A. Vladimirov, Theor. Math. Phys. 43 (1980) 417.
[17] K.G. Chetyrkin and V.A. Smirnov, Phys. Lett. B144 (1984) 419.
[18] K.G. Chetyrkin, Phys. Lett. B391 (1997) 402, hep-ph/9608480.
[19] P.A. Baikov, Phys. Lett. B385 (1996) 404, hep-ph/9603267.
[20] P.A. Baikov, Nucl. Instrum. Meth. A389 (1997) 347, hep-ph/9611449.
[21] P.A. Baikov, Phys. Lett. B634 (2006) 325, hep-ph/0507053.
[22] M. Tentyukov et al., (2004), cs.sc/0407066.
[23] J.A.M. Vermaseren, (2000), math-ph/0010025.
[24] G. Grunberg, Phys. Rev. D29 (1984) 2315.
[25] P.M. Stevenson, Phys. Rev. D23 (1981) 2916.
[26] A.L. Kataev and V.V. Starshenko, Mod. Phys. Lett. A10 (1995) 235, hep-ph/9502348.
[27] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. D67 (2003) 074026, hep-ph/0212299.
[28] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 96 (2006) 012003, hep-ph/0511063.
[29] J.A.M. Vermaseren, S.A. Larin and T. van Ritbergen, Phys. Lett. B405 (1997) 327, hep-ph/9703284.
[30] M. Beneke, Nucl. Phys. B405 (1993) 424.
[31] J.L. Rosner, Phys. Rev. Lett. 17 (1966) 1190.
[32] J.L. Rosner, Annals Phys. 44 (1967) 11.
[33] S.G. Gorishnii et al., Phys. Lett. B256 (1991) 81.
[34] K. Johnson, R. Willey and M. Baker, Phys. Rev. 163 (1967) 1699.
[35] K. Johnson and M. Baker, Phys. Rev. D8 (1973) 1110.
[36] D.J. Broadhurst, R. Delbourgo and D. Kreimer, Phys. Lett. B366 (1996) 421, hep-ph/9509296.
[37] D.J. Broadhurst, Phys. Lett. B466 (1999) 319, hep-ph/9909336.
[38] D.J. Broadhurst, (1999), hep-th/9909185.
[39] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002, arXiv:0801.1821 [hep-ph].