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Applying Mellin-Barnes technique and Gröbner bases to the three-loop static potential

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The Mellin–Barnes technique to evaluate master integrals and the algorithm called FIRE to solve IBP relations with the help of Gröbner bases are briefly reviewed. In FIRE, an extension of the classical Buchberger algorithm to construct Gröbner bases is combined with the well-known Laporta algorithm. It is explained how both techniques are used when evaluating the three-loop correction to the static QCD quark potential. First results are presented: the coefficients of n_l^3 and n_l^2, where n_l is the number of light quarks.

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1. Introduction

At the modern level of analytic calculations in elementary particle physics one needs to evaluate thousands and millions of multiloop Feynman integrals. To evaluate a family of Feynman integrals which have the same structure of the integrand and differ by powers of propagators (indices) the standard strategy is to apply integration-by-parts (IBP) relations and solve the problem in two steps: a reduction of any given Feynman integral to so-called master integrals and the evaluation of the master integrals. In this contribution we describe the computation of Feynman integrals needed for the evaluation of the three-loop corrections to the static QCD quark potential. In particular we describe the use of the Mellin–Barnes technique to evaluate master integrals and the application of Gröbner bases to solve the IBP relations.

In Section 2 we present a brief review of the method of Mellin–Barnes (MB) representation and exemplify it by the evaluation of a non-trivial integral contributing to the three-loop potential. Afterwards we explain in Section 3 the main features of the algorithm called `FIRE` (Feynman Integral REduction) which is based on an extension of the classical Buchberger algorithm to construct Gröbner bases (see, e.g., Ref. [1]). In `FIRE` Gröbner bases are naturally combined with the well-known Laporta algorithm. Finally, in Section 4 we present the first results of this evaluation: the contributions proportional to n_l^3 and n_l^2, where n_l is the number of massless quarks.

2. Mellin–Barnes technique

The MB representation

$$\frac{1}{(X+Y)^{\lambda}} = \int_{-i\infty}^{+i\infty} \frac{Y^z}{X^{\lambda+z}} \frac{\Gamma(\lambda+z)\Gamma(-z)}{\Gamma(\lambda)} \frac{\mathrm{d}z}{2\pi i} \tag{2.1}$$

can be applied to replace a sum of two terms raised to some power by their products in some powers. For planar diagrams, experience shows that a minimal number of MB integrations is achieved if one introduces them loop by loop, i.e. one derives a MB representation for a one-loop subintegral, inserts it into a higher two-loop integral, etc. Consider, for example, the dimensionally regularized Feynman integral of Fig. 1 which we denote by $F(a_1,\ldots,a_{11})$. A straightforward implementation of the loop-by-loop strategy leads to a six-fold MB representation which reads

$$F(a_1,\ldots,a_{11}) = \frac{\left(i\pi^{d/2}\right)^3 (-q^2)^{6-a_{1,\ldots,8}-a_{9,10,11}/2-3\varepsilon} 2^{a_{9,11}-2}}{(v^2)^{a_{9,10,11}/2}\sqrt{\pi}\prod_{i=1,3,4,6,7,8,9,11}\Gamma(a_i)\Gamma(4-a_{3,4,8,11}-2\varepsilon)\Gamma(4-a_{1,6,7,9}-2\varepsilon)}$$

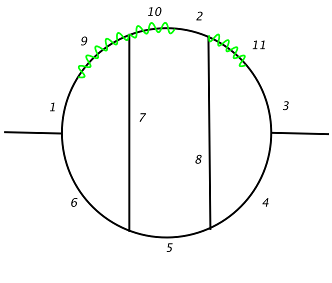

Figure 1: Feynman integral appearing in the calculation of the three-loop corrections to the static potential. The straight and wiggled lines correspond to massless scalar and static propagators, respectively. The numbers next to the lines refers to the corresponding index a_i.

$$\times \frac{1}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} \prod_{j=1}^{6} (\Gamma(-z_j) \mathrm{d}z_j) \frac{\Gamma(a_4 + z_{1,2})\Gamma(a_6 + z_{4,5})\Gamma(1/2 - z_3)\Gamma(1/2 - z_6)}{\Gamma(a_5 - z_{2,5})\Gamma(a_{1,2,3,4,6,7,8} + a_{9,11}/2 + 2\varepsilon - 4 + z_{1,\ldots,6})}$$
$$\times \frac{\Gamma(a_9/2 + z_6)\Gamma(a_{11}/2 + z_3)\Gamma(2 - a_{3,4} - a_{11}/2 - \varepsilon - z_1 + z_3)\Gamma(a_{11}/2 - 2 + a_{3,4,8} + \varepsilon + z_{1,2,3})}{\Gamma(a_{10}/2 + 1/2 - z_{3,6})\Gamma(8 - a_{1,\ldots,8} - a_{9,10,11}/2 - 4\varepsilon - z_{1,4} + z_{3,6})}$$
$$\times \Gamma(a_{1,\ldots,8} + a_{9,10,11}/2 - 6 + 3\varepsilon + z_{1,4})\Gamma(6 - a_{1,2,3,4,6,7,8} - a_{9,10,11}/2 - 3\varepsilon - z_{1,2,4,5})$$
$$\times \Gamma(2 - a_{4,8} - a_{11}/2 - \varepsilon - z_{2,3})\Gamma(2 - a_{1,6} - a_9/2 - \varepsilon - z_4 + z_6)\Gamma(2 - a_5 - a_{10}/2 - \varepsilon + z_{2,3,5,6})$$
$$\times \Gamma(2 - a_{6,7} - a_9/2 - \varepsilon - z_{5,6})\Gamma(a_{1,6,7} + a_9/2 - 2 + \varepsilon + z_{4,5,6}) , \qquad (2.2)$$

where $a_{3,4,8,11} = a_3 + a_4 + a_8 + a_{11}$, $z_{1,2,3} = z_1 + z_2 + z_3$, etc. By definition, any integration contour over z_i should go to the right (left) of poles of Gamma functions with $+z$-dependence ($-z$-dependence).

There are two strategies for resolving the singularities in ε in MB integrals suggested in Refs. [2, 3] (see also Chapter 4 of [4]). The second one was formulated algorithmically [5, 6], and the corresponding public code `MB.m` [6] has become by now a standard way to evaluate MB integrals in an expansion in ε. It can be combined with the program `AMBRE` [7] which can be used to derive MB representations in the loop-by-loop approach. Using `MB.m` and evaluating the resulting finite MB integrals by corollaries of Barnes lemmas we have obtained, for example, the following result for one of the master integrals occuring in the reduction of $F(a_1, \ldots, a_{11})$

$$F(1, \ldots, 1, 0, 1) = -\frac{(i\pi^{d/2})^3}{(-q^2)^{3+3\varepsilon} v^2} \left[\frac{56\pi^4}{135\varepsilon} + \frac{112\pi^4}{135} + \frac{16\pi^2 \zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\varepsilon) \right] . \qquad (2.3)$$

At the three- and four-loop level, the method of MB representation was successfully applied in [8, 9, 10, 11].

The loop-by-loop approach becomes problematic for non-planar diagrams. A typical phenomenon is that factors such as $(-1)^{z_i}$ arise. This means that the convergence of MB integrals at large values of $\mathrm{Im}(z_i)$ is no longer guaranteed. Moreover, poles in ε arise not only due to "gluing" of poles of different nature (a typical example is the product $\Gamma(\varepsilon + z)\Gamma(-z)$; there is no space between the first poles of the two gamma functions if $\varepsilon \to 0$) but also from the integration over large $\mathrm{Im}(z_i)$. A safe way to proceed with non-planar diagrams is to start from an alpha (Feynman) parametric representation and apply (2.1) to the two basic functions in this representation. (See also Ref. [12] for a discussion of this problem.)

3. `FIRE`: Feynman Integral REduction

There are three approaches to solve IBP relations [13] in a systematic way: Laporta's algorithm[14], Baikov's method [15] and two approaches using Gröbner bases which are described in Refs. [16] and [17, 18, 19, 20], respectively. To solve the reduction problems arising in the evaluation of the three-loop potential we use the latter algorithm whose computer implementation is called `FIRE`. It is based on a generalized Buchberger algorithm for constructing Gröbner-type bases associated with polynomials of shift operators. This method was recently used to evaluate a family of nontrivial three-loop Feynman integrals [21]. Let us in the following describe some new features of this algorithm.

Similarly to Laporta's algorithm, in our approach we work in a given *sector*, i.e. a domain of integer indices a_i where some indices are positive and the rest of the indices are non-positive. The aim is to express any integral from the sector in terms of master integrals of this sector *and* integrals from lower sectors, where at least one more index is non-positive. It turns out that in the higher sectors (with a small number of non-positive indices) the corresponding s-basis [18, 20, 19] (a kind of a Gröbner basis) can be constructed easily (and, in most cases, even automatically).

In the opposite situation where a lot of non-positive indices occur, s-bases are constructed not so easily. Usually there is the possibility to explicitly perform an integration over some loop momentum for general value of ε with results in terms of gamma functions. A straightforward way to do this leads to multiple summations and turns out to be impractical. However, there is an alternative approach which can be illustrated using the example of diagram of Fig. 1: consider the region $a_2, a_5, a_{10} \leq 0$ and $a_7, a_8 > 0$ (i.e. the union of the sectors with such restrictions). In this situation one can integrate over the middle loop momentum l which enters the propagators of the central subgraph with five lines. By constructing an s-basis for this region, it turns out possible to solve the IBP relations for the corresponding subintegral over l in order to express any such subintegral in terms of master integrals. In other words, the indices $a_2, a_5, a_{10}, a_7, a_8$ can be reduced to their boundary values, i.e. $a_2, a_5, a_{10} = 0,\ a_7, a_8 = 1$, up to integrals that drop out from this region. Then, after using this reduction procedure, it will be sufficient to use explicit integration formulae only for the boundary values of the indices. This replacement is very simple, without multiple summations.

It turned out possible to implement the solution of the recursive problem for the subgraph in terms of the Feynman integrals for the whole graph. In this reduction, pure powers of the parameters which are external for the subgraph transform naturally into the corresponding shift operators and their inverse. Integrals which are obtained from initial integrals by an explicit integration over a loop momentum in terms of gamma function usually involve a propagator with an analytic regularization by an amount proportional to ε (and, sometimes, 2ε). After this integration we obtain a two-loop reduction problem with seven indices which is then solved by `FIRE`.

After using Gröbner bases in higher sectors and an explicit integration in lower sectors, it is still necessary to solve the reduction problem in a relatively small number of intermediate sectors. In these cases we turn to Laporta's algorithm implemented as part of `FIRE`.

4. Evaluating three-loop static quark potential

The QCD potential between a static quark and its antiquark can be cast in the form

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s}{\vec{q}^2}\left[1 + \frac{\alpha_s}{4\pi}a_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^2}\right) + \cdots\right], \quad (4.1)$$

where the renormalization scale of α_s is set to $\vec{q}^2$. The one-loop contribution a_1 is known since almost 30 years and also the two-loop term has already been computed end of the nineties. Furthermore logarithmic contributions are known at three- and four-loop level. Explicit results and the references are nicely summarized in the review [22]. The non-logarithmic third-order term, a_3, is still unknown. It is conveniently be parametrized in the form

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}\,, \quad (4.2)$$

where n_l denotes the number of massless quarks. Using the techniques described above we evaluated the coefficients $a_3^{(3)}$ and the $C_A T_F^2$ part of $a_3^{(2)}$ which read

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3\,, \quad a_3^{(2)}\Big|_{C_A T^2} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2\,. \tag{4.3}$$

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References

[1] B. Buchberger and F. Winkler (eds.), *Gröbner Bases and Applications*, Cambridge University Press, 1998.

[2] V.A. Smirnov, Phys. Lett. B **460** (1999) 397.

[3] J.B. Tausk, Phys. Lett. B **469** (1999) 225.

[4] V. A. Smirnov, "Evaluating Feynman Integrals," Springer Tracts Mod. Phys. **211** (2004) 1; "Feynman integral calculus," *Berlin, Germany: Springer (2006) 283 p.*

[5] C. Anastasiou and A. Daleo, JHEP **0610** (2006) 031.

[6] M. Czakon, Comput. Phys. Commun. **175** (2006) 559.

[7] J. Gluza, K. Kajda and T. Riemann, Comput. Phys. Commun. **177** (2007) 879; J. Gluza, F. Haas, K. Kajda and T. Riemann, arXiv:0707.3567.

[8] V.A. Smirnov, Phys. Lett. B **547** (2002) 239; Phys. Lett. B **567** (2003) 193.

[9] Z. Bern, L.J. Dixon, and V.A. Smirnov, Phys. Rev. D **72** (2005) 085001.

[10] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, Phys. Rev. D **75** (2007) 085010.

[11] F. Cachazo, M. Spradlin and A. Volovich, JHEP **0607** (2006) 007; Phys. Rev. D **75** (2007) 105011; Phys. Rev. D **76** (2007) 106004.

[12] M. Czakon, A. Mitov and S. Moch, arXiv:0707.4139 [hep-ph].

[13] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B **192** (1981) 159.

[14] S. Laporta, Int. J. Mod. Phys. A15 (2000) 5087; T. Gehrmann and E. Remiddi, Nucl. Phys. B **601** (2001) 248, 287.

[15] P.A. Baikov, Phys. Lett. B **385** (1996) 404; Nucl. Instrum. Methods A **389** (1997) 347; Phys. Lett. B **474** (2000) 385; Nucl. Phys. Proc. Suppl. **116** (2003) 378; Phys. Lett. B **634** (2006) 325; V.A. Smirnov and M. Steinhauser, Nucl. Phys. B **672** (2003) 199.

[16] O.V. Tarasov, Acta Phys. Polon. B **29** (1998) 2655; Nucl. Instrum. Meth. A **534** (2004) 293.

[17] A.V. Smirnov and V.A. Smirnov, JHEP **0601**, 001 (2006).

[18] A.V. Smirnov and V.A. Smirnov, Nucl. Phys. Proc. Suppl. **160**, 80 (2006); V.A. Smirnov, Nucl. Phys. Proc. Suppl. **157**, 131 (2006).

[19] A.V. Smirnov, JHEP **0604** (2006) 026.

[20] A.V. Smirnov and V.A. Smirnov, PoS(ACAT) (2007) 085 [arXiv:0707.3993].

[21] A.G. Grozin, A.V. Smirnov and V.A. Smirnov, JHEP **0611**, 022 (2006).

[22] A. Vairo, AIP Conf. Proc. **964** (2007) 102.