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Hadronic Z- and τ-Decays in Order α_s^4

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Using recently developed methods for the evaluation of five-loop amplitudes in perturbative QCD, corrections of order α_s^4 for the cross section of electron-positron annihilation into hadrons and for the decay rates of the Z-boson and the τ-lepton into hadrons are evaluated. The new terms lead to a significant stabilization of the perturbative series, to a reduction of the theory uncertainly in the strong coupling constant α_s, as extracted from these measurements, and to a small shift of the central value, moving two central values closer together. The agreement between two values of α_s measured at vastly different energies constitutes a striking test of asymptotic freedom. Combining the results from Z and τ decays we find $\alpha_s(M_Z) = 0.1198 \pm 0.0015$ as one of the most precise and presently only NNNLO result for the strong coupling constant.

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The strong coupling constant α_s is one of the three fundamental gauge couplings constants of the Standard Model (SM) of particle physics. Its precise determination is one of the most important aims of particle physics. Experiments at different energies allow to test the predictions for its energy dependence based on the renormalization group equations, the comparison of the results obtained from different processes leads to critical tests of the theory and potentially to the discovery of physics beyond the Standard Model. Last not least, the convergence of the three gauge coupling constants related by SU(3)xSU(2)xU(1) to a common value, after evolving them to high energies, allows to draw conclusions about the possibility of embedding the SM in the framework of a Grand Unified Theory.

One of the most precise and theoretically safe determination of α_s is based on measurements of the cross section for electron-positron annihilation into hadrons. These have been performed in the low-energy region between 2 GeV and 10 GeV and, in particular, at and around the Z resonance at 91.2 GeV. Conceptually closely related is the measurement of the semileptonic decay rate of the τ-lepton, leading to a determination of α_s at a scale below 2 GeV.

From the theoretical side, in the framework of perturbative QCD, these rates and cross sections are evaluated as inclusive rates into massless quarks and gluons [1, 2]. (Power suppressed mass effects are well under control for e^+e^--annihilation, both at low energies and around the Z resonance, and for τ decays [3–8], and the same applies to mixed QCD and electroweak corrections [9, 10]).

The ratio $R(s) \equiv \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ is expressed through the absorptive part of the correlator of the electromagnetic current j_μ:

$$R(s) \;=\; 12\pi\, \Im\Pi(-s - i\epsilon), \tag{1}$$

$$3\, Q^2\, \Pi(Q^2) \;=\; i \int \mathrm{d}^4 x e^{iq\cdot x} \langle 0|\mathrm{T} j_\mu(x) j^\mu(0)|0\rangle, \tag{2}$$

with $Q^2 = -q^2$. It is also convenient to introduce the Adler function as

$$D(Q^2) = -12\, \pi^2 Q^2 \tfrac{\mathrm{d}}{\mathrm{d}Q^2} \Pi(Q^2) = \int_0^\infty \tfrac{Q^2\;R(s) ds}{(s+Q^2)^2},$$
$$R(s) = D(s) - \pi^2 \beta_0^2 \{ \tfrac{d_1}{3} a_s^3 + (d_2 + \tfrac{5}{6\,\beta_0} d_1\, \beta_1)\, a_s^4 \} + \dots\ .$$

We define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} \, d_i a_s^i(Q^2), \quad R(s) = \sum_{i=0}^{\infty} \, r_i a_s^i(s), \tag{3}$$

where $a_s \equiv \alpha_s/\pi$ and we have set the normalization scale $\mu^2 = Q^2$ or to $\mu^2 = s$ for the Euclidian and Minkowskian functions respectively. The results for generic values of μ can be easily recovered with standard RG techniques.

Note that the first three terms of the perturbative series for D and R coincide. Starting from r_3, terms proportional π^2 arise which can be predicted from those of lower order. It has been speculated that these "π^2-terms", also called "kinematical terms", might constitute a major part of the full higher order corrections (see, e.g. [11]); however, the validity of this hypothesis can only be established by the full calculation. Indeed, for the scalar correlator this assumption has been shown to fail completely [12].

For the vector correlator the terms of order a_s^2 and a_s^3 have been evaluated nearly thirty and about fifteen years ago [13, 14], respectively. The a_s^4 corrections are conveniently classified according to their power of n_f, with n_f denoting the number of light quarks. The $a_s^4 n_f^3$ term is part of the "renormalon chain", the evaluation of the next term, of order $a_s^4 n_f^2$, was a test case for the techniques used extensively in this paper and, furthermore, led to useful insights into the structure of the perturbative series already [15].

The complete five-loop calculation requires the evaluation of about twenty thousand diagrams (we have used

QGRAF [16] for their automatic generation). Using "infrared rearrangement" [17], the R^* operation [18] and the prescriptions formulated in [19] to algorithmically resolve the necessary combinatorics, it is possible to express the absorptive part of the five-loop diagrams in terms of four-loop massless propagator integrals. Using then a representation for Feynman integrals proposed in [20, 21], these integrals can be reduced to a sum of 28 master integrals, with coefficients which are rational functions of the space-time dimension D. These coefficients were, in a first step, evaluated in the large-D limit, and, after evaluating sufficiently many terms of the $1/D$ expansion, fully reconstructed [22]. This direct and largely automatic procedure required enormous computing resources and was performed on a 32+8-node SGI parallel computer with SMP architecture and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using a parallel version [23] of FORM [24].

In this paper we present the results for the so-called "non-singlet" diagrams, where one and the same closed quark line is connected to the external currents. These are sufficient for a complete description of τ-decays. For e^+e^- annihilation they correspond to the dominant terms proportional $\sum_i Q_i^2$. The singlet contributions proportional $(\sum_i Q_i)^2$ arise for the first time in $\mathcal{O}(\alpha_s^3)$. They are known to be small, and will be evaluated at a later point. Similar comments apply to the singlet contributions in Z decays.

The analytic result for the five loop term in the Adler function is given by (we suppress the trivial factor $3\sum_f Q_f^2$ throughout)

$$
\begin{aligned}
d_4 = {} & n_f^3\left[-\tfrac{6131}{5832} + \tfrac{203}{324}\zeta_3 + \tfrac{5}{18}\zeta_5\right] \\
& + n_f^2\left[\tfrac{1045381}{15552} - \tfrac{40655}{864}\zeta_3 + \tfrac{5}{6}\zeta_3^2 - \tfrac{260}{27}\zeta_5\right] \\
& + n_f\left[-\tfrac{13044007}{10368} + \tfrac{12205}{12}\zeta_3 - 55\zeta_3^2 + \tfrac{29675}{432}\zeta_5 + \tfrac{665}{72}\zeta_7\right] \\
& + \tfrac{144939499}{20736} - \tfrac{5693495}{864}\zeta_3 + \tfrac{5445}{8}\zeta_3^2 + \tfrac{65945}{288}\zeta_5 - \tfrac{7315}{48}\zeta_7 .
\end{aligned}
$$

The result contains transcendental numbers up to ζ_7, as expected from the structure of the masters. Although present in individual diagrams, ζ_4 and ζ_6 terms are absent in the complete, physical result.

The knowledge of d_4 leads straightforwardly to R at order α_s^4, for brevity given below in numerical form

$$
\begin{aligned}
R &= 1 + a_s + (1.9857 - 0.1152\, n_f)\, a_s^2 \\
&+ (-6.63694 - 1.20013 n_f - 0.00518 n_f^2)\, a_s^3 \qquad (4) \\
&+ (-156.61 + 18.77\, n_f - 0.7974\, n_f^2 + 0.0215\, n_f^3)\, a_s^4 .
\end{aligned}
$$

It is also instructive to explicitly display the genuine five-loop contributions to d_4 (underlined in (5,6) below) and the "kinematical" terms originating from the analytic continuation:

$$
\begin{aligned}
r_3 &= \underline{18.2} - 24.9 + (\underline{-4.22} + 3.02)\, n_f \qquad (5) \\
&+ (\underline{-0.086} + 0.091)\, n_f^2, \\
r_4 &= \underline{135.8} - 292.4 + (\underline{-34.4} + 53.2)\, n_f \qquad (6) \\
&+ (\underline{1.88} - 2.67)\, n_f^2 + (\underline{-0.010} + 0.032)\, n_f^3 .
\end{aligned}
$$

Since it will presumably take a long time until the next term of the perturbative series will be evaluated, it is of interest to investigate the predictive power of various optimization schemes empirically. Using the principles of "Fastest Apparent Convergence" (FAC) [25] or of "Minimal Sensitivity" (PMS) [26], which happen to coincide in this order, the central values of the predictions [27, 28]

$$d_4^{\rm pred}(n_f = 3,4,5) = 27 \pm 16, 8 \pm 18, -8 \pm 44$$

differ significantly from the exact result

$$d_4^{\rm exact}(n_f = 3,4,5) = 49.08, \quad 27.39, \quad 9.21. \qquad (7)$$

However, within the error estimates [28] predicted and exact values are in agreement. The picture changes, once these estimates are used to predict the coefficient r_4. Although sizable cancellations between "dynamical" and "kinematical" terms are observed for the individual n_f coefficients in (6) the predictions for the final results are significantly closer (in relative sense) to the results of the exact calculation:

$$
\begin{aligned}
r_4^{\rm pred}(n_f = 3,4,5) &= -129 \pm 16, \ -112 \pm 30, -97 \pm 44, \\
r_4^{\rm exact}(n_f = 3,4,5) &= -106.88, \quad -92.898, \quad -79.98.
\end{aligned}
$$

(This is in striking contrast to the case of the scalar correlator, where the predictions for the dynamical terms work well, but, as a consequence of the strong cancellations between dynamical and kinematical terms fail completely in the Minkowskian region.) Using FAC and the exact result for d_4, the coefficients d_5 and r_5 can be predicted (following [28, 29]) for $n_f = 3, 4, 5$, namely

$$
\begin{aligned}
d_5^{\rm pred}(n_f = 3,4,5) &= 275, \quad 152, \quad 89, \qquad (8) \\
r_5^{\rm pred}(n_f = 3,4,5) &= -505, \quad -134, \quad 168. \qquad (9)
\end{aligned}
$$

These terms may become of relevance for the International Linear Collider (ILC) running in the GIGA-Z mode with an anticipated precision of $\delta\alpha_s = 0.0005 — 0.0007$ [30], and already today for the analysis of τ-decays.

From the combined analysis of data for $\sigma(e^+e^- \to$ hadrons) in the region between 3 and 10 GeV a value

$$\alpha_s(9\ {\rm GeV}) = 0.160 \pm 0.034 \qquad (10)$$

has been obtained recently [31]. The shift in α_s from the inclusion of the α_s^4 term amounts to $\delta\alpha_s(5{\rm GeV}) = 0.002$ and is thus irrelevant compared to the large experimental error.

The situation is different for Z decays. The analysis of the electroweak working group [32] is based on eq. (4) with $n_f = 5$, including term up to $\mathcal{O}(\alpha_s^3)$ and leads to

$$\alpha_s(M_Z)^{NNLO} = 0.1185 \pm 0.0026 \qquad (11)$$

Since additional corrections (mixed QCD/electroweak or mass terms) are only weekly α_s-dependent we may consider $R(s = M_Z^2)$ as a pseudo-observable:

$$R(s = M_Z) = 1.03904 \pm 0.00087. \qquad (12)$$

Including the α_s^4 term leads to a shift $\delta\alpha_s(M_Z) = 0.0005$

$$\alpha_s(M_Z)^{NNNLO} = 0.1190 \pm 0.0026^{\rm exp}, \qquad (13)$$

The theory error may either be conservatively based on the shift produced by the last term (0.0005) or on the scale variation with $\mu/M_Z = \frac{1}{3} \div 3$, leading to ± 0.0002 and can be neglected in both cases.

Higher orders are of particular relevance in the low-energy region, for example in τ decays. The correction from perturbative QCD to the ratio

$$\begin{aligned} R_{\tau,V+A} &= \frac{\Gamma(\tau \to {\rm hadrons}_{S=0} + \nu_\tau)}{\Gamma(\tau \to l + \bar{\nu}_l + \nu_\tau)} \\ &= 3|V_{ud}|^2 S_{EW}(1 + \delta_0 + \delta'_{EW} + \delta_2 + \delta_{NP}) \end{aligned} \qquad (14)$$

is given by

$$1+\delta_0 = 2\int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) R(s). \qquad (15)$$

In the subsequent analysis we will use $S_{EW} = 1.0198 \pm 0.0006$ and $\delta'_{EW} = 0.001$ for the electroweak corrections, $\delta_2 = (-4.4 \pm 2.0) \times 10^{-4}$ for light quark mass effects, $\delta_{NP} = (-4.8 \pm 1.7) \times 10^{-3}$ for the nonperturbative effects [2, 33] and $V_{ud} = 0.97418 \pm 0.00027$ [34].

The perturbative quantity δ_0 can be evaluated in Fixed Order perturbation theory or with "Contour Improvement" as proposed in [35, 36]

$$\begin{aligned} \delta_0^{FO} &= a_s + 5.202\, a_s^2 + 26.366\, a_s^3 + 127.079\, a_s^4, \quad (16) \\ \delta_0^{CI} &= 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + 64.29\, a_s^4. \quad (17) \end{aligned}$$

(To obtain the coefficients in eq. (17) we have used $\alpha_s(M_\tau) = 0.334$ as a reference value.)

For the subsequent analysis we will use as starting point

$$\delta_0^{\rm exp} = 0.1998 \pm 0.0043_{\rm exp} \qquad (18)$$

as obtained from [37]

$$R_{\tau,V+A} = 3.471 \pm 0.011$$

which in turn is based on the "universality-improved" electronic branching ratio $B_e = (17.818 \pm 0.032)\%$ and the world average of the ratio of strange hadronic and electronic widths 0.1686 ± 0.0047. Starting from (18) one easily obtains the new values of $\alpha_s(M_\tau)$. Its dependence on the choice of d_4 (previously estimate and the new exact result) are summarized in Table I.

TABLE I: Results for $\alpha_s(M_\tau)$ for different values of d_4. The first line displays the α_s^3 results (with the α_s^4 terms set to zero in eqs. (16,17)). The second line uses the the previously predicted value for d_4, the last one uses the exact result (4). The first error is the experimental one; the second (theoretical) uncertainty in the value of α_s corresponds to changing the renormalization scale μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$.

	$\alpha_s^{FO}(M_\tau)$	$\alpha_s^{CI}(M_\tau)$
-	$0.337 \pm 0.004 \pm 0.03$	$0.354 \pm 0.006 \pm 0.02$
$d_4 = 25$	$0.325 \pm 0.004 \pm 0.02$	$0.347 \pm 0.006 \pm 0.009$
$d_4 = 49.08$	$0.322 \pm 0.004 \pm 0.02$	$0.342 \pm 0.005 \pm 0.01$

As stated above the theory error for α_s from Z decays is small compared to the experimental uncertainties. The situation is more problematic for τ decays and to some extent the theory error remains to be matter of choice. As anticipated in [28] it decreases significantly, once α_s^4 terms are included. However, the difference between the two methods stabilizes (this was checked in [28] by adding an estimate for the α_s^5 term) and must be considered as irreducible uncertainty. Given the input specified above we obtain as our final result

$$\alpha_s(M_\tau) = 0.332 \pm 0.005_{\rm exp} \pm 0.015_{\rm th}. \qquad (19)$$

For the central value we take mean value of FO and CI. For the theory error we take half of the difference between two methods (0.01) plus (module of) the estimated correction from α_s^5 term (-0.005), the latter being based on $d_5 = 275$ (see eq. (8)).

Applying four-loop running and matching [38–41] to (19) we arrive

$$\begin{aligned} \alpha_s(M_Z) &= 0.1202 \pm 0.0006_{\rm exp} \pm 0.0018_{\rm th} \pm 0.0003_{\rm evol}\,, \\ &= 0.1202 \pm 0.0019. \end{aligned} \qquad (20)$$

Here the evolution error receives contributions from the uncertainties in the c-quark mass (0.00003, $m_c(m_c) = 1.286(13)$ GeV [42]) and the b-quark mass (0.00001, $m_b(m_b) = 4.164(25)$ GeV [42]), the matching scale (0.0001, μ varied between $0.7\, m_q(m_q)$ and $3.0\, m_q(m_q)$), the four-loop truncation in the matching expansion (0.0001) and the four-loop truncation in the RGE equation (0.0003). (For the last two errors the size of the shift due the highest known perturbative term was treated as systematic uncertainty.) The errors had been added in quadrature.

Summary: The exact result for the α_s^4 term in the Adler function allows to extract the strong coupling constant from Z and τ decays with high precision. Including the exact α_s^4 leads to small shifts of the central value and to a significant reduction of the theory uncertainty. Note that the shifts in $\alpha_s(M_Z)$ from Z- and τ-decays, are opposite in sign and move the values in the *proper* direction, decreasing, thus, the current slight mismatch between two independent determinations of α_s.

The final results

$$\alpha_s(M_Z)|_Z = 0.1190 \pm 0.0026 \tag{21}$$

and

$$\alpha_s(M_Z)|_\tau = 0.1202 \pm 0.0019 \tag{22}$$

from these two observables, although based on measurements of vastly different energy scales, are in remarkable agreement. This constitutes a striking test of asymptotic freedom in QCD. The two values can be combined to

$$\alpha_s(M_Z) = 0.1198 \pm 0.0015.$$

This is one of the most precise and presently only N^3LO result.

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