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Three-Loop Chromomagnetic Interaction in HQET

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Abstract

We compute the three-loop QCD corrections to the quark chromomagnetic moment and thus obtain the matching coefficient and the anomalous dimension of the chromomagnetic interaction in HQET. As a byproduct we obtain the three-loop corrections to the quark anomalous magnetic moment.

Key words: Heavy Quark Effective Theory, radiative corrections
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1 Introduction

We consider Quantum Chromodynamics (QCD) with n_l light flavours and one heavy flavour Q. The interaction of a single heavy quark having momentum $m_Q v + k$ (m_Q is the on-shell mass and $v^2 = 1$) with gluons and light quarks in the situation when the residual momentum $k \ll m_Q$ (and momenta of light fields are also small) is described by the Heavy Quark Effective Theory (HQET) Lagrangian [1,2]

$$L = \bar{Q}_v iv \cdot DQ_v + \frac{1}{2m_Q} \left(O_k + C_{cm}(\mu) O_{cm}(\mu) \right) + \mathcal{O}\left(\frac{1}{m_Q^2}\right) , \tag{1}$$

where $\not{v}Q_v = Q_v$ is the HQET quark field and D denotes the covariant derivative (see the books [3,4] for more details). The kinetic energy operator

$$O_k = -\bar{Q}_v D_\perp^2 Q_v \,, \qquad D_\perp = D - v(v \cdot D) \,, \tag{2}$$

does not renormalize and its coefficient is equal to 1, to all orders, due to reparametrization invariance [5]. The chromomagnetic interaction operator is defined as

$$O_{cm} = \frac{1}{2}\bar{Q}_v G_{\mu\nu}\sigma^{\mu\nu} Q_v \,, \tag{3}$$

where $G_{\mu\nu} = g_s G^a_{\mu\nu} t_a$ is the gluon field strength tensor, t_a is a colour matrix, $\alpha_s = g_s^2/(4\pi)$ the strong coupling and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. It is responsible for the violation of the heavy-quark spin symmetry and thus, e.g., for the B–B^* mass splitting. The coefficient $C_{cm}(\mu)$ is obtained by matching the scattering amplitudes of an on-shell heavy quark in an external chromomagnetic field, expanded in the momentum transfer q up to the linear term, in the full theory (QCD) and the effective theory (HQET). If all flavours except Q are massless, all loop corrections in HQET vanish. It is most convenient to calculate the QCD scattering amplitude using the background field method [6].

The chromomagnetic interaction coefficient $C_{cm}(\mu)$ has been calculated at one-loop order in Ref. [1]. The one-loop anomalous dimension γ_{cm} of the chromomagnetic operator (3) follows from its μ dependence; it has also been found in Ref. [2]. The two-loop anomalous dimension has been obtained in Refs. [7,8], and the two-loop matching coefficient $C_{cm}(\mu)$ in Ref. [8]. All orders of perturbation theory in the large-β_0 limit were summed in Ref. [9]. The effect of a non-zero charm quark mass m_c on the bottom-quark chromomagnetic interaction at two loops has been investigated in Ref. [10]. In this paper (cf. Section 3) we calculate γ_{cm}, as well as $C_{cm}(\mu)$ at three loops, provided that all light flavours are massless. Using these results, we obtain the next-to-next-to-leading perturbative correction to the ratio

$$R = \frac{m_{B^*}^2 - m_B^2}{m_{D^*}^2 - m_D^2} \,, \tag{4}$$

which we discuss in Section 4.

The mass difference between the vector and pseudo-scalar B meson is also often studied with lattice gauge theory simulations where non-perturbative results for the operator matrix element can be obtained (see, e.g., Ref. [11] for an introduction to lattice HQET and Ref. [12] for a recent study). In principle there are two possibilities to make contact with the experimentally measured result: the matching can be performed perturbatively and non-pertubatively.

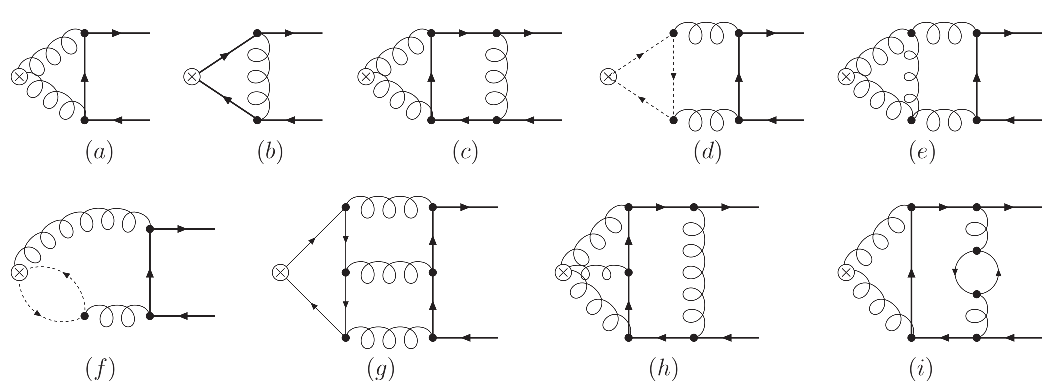

Fig. 1. Sample diagrams contributing to the quark chromomagnetic moment. Solid, curly and dotted lines denote quarks, gluons and ghosts, respectively. $\otimes$ represents the coupling of the background field. In the closed quark loops all flavours have to be considered.

In the first case the perturbatively computed n-loop matching coefficient and the $(n+1)$-loop result for the corresponding anomalous dimension have to be combined with the non-perturbative lattice results. Currently this is done for $n = 1$ which according to Ref. [12] induces an uncertainty of about 4%. With the new results of this paper this uncertainty can be significantly reduced. This topic is dicussed in Section 5.

We also investigate the heavy-quark magnetic moments. They have not yet been measured experimentally, however, for the bottom and the lighter quarks there are upper limits from LEP1 data [13]. For the bottom quark this limit is close to the Standard Model (SM) prediction including the two-loop QCD correction [14]. Thus, a more precise measurement at a future linear collider should be able to determine the bottom-quark anomalous magnetic moment and probe possible deviations from the SM.

The top-quark magnetic moment has not been measured so far. However, such a measurement would be very interesting since the top-quark couplings to photons or Z bosons are very sensitive to contributions from physics beyond the SM. Having this in mind, it is mandatory to have precise SM predictions for these couplings. In Section 6 we provide the three-loop QCD corrections to the coupling of the photon to heavy quarks.

2 The Calculation

To calculate the chromomagnetic moment we have to consider the quark–anti-quark–gluon vertex in the background-field formalism in QCD. Sample diagrams are depicted in Fig. 1. When both the quark and anti-quark are on the (renormalized) mass shell and have physical polarizations, the vertex

$\Gamma^\mu_a = \Gamma^\mu t_a$ can be decomposed into two form factors,

$$\Gamma^\mu = \gamma^\mu F_1(q^2) - \frac{i}{2m_Q}\sigma^{\mu\nu} q_\nu F_2(q^2)\,, \tag{5}$$

where $q = p_1 - p_2$ is the gluon momentum and p_1 and p_2 are the momenta of the quark and anti-quark, respectively.

The anomalous chromomagnetic moment is given by $\mu_c = Z_2^{\rm OS} F_2(0)$, where $Z_2^{\rm OS}$ is the quark wave function renormalization constant in the on-shell scheme. The total quark colour charge is given by $Z_2^{\rm OS} F_1(0) = 1$. Thus, $F_1(0)$ is the inverse of the on-shell wave function renormalization constant, which has been calculated to three-loops in Ref. [15] (see also Ref. [16]). Therefore, the calculation of $F_1(0)$ provides a strong check on the correctness of our result.

In order to extract the form factors, we use projection operators. They are conveniently obtained by introducing the momentum $p = (p_1 + p_2)/2$, since $p \cdot q = 0$. With this definition we have

$$\begin{aligned}
F_1(q^2) =& \frac{1}{2(d-2)(q^2 - 4m_Q^2)} \\
&\times \mathrm{Tr}\left\{ (\not{p}_1 + m_Q)\left(\gamma_\mu + \frac{4m_Q(d-1)}{q^2 - 4m_Q^2}\, p_\mu\right)(\not{p}_2 + m_Q)\,\Gamma^\mu \right\}\,, \qquad (6)\\
F_2(q^2) =& -\frac{2m_Q^2}{(d-2)q^2(q^2 - 4m_Q^2)} \\
&\times \mathrm{Tr}\left\{ (\not{p}_1 + m_Q)\left(\gamma_\mu + \frac{4m_Q^2 + (d-2)q^2}{m_Q(q^2 - 4m_Q^2)}\, p_\mu\right)(\not{p}_2 + m_Q)\,\Gamma^\mu \right\}. \qquad (7)
\end{aligned}$$

Since the projector for F_2 develops a pole for $q^2 = 0$, we cannot set $q^2 = 0$ from the beginning. Instead, we expand in q and keep all terms which are at most quadratic in q. In the final result the limit $q^2 = 0$ can be taken. Due to the expansion in q all occurring integrals are on-shell propagator-type integrals.

All Feynman diagrams are generated with `QGRAF` [17] and the various topologies are identified with the help of `q2e` and `exp` [18,19]. In a next step the reduction of the various functions to so-called master integrals has to be achieved. For this step we use the so-called Laporta method [20,21] which reduces the three-loop integrals to 19 master integrals. We use the implementation of Laporta's algorithm in the program `Crusher` [22]. It is written in `C++` and uses `GiNaC` [23] for simple manipulations like taking derivatives of polynomial quantities. In the practical implementation of the Laporta algorithm one of the most time-consuming operations is the simplification of the coefficients appearing in front of the individual integrals. This task is performed with the help of `Fermat` [24] where a special interface has been used (see Ref. [25]). The main features of the implementation are the automated generation of the integration-by-parts (IBP) identities [26] and a complete symmetrization of the diagrams. The master integrals are known from [15] (see also comments in Ref. [16]). To calculate the colour factors, we have used the program described in Ref. [27]. [1]

The calculation is performed for an arbitrary gauge parameter in order to use its cancellation as a check. However, at three-loop level the expressions for the individual diagrams become very big. Thus we discard all terms with more than linear ξ dependence. This also concerns the factors $1/(1-\xi)$ appearing in the vertex of a background field with two quantum gluons, which does not appear in the usual formulation of QCD. If this is done, our final result is gauge-parameter independent up to terms which are quadratic in ξ. Furthermore, our calculation of F_1 reproduces Z_2^{OS} — including the gauge-dependent terms [15,16].

3 Chromomagnetic moment

The renormalized scattering amplitude of an on-shell heavy quark with initial momentum $p_1 = m_Q v$ and final momentum $p_2 = m_Q v - q$ in an external gluon field is given by the vertex (5) sandwiched between $\bar{u}(p_2)$ and $u(p_1)$ and multiplied by Z_2^{OS}. We expand this amplitude in q up to linear terms, and re-express (relativistic) QCD spinors via HQET (non-relativistic) spinors:

$$u(m_Q v + k) = \left[1 + \frac{\not{k}}{2m_Q} + \mathcal{O}\left(\frac{k^2}{m_Q^2}\right)\right] u_v(k)\,. \tag{8}$$

Then the QCD scattering amplitude reads

$$\bar{u}_v(-q)\left[v^\mu - \frac{q^\mu}{2m_Q} - \frac{i}{2m_Q}\sigma^{\mu\nu}q_\nu(1+\mu_c)\right] t_a u_v(0)\,, \tag{9}$$

(we have used $\bar{u}(p_2)\not{q}u(p_1) = 0$). It must be reproduced by the HQET Lagrangian of Eq. (1). If all flavours except Q are massless, all loop corrections vanish, [2] and the scattering amplitude is given by the Born approximation:

$$\bar{u}_v(-q)\left[v^\mu - \frac{q^\mu}{2m_Q} - \frac{i}{2m_Q}\sigma^{\mu\nu}q_\nu C_{cm}^0\right] t_a u_v(0)\,, \tag{10}$$

[1] We thank Philipp Kant for providing his interface for the program of Ref. [27].
[2] We imply the use of Dimensional Regularization.

where $C^0_{cm} = Z^{-1}_{cm}(\mu)C_{cm}(\mu)$ is the bare chromomagnetic interaction coefficient, and $Z_{cm}(\mu)$ is the $\overline{\text{MS}}$ renormalization constant of the chromomagnetic operator given in Eq. (3). Note that both scattering amplitudes (9) and (10) are renormalized and hence ultraviolet-finite, however, both have infrared divergences. These divergences are the same, because HQET has been constructed to reproduce the infrared behaviour of QCD. Vanishing loop correction in HQET have ultraviolet and infrared divergences which cancel each other. The ultraviolet divergences of C^0_{cm} are removed by $Z^{-1}_{cm}(\mu)$; the infrared ones match those of $1 + \mu_c$ (cf. Eq. (9)).

In order to find the chromomagnetic interaction coefficient $C_{cm}(\mu)$ in the HQET Lagrangian (1), we calculate the anomalous chromomagnetic moment $\mu_c = Z_2^{\text{OS}} F_2(0)$ and re-express it in terms of $\alpha_s^{(n_l)}(\mu)$, where the superscript denotes the number of active flavours. Then

$$C_{cm}(\mu) = Z_{cm}(\alpha_s^{(n_l)}(\mu)) \left[1 + \mu_c(\alpha_s^{(n_l)}(\mu))\right] , \tag{11}$$

where $n_l = n_f - 1$ is the number of light-quark flavours, which are considered to be massless in our calculation, and n_f is the total number of quark flavours.

The coupling constants $\alpha_s^{(n_l+1)}(\mu)$ in QCD (with $n_l + 1$ flavours) and $\alpha_s^{(n_l)}(\mu)$ in HQET (with n_l flavours) are related by [28]

$$\begin{aligned}
\frac{\alpha_s^{(n_l+1)}(\mu)}{\pi} &= \frac{\alpha_s^{(n_l)}(\mu)}{\pi} + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^2 T_F \Bigg[\frac{1}{3}L + \left(\frac{1}{6}L^2 + \frac{1}{36}\pi^2\right)\varepsilon \\
&\quad + \left(\frac{1}{18}L^3 + \frac{1}{36}\pi^2 L - \frac{1}{9}\zeta_3\right)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right)\Bigg] \\
&+ \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi}\right)^3 T_F \Bigg\{\left(\frac{1}{4}L + \frac{15}{16}\right) C_F + \left(\frac{5}{12}L - \frac{2}{9}\right) C_A + \frac{1}{9}T_F L^2 \\
&+ \Bigg[\left(\frac{1}{4}L^2 + \frac{15}{8}L + \frac{1}{48}\pi^2 + \frac{31}{32}\right) C_F + \left(\frac{5}{12}L^2 - \frac{4}{9}L + \frac{5}{144}\pi^2 + \frac{43}{108}\right) C_A \\
&\quad + \left(\frac{1}{9}L^3 + \frac{1}{54}\pi^2 L\right) T_F\Bigg]\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\Bigg\} + \mathcal{O}\left(\alpha_s^4\right) ,
\end{aligned} \tag{12}$$

where $L = \ln(\mu^2/m_Q^2)$ and m_Q is the on-shell mass of the heavy quark. $C_F = (N_c^2 - 1)/(2N_c)$ and $C_A = N_c$ are the eigenvalues of the quadratic Casimir operators of the fundamental and adjoint representation for the $\text{SU}(N_c)$ colour group, respectively. In the case of QCD we have $N_c = 3$ and $T_F = 1/2$. ζ_n denotes Riemann's zeta function with integer argument n.

The ultraviolet divergences contained in Z_{cm} can be transformed into an anomalous dimension which is given by

$$
\begin{aligned}
\gamma_{cm} = \frac{\mathrm{d}\ln Z_{cm}}{\mathrm{d}\ln\mu} &= \frac{\alpha_s^{(n_l)}}{\pi}\frac{1}{2}C_A + \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^2 C_A\left(\frac{17}{36}C_A - \frac{13}{36}T_F n_l\right) \\
&+ \left(\frac{\alpha_s^{(n_l)}}{\pi}\right)^3 \left\{ \left(\frac{1}{8}\zeta_3 + \frac{899}{1728}\right) C_A^3 + \frac{1}{2}\pi^2 \frac{d_F^{abcd} d_A^{abcd}}{C_F N_F} \right. \\
&- \left[\left(\frac{1}{2}\zeta_3 + \frac{65}{216}\right) C_A^2 - \left(\frac{1}{2}\zeta_3 - \frac{49}{96}\right) C_A C_F + \frac{1}{36} C_A T_F n_l \right] T_F n_l \\
&\left. - \frac{2}{3}\pi^2 \frac{d_F^{abcd} d_F^{abcd}}{C_F N_F} n_l \right\} + \mathcal{O}\left(\alpha_s^4\right) ,
\end{aligned}
\tag{13}
$$

where d_F^{abcd} and d_A^{abcd} are the symmetrized traces of four generators in the fundamental and adjoint representation, respectively (for $\mathrm{SU}(N_c)$, $d_F^{abcd} d_F^{abcd} = (N_c^2-1)(N_c^4-6N_c^2+18)/(96N_c^2)$, $d_F^{abcd} d_A^{abcd} = N_c(N_c^2-1)(N_c^2+6)/48$). $N_F = N_c$ is the dimension of the fundamental representation. The two-loop result agrees with [7,8], and the n_l^2 part of the three-loop one with [9].

Our result for C_{cm} reads

$$
\begin{aligned}
C_{cm}(\mu) = 1 &+ \frac{\alpha_s^{(n_l)}(m_Q)}{\pi} \left[\left(\frac{1}{4}L + \frac{1}{2}\right) C_A + \frac{1}{2} C_F \right] \\
&+ \left(\frac{\alpha_s^{(n_l)}(m_Q)}{\pi}\right)^2 \left[\left(-\frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta_3 + \frac{5}{12}\pi^2 - \frac{31}{16}\right) C_F^2 \right. \\
&+ \left(\frac{1}{8}L + \frac{1}{12}\pi^2 \ln 2 - \frac{1}{8}\zeta_3 + \frac{1}{12}\pi^2 + \frac{269}{144}\right) C_F C_A \\
&+ \left(-\frac{1}{12}L^2 + \frac{13}{36}L + \frac{1}{12}\pi^2 \ln 2 - \frac{1}{8}\zeta_3 - \frac{17}{144}\pi^2 + \frac{805}{432}\right) C_A^2 \\
&- \frac{25}{36} C_F T_F n_l + \left(\frac{1}{24}L^2 - \frac{13}{72}L - \frac{1}{36}\pi^2 - \frac{299}{432}\right) C_A T_F n_l \\
&\left. + \left(-\frac{1}{3}\pi^2 + \frac{119}{36}\right) C_F T_F + \left(\frac{5}{72}\pi^2 - \frac{149}{216}\right) C_A T_F \right] \\
&+ \left(\frac{\alpha_s^{(n_l)}(m_Q)}{\pi}\right)^3 c_{cm}^{(3)} + \mathcal{O}\left(\alpha_s^4\right) .
\end{aligned}
\tag{14}
$$

The two-loop corrections were already calculated in Ref. [8]. [3] It is convenient to decompose the three-loop contribution in terms of the different colour structures as

$$
\begin{aligned}
c_{cm}^{(3)} = &X_{FFF} C_F^3 + X_{FFA} C_F^2 C_A + X_{FAA} C_F C_A^2 + X_{AAA} C_A^3 \\
&+ X_{dd}^{FA} \frac{d_F^{abcd} d_A^{abcd}}{C_F N_F} + \left(X_{FFl} C_F^2 + X_{FAl} C_F C_A + X_{AAl} C_A^2 \right) T_F n_l
\end{aligned}
$$

[3] Note that in Ref. [8] the term $\pi^2 C_A T_F$ is not correct, since the $\mathcal{O}(\varepsilon)$ term in the one-loop decoupling relation for α_s (12) has not been taken into account.

$$+ \left(X_{Fll} C_F + X_{All} C_A\right) T_F^2 n_l^2 + \left(X_{Flh} C_F + X_{Alh} C_A\right) T_F^2 n_l$$
$$+ \left(X_{FFh} C_F^2 + X_{FAh} C_F C_A + X_{AAh} C_A^2\right) T_F$$
$$+ \left(X_{Fhh} C_F + X_{Ahh} C_A\right) T_F^2 + \left(X_{\sin}^l n_l + X_{\sin}^h\right) \frac{d_F^{abcd} d_F^{abcd}}{C_F N_F} \,. \tag{15}$$

Our results for the individual terms read

$$X_{FFF} = \frac{20}{3} a_4 + \frac{5}{18} \ln^4 2 - \frac{5}{18} \pi^2 \ln^2 2 - \frac{22}{3} \pi^2 \ln 2 - \frac{235}{24} \zeta_5 + \frac{103}{72} \pi^2 \zeta_3 - \frac{139}{2160} \pi^4 + \frac{241}{24} \zeta_3 + \frac{23}{6} \pi^2 - \frac{101}{64} \,, \tag{16}$$

$$X_{FFA} = \left(- \frac{1}{8} \pi^2 \ln 2 + \frac{3}{16} \zeta_3 + \frac{5}{48} \pi^2 - \frac{31}{64} \right) L - \frac{35}{3} a_4 - \frac{35}{72} \ln^4 2 + \frac{13}{9} \pi^2 \ln^2 2 - \frac{101}{24} \pi^2 \ln 2 + \frac{115}{6} \zeta_5 - \frac{17}{9} \pi^2 \zeta_3 - \frac{209}{2880} \pi^4 - \frac{847}{96} \zeta_3 + \frac{9767}{1728} \pi^2 - \frac{2803}{288} \,, \tag{17}$$

$$X_{FAA} = - \frac{1}{24} L^2 + \left(\frac{1}{48} \pi^2 \ln 2 - \frac{1}{32} \zeta_3 + \frac{1}{48} \pi^2 + \frac{337}{576} \right) L + \frac{191}{18} a_4 + \frac{191}{432} \ln^4 2 - \frac{169}{216} \pi^2 \ln^2 2 + \frac{1745}{432} \pi^2 \ln 2 - \frac{265}{16} \zeta_5 + \frac{161}{72} \pi^2 \zeta_3 + \frac{491}{10368} \pi^4 + \frac{2951}{288} \zeta_3 - \frac{17375}{3456} \pi^2 + \frac{122971}{10368} \,, \tag{18}$$

$$X_{AAA} = \frac{19}{432} L^3 - \frac{497}{1728} L^2 + \left(\frac{1}{48} \pi^2 \ln 2 + \frac{1}{32} \zeta_3 - \frac{17}{576} \pi^2 + \frac{2917}{3456} \right) L - \frac{8}{3} a_4 - \frac{1}{9} \ln^4 2 - \frac{1}{36} \pi^2 \ln^2 2 - \frac{317}{864} \pi^2 \ln 2 + \frac{925}{192} \zeta_5 - \frac{653}{864} \pi^2 \zeta_3 - \frac{17}{810} \pi^4 - \frac{6079}{1728} \zeta_3 + \frac{1585}{1296} \pi^2 + \frac{1302797}{186624} \,, \tag{19}$$

$$X_{dd}^{FA} = \frac{1}{4} \pi^2 L - \frac{40}{3} a_4 - \frac{5}{9} \ln^4 2 + \frac{20}{9} \pi^2 \ln^2 2 + \frac{91}{12} \pi^2 \ln 2 - 10 \zeta_5 + \frac{97}{72} \pi^2 \zeta_3 + \frac{7}{270} \pi^4 - \frac{73}{24} \zeta_3 - \frac{151}{27} \pi^2 - \frac{5}{18} \,, \tag{20}$$

$$X_{FFl} = - \frac{8}{3} a_4 - \frac{1}{9} \ln^4 2 - \frac{2}{9} \pi^2 \ln^2 2 + \frac{5}{3} \pi^2 \ln 2 + \frac{11}{216} \pi^4 - 3 \zeta_3 - \frac{79}{54} \pi^2 + \frac{125}{32} \,, \tag{21}$$

$$X_{FAl} = \frac{5}{96} L^2 + \left(\frac{1}{4} \zeta_3 - \frac{299}{576} \right) L + \frac{4}{9} a_4 + \frac{1}{54} \ln^4 2 + \frac{1}{27} \pi^2 \ln^2 2 - \frac{1}{108} \pi^2 \ln 2 - \frac{23}{3240} \pi^4 + \frac{2}{3} \zeta_3 - \frac{23}{72} \pi^2 - \frac{88351}{10368} \,, \tag{22}$$

$$X_{AAl} = - \frac{35}{864} L^3 + \frac{235}{864} L^2 - \left(\frac{1}{4} \zeta_3 + \frac{1}{144} \pi^2 + \frac{715}{1728} \right) L + \frac{4}{9} a_4 + \frac{1}{54} \ln^4 2 + \frac{1}{27} \pi^2 \ln^2 2 - \frac{89}{216} \pi^2 \ln 2 + \frac{1}{180} \pi^4 - \frac{101}{432} \zeta_3 + \frac{35}{864} \pi^2$$

$$
-\frac{236801}{46656}\,, \tag{23}
$$

$$
X_{Fll}=\frac{1}{27}\pi^2+\frac{317}{324}\,, \tag{24}
$$

$$
X_{All}=\frac{1}{108}L^3-\frac{13}{216}L^2-\frac{1}{72}L+\frac{7}{54}\zeta_3+\frac{25}{324}\pi^2+\frac{3535}{5832}\,, \tag{25}
$$

$$
X_{Flh}=\frac{1}{27}\pi^2-\frac{61}{162}\,, \tag{26}
$$

$$
X_{Alh}=-\frac{11}{108}\pi^2+\frac{167}{162}\,, \tag{27}
$$

$$
\begin{aligned}
X_{FFh}=&\frac{32}{3}a_4+\frac{4}{9}\ln^4 2-\frac{4}{9}\pi^2\ln^2 2-\frac{16}{9}\pi^2\ln 2+\frac{4}{135}\pi^4-\frac{263}{72}\zeta_3\\
&+\frac{11}{162}\pi^2+\frac{2027}{216}\,,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
X_{FAh}=&\left(-\frac{1}{12}\pi^2+\frac{119}{144}\right)L-10a_4-\frac{5}{12}\ln^4 2+\frac{5}{12}\pi^2\ln^2 2+\frac{83}{27}\pi^2\ln 2\\
&-\frac{25}{24}\zeta_5+\frac{1}{8}\pi^2\zeta_3-\frac{101}{1440}\pi^4-\frac{6937}{864}\zeta_3-\frac{22241}{19440}\pi^2+\frac{8447}{864}\,,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
X_{AAh}=&\left(\frac{5}{288}\pi^2-\frac{149}{864}\right)L+a_4+\frac{1}{24}\ln^4 2-\frac{1}{24}\pi^2\ln^2 2+\frac{1211}{432}\pi^2\ln 2\\
&-\frac{65}{144}\zeta_5+\frac{65}{432}\pi^2\zeta_3+\frac{53}{5184}\pi^4+\frac{4423}{1728}\zeta_3-\frac{283429}{155520}\pi^2\\
&-\frac{71965}{10368}\,,
\end{aligned} \tag{30}
$$

$$
X_{Fhh}=\frac{8}{3}\zeta_3-\frac{4}{135}\pi^2-\frac{943}{324}\,, \tag{31}
$$

$$
X_{Ahh}=-\frac{4}{9}\zeta_3+\frac{1}{270}\pi^2+\frac{487}{972}\,, \tag{32}
$$

$$
X^l_{\sin}=-\frac{1}{3}\pi^2L+\frac{29}{270}\pi^4-3\zeta_3-\frac{44}{27}\pi^2+\frac{2}{3}\,, \tag{33}
$$

$$
\begin{aligned}
X^h_{\sin}=&16a_4+\frac{2}{3}\ln^4 2-\frac{2}{3}\pi^2\ln^2 2-24\pi^2\ln 2+\frac{5}{6}\zeta_5-\frac{5}{18}\pi^2\zeta_3-\frac{41}{540}\pi^4\\
&-\frac{4}{3}\zeta_3+\frac{931}{54}\pi^2+\frac{5}{9}\,,
\end{aligned} \tag{34}
$$

with $a_4=\mathrm{Li}_4(1/2)$. The n_l^2 part agrees with the result obtained in Ref. [9].

Substituting numerical values of the constants, we obtain, for the physical SU(3) colour group,

$$
\begin{aligned}
C_{cm}(m_Q)=&1+0.6897\,\alpha_s^{(n_l)}(m_Q)+(2.2186-0.1938\,n_l)\left[\alpha_s^{(n_l)}(m_Q)\right]^2\\
&+\left(11.079-1.7490\,n_l+0.0513\,n_l^2\right)\left[\alpha_s^{(n_l)}(m_Q)\right]^3+\mathcal{O}\left(\alpha_s^4\right)\\
=&1+0.6897\,\alpha_s^{(n_l)}(m_Q)+(1.1626\,\beta_0-0.9786)\left[\alpha_s^{(n_l)}(m_Q)\right]^2
\end{aligned}
$$

$$+\left(1.8468\,\beta_0^2+0.3370\,\beta_0-3.8137\right)\left[\alpha_s^{(n_l)}(m_Q)\right]^3+\mathcal{O}\left(\alpha_s^4\right)\,, \tag{35}$$

with $\beta_0=(11C_A/3-4T_Fn_l/3)/4$. The first-order $1/\beta_0$ result [9] contains the highest powers of β_0 in each term. For $n_l=4$, for example, the coefficient of $\left[\alpha_s^{(n_l)}(m_Q)\right]^2$ is $2.4221-0.9786=1.4435$, and that of $\left[\alpha_s^{(n_l)}(m_Q)\right]^3$ is $8.0155+0.7020-3.8137=4.9039$; the large-$\beta_0$ approximation of Ref. [9], which only includes the first terms in these sums, overestimates these two coefficients by 68% and 63%, correspondingly.

For the numerical evaluation of $C_{cm}(m_Q)$ and γ_{cm}, we use the values $m_c=1.6$ GeV, $m_b=4.7$ GeV and $m_t=175$ GeV. The number of light-quark flavours n_l is three, four and five for the charm, bottom and top quark, respectively. To evaluate $\alpha_s^{(n_l)}(m_Q)$, defined with n_l active flavours, from $\alpha_s^{(5)}(m_Z)=0.118$, we use the program `RunDec` [29] and obtain $\alpha_s^{(3)}(m_c)=0.3348$, $\alpha_s^{(4)}(m_b)=0.2163$ and $\alpha_s^{(5)}(m_t)=0.1074$, which leads to

$$\begin{aligned}
C_{cm}(m_c) &= 1+0.2309+0.1835+0.2362=1.6506\,, && (36)\\
C_{cm}(m_b) &= 1+0.1492+0.0676+0.0497=1.2664\,, && (37)\\
C_{cm}(m_t) &= 1+0.0741+0.0144+0.0045=1.0930\,. && (38)
\end{aligned}$$

In the case of the charm quark, we see that the perturbative series does not converge which is probably connected to the relatively light scale of 1.6 GeV at which the strong coupling is evaluated. Thus, it seems that there are potentially large non-perturbative corrections. While the situation is better in the case of the bottom quark, the corrections are still very large. The three-loop correction amounts to about 30% of the one-loop contribution. For the top quark, we find that our new term contributes about 6% of the one-loop correction leading to a fairly reliable prediction of $C_{cm}(m_t)$.

The numerical evaluation of the anomalous dimension (13) gives

$$\begin{aligned}
\gamma_{cm} &= 0.4775\,\alpha_s^{(n_l)}+(0.4306-0.0549\,n_l)\left[\alpha_s^{(n_l)}\right]^2\\
&\quad+\left(0.8823-0.1472\,n_l-0.0007\,n_l^2\right)\left[\alpha_s^{(n_l)}\right]^3+\mathcal{O}\left(\alpha_s^4\right)\,. && (39)
\end{aligned}$$

For the individual quark flavours this leads to

$$\begin{aligned}
\gamma_{cm}(m_c) &= 0.1599+0.0298+0.0163=0.2060\,,\\
\gamma_{cm}(m_b) &= 0.1033+0.0099+0.0029=0.1160\,,\\
\gamma_{cm}(m_t) &= 0.0513+0.0018+0.0002=0.0533\,. && (40)
\end{aligned}$$

Thus, as far as the anomalous dimension is concerned the convergence behaviour is acceptable even for the charm quark.

These observations are in good agreement with the analysis of Ref. [9]. The chromomagnetic interaction coefficient $C_{cm}(m_Q)$ has the leading renormalon singularity (namely, a branch point) at the Borel parameter $u = 1/2$, quite close to the origin; it leads to a very fast growth of coefficients of the perturbative series, $\sim n!(\beta_0/2)^n$. It also means that the leading non-perturbative correction is only suppressed by the first power of $1/m_Q$, and is thus important, especially for the charm quark. On the other hand, the perturbative series for the anomalous dimension has a finite radius of convergence.

4 Application: heavy-meson mass splittings

The most prominent physical effect caused by the chromomagnetic interaction is the mass splittings of hadronic doublets which are degenerate at $m_Q = \infty$ due to the heavy-quark spin symmetry. For example, for the bottom mesons B and B^* one has [30],

$$m_{B^*}^2 - m_B^2 = \frac{4}{3} C_{cm}^{(4)}(\mu)\mu_{G(4)}^2(\mu) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_b}\right) , \tag{41}$$

where the index "(4)" means that we are considering $n_l = 4$ flavour HQET, and $\mu_{G(4)}^2(\mu)$ is the matrix element of $O_{cm}(\mu)$ (cf. Eq. (3)) over the ground-state meson. It is most natural to choose $\mu = m_b$ in Eq. (41), because then C_{cm} contains no large logarithms. A similar formula can be written down for the D mesons where $C_{cm}^{(3)}(\mu)$ and $\mu_{G(3)}^2(\mu)$ appear in the corresponding expression for $m_{D^*}^2 - m_D^2$. The running of $\mu_{G(n_l)}^2(\mu)$ is governed by the anomalous dimension given in Eq. (13). Furthermore, it is necessary to relate the matrix elements in the two theories via the following decoupling relation

$$\mu_{G(4)}^2(m_c) = \mu_{G(3)}^2(m_c) \left[1 + z_2 \left(\frac{\alpha_s^{(4)}(m_c)}{\pi} \right)^2 + z_3 \left(\frac{\alpha_s^{(4)}(m_c)}{\pi} \right)^3 + \mathcal{O}\left(\alpha_s^4\right) \right] , \tag{42}$$

with $z_2 = -71 C_A T_F/432$ [31,4]. We introduce the unknown[4] coefficient z_3 since it appears in estimates of higher order effects which are presented below.

In the formulation with resummed logarithms one combines for consistency Eq. (42) with the two-loop result for C_{cm} and the three-loop anomalous di-

[4] z_3 can be calculated using 3-loop HQET integrals considered in [32,33].

mension. This leads to the ratio R of Eq. (4) to the next-to-next-to-leading (NNL) order approximation. For later use we extend the formalism to NNNL order where we obtain

$$
\begin{aligned}
R = x^{-\gamma_0/(2\beta_0)} \Bigg\{ & 1 + r_1 (x-1) \frac{\alpha_s^{(4)}(m_b)}{\pi} \\
& + \left[r_{20} + r_{21}(x^2-1) + \frac{r_1^2}{2}(x-1)^2 \right] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi} \right)^2 \\
& + \Bigg[r_{30} + r_{31}(x-1)(x^2+x+1) + \frac{r_1^3}{6}(x-1)^3 + r_1 r_{20}(x-1) \\
& \quad + r_1 r_{21}(x-1)^2(x+1) \Bigg] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi} \right)^3 + \mathcal{O}\left(\alpha_s^4, \frac{\Lambda_{\mathrm{QCD}}}{m_{b,c}} \right) \Bigg\} , \qquad (43)
\end{aligned}
$$

with

$$
\begin{aligned}
x &= \frac{\alpha_s^{(4)}(m_c)}{\alpha_s^{(4)}(m_b)} , \\
r_1 &= -c_{cm}^{(1)} - \frac{\gamma_0}{2\beta_0} \left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) , \\
r_{20} &= c_{cm}^{(2)}(n_l = 4) - c_{cm}^{(2)}(n_l = 3) + z_2 , \\
r_{21} &= -c_{cm}^{(2)}(n_l = 3) + \frac{\left(c_{cm}^{(1)} \right)^2}{2} + z_2 \\
& \quad + \frac{\gamma_0}{4\beta_0} \left[-\frac{\gamma_2}{\gamma_0} + \frac{\beta_1}{\beta_0}\frac{\gamma_1}{\gamma_0} + \frac{\beta_2}{\beta_0} - \left(\frac{\beta_1}{\beta_0} \right)^2 \right] , \\
r_{30} &= c_{cm}^{(3)}(n_l = 4) - c_{cm}^{(3)}(n_l = 3) - c_{cm}^{(1)} \left(c_{cm}^{(2)}(n_l = 4) - c_{cm}^{(2)}(n_l = 3) + d_2 \right) \\
& \quad + z_3 , \\
r_{31} &= -c_{cm}^{(3)}(n_l = 3) + c_{cm}^{(1)} \left(c_{cm}^{(2)}(n_l = 3) - d_2 \right) - \frac{\left(c_{cm}^{(1)} \right)^3}{3} + z_3 + \frac{\gamma_0}{6\beta_0} \Bigg[-\frac{\gamma_3}{\gamma_0} \\
& \quad + \frac{\beta_1}{\beta_0}\frac{\gamma_2}{\gamma_0} + \frac{\beta_2}{\beta_0}\frac{\gamma_1}{\gamma_0} - \left(\frac{\beta_1}{\beta_0} \right)^2 \frac{\gamma_1}{\gamma_0} + \frac{\beta_3}{\beta_0} - 2\frac{\beta_1}{\beta_0}\frac{\beta_2}{\beta_0} + \left(\frac{\beta_1}{\beta_0} \right)^3 \Bigg] . \qquad (44)
\end{aligned}
$$

$c_{cm}^{(n)}$ denotes the coefficient of $(\alpha_s(m_Q)/\pi)^n$ in $C_{cm}(\mu = m_Q)$. The terms z_2 and z_3 stem from the decoupling of the matrix element and are introduced in Eq. (42) and $d_2 = [(2/9)C_A - (15/16)C_F]T_F$ stems from the decoupling of α_s (cf. Eq. (12)). γ_n are the coefficients of $(\alpha_s/\pi)^{n+1}$ in the anomalous dimension and the coefficients of the β function are used in the form $\beta_0 = (11C_A/3 - 4T_F n_l/3)/4$; see Refs. [34,35] for the remaining β_i.

Both for γ_n and β_n $n_l = 4$ active flavours have to be chosen. Let us mention that the next-to-leading (NL) order result of Eq. (43) has been obtained in Ref. [7]. Inserting the numerical values given above and displaying the contributions from the individual orders separately, we find

$$\begin{aligned} R &= 0.8517 - 0.0696 - 0.0908 + [-0.1285] + \ldots \\ &= 0.6914 + [-0.1285] + \ldots\,, \end{aligned} \tag{45}$$

where the ellipses denote terms of higher order and power corrections. The term in square brackets is our estimate of the fourth order contribution, where we assume that the four-loop coefficient of the anomalous dimension is negligible. This is justified by the rapid convergence of γ_{cm} in the case of the bottom quark as can be seen in Eq. (40). Furthermore, we set the unknown coefficient z_3 of Eq. (42) to zero which is a good approximation since it enters with a small coefficient.

The experimental value is $R_{\rm exp} = 0.88$ [36] with a negligible uncertainty. The NNL correction amounts to 10% of the LO contribution, however, it is larger than the NL one. Furthermore it is negative and thus increases the difference of the perturbative result and the experimental value. The estimated third-order correction is even larger than the NL and NNL one and contributes also with a negative sign. This indicates that the $\Lambda_{\rm QCD}/m_c$ correction may be quite substantial.

It is interesting to consider the quantity R also without performing the resummation of the logarithms. In this way the three-loop result for the coefficient can be incorporated in a consistent way. The starting point is Eq. (4) where quantities defined for $n_l = 4$ are present in the numerator and the ones defined for $n_l = 3$ in the denominator. Using Eq. (42) for the decoupling of the matrix element and running from $\mu = m_c$ to $\mu = m_b$ cancels $\mu^2_{G(4)}$. Afterwards, we can replace $\alpha_s^{(3)}(m_c)$ by $\alpha_s^{(4)}(m_b)$, using decoupling and renormalization group running, and perform a consistent expansion of R in $\alpha_s^{(4)}(m_b)$. As a result we obtain

$$\begin{aligned} R = {} & 1 - \gamma_0\, l\, \frac{\alpha_s^{(4)}(m_b)}{\pi} \\ & + \Big[\gamma_0 \left(\frac{1}{2}\gamma_0 - \beta_0 \right) l^2 - \left(2\beta_0 c_{cm}^{(1)} + \gamma_1 \right) l \\ & \qquad + c_{cm}^{(2)}(n_l = 4) - c_{cm}^{(2)}(n_l = 3) + z_2 \Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi} \right)^2 \\ & + \Big[\gamma_0 \left(-\frac{1}{6}\gamma_0^2 + \beta_0\gamma_0 - \frac{4}{3}\beta_0^2 \right) l^3 + \left((\gamma_0 - 2\beta_0)(2\beta_0 c_{cm}^{(1)} + \gamma_1) - \beta_1\gamma_0 \right) l^2 \\ & \qquad + \Big(\gamma_0 (c_{cm}^{(2)}(n_l = 3) - c_{cm}^{(2)}(n_l = 4)) - 4\beta_0 c_{cm}^{(2)}(n_l = 3) + 2\beta_0 (c_{cm}^{(1)})^2 \end{aligned}$$

$$
\begin{aligned}
&\quad - 2\beta_1 c_{cm}^{(1)} + (4\beta_0 - \gamma_0) z_2 - \gamma_2\Big) l + c_{cm}^{(3)}(n_l = 4) - c_{cm}^{(3)}(n_l = 3) \\
&\quad - c_{cm}^{(1)}(c_{cm}^{(2)}(n_l = 4) - c_{cm}^{(2)}(n_l = 3) + d_2) + z_3\Big] \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 \\
&+ \mathcal{O}\left(\alpha_s^4\right) \,, \qquad (46)
\end{aligned}
$$

where $l = \ln(m_b/m_c)$. In our numerical evaluation we set the decoupling coefficient z_3 to zero. Since similar three-loop decoupling effects are small we expect the same in our case. Furthermore, note that $c_{cm}^{(3)}$ is numerically rather large. Inserting the numerical values yields

$$
R = 1 - 0.1113 - 0.0780 - 0.0755 + \ldots = 0.7352 + \ldots \,. \qquad (47)
$$

Comparing Eqs. (45) and (47), we find that the convergence of R without resummation behaves slightly better. However, the coefficients of the perturbative series are still large.

5 Matching coefficient and renormalization group invariant quark mass

In this section we would like to discuss the result of the matching coefficient in the form which is often used in lattice simulations of the mass difference $m_{B^*}^2 - m_B^2$. In doing so we follow the procedure outlined in Ref. [37].

In lattice simulations one usually determines the renormalization group invariant (RGI) matrix element of the operator O_{cm} which has to be multiplied by the corresponding matching coefficient. For its derivation one considers in a first step

$$
C_{\rm mag} = \left(2\beta_0 \frac{\alpha_s(\bar{m}_*)}{\pi}\right)^{\frac{\gamma_0^{\rm mag}}{2\beta_0}} \exp\left[\int\limits_0^{\alpha_s(\bar{m}_*)} \left(\frac{\gamma_{\rm mag}}{2\beta} - \frac{\gamma_0^{\rm mag}}{2\beta_0}\right) \frac{d\alpha_s}{\alpha_s}\right], \qquad (48)
$$

where $\bar{m}_* = \bar{m}_Q(\bar{m}_*)$ is the scale invariant $\overline{\rm MS}$ mass. The anomalous dimension $\gamma_{\rm mag}$ is given by a combination of $\gamma_{cm}(\alpha_s)$, $\beta(\alpha_s)$ and $C_{cm}(\mu)$ and reads

$$
\begin{aligned}
\gamma_{\rm mag} &= \frac{\alpha_s}{\pi}\left(\gamma_0^{\rm mag} + \gamma_1^{\rm mag}\frac{\alpha_s}{\pi} + \gamma_2^{\rm mag}\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right) \qquad (49) \\
&= \gamma_{cm} + 2\beta_0 c_{cm}^{(1)}\left(\frac{\alpha_s}{\pi}\right)^2 + \left(4\beta_0 c_{cm}^{(2)} + 2\beta_1 c_{cm}^{(1)} - 2\beta_0 (c_{cm}^{(1)})^2\right)\left(\frac{\alpha_s}{\pi}\right)^3
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(2\beta_2 c_{cm}^{(1)} - 2\beta_1 (c_{cm}^{(1)})^2 + 2\beta_0 (c_{cm}^{(1)})^3 + 4\beta_1 c_{cm}^{(2)} \\
&- 6\beta_0 c_{cm}^{(1)} c_{cm}^{(2)} + 6\beta_0 c_{cm}^{(3)}\Big) \left(\frac{\alpha_s}{\pi}\right)^4 + \ldots
\end{aligned}
$$

The terms containing the β-function stem from the running of α_s.

Since $C_{\rm mag}$ is still multiplied by the inverse pole mass, $1/m_Q$, in a further step the renormalization group invariant mass $M^{\rm RGI}$ defined by [37]

$$
M^{RGI} = \bar{m}_* \left(2\beta_0 \frac{\alpha_s(\bar{m}_*)}{\pi}\right)^{-\frac{\gamma_{m,0}}{\beta_0}} \exp\left[-\int\limits_0^{\alpha_s(\bar{m}_*)} \left(\frac{\gamma_m}{\beta} - \frac{\gamma_{m,0}}{\beta_0}\right) \frac{d\alpha_s}{\alpha_s}\right], \tag{50}
$$

can be used, where we introduced the quark mass anomalous dimension γ_m

$$
\gamma_m = \frac{\alpha_s}{\pi}\left(\gamma_{m,0} + \gamma_{m,1}\frac{\alpha_s}{\pi} + \gamma_{m,2}\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right), \tag{51}
$$

with $\gamma_{m,0} = 3C_F/4$; the other coefficients can be found in Refs. [38,39]. Using Eq. (50) in addition to the $\overline{\rm MS}$–on-shell relation the overall and logarithmic dependence on the pole mass can be replaced by the RGI mass. This procedure turns $C_{\rm mag}$ into $C_{\rm spin}$ and one obtains an equation analog to Eq. (48) for $C_{\rm spin}$ with the anomalous dimension given by

$$
\gamma_{\rm spin} = \tilde{\gamma}_{\rm mag} - 2\gamma_m \tag{52}
$$

with

$$
\begin{aligned}
\tilde{\gamma}_0^{\rm mag} &= \gamma_0^{\rm mag}, \\
\tilde{\gamma}_1^{\rm mag} &= \gamma_1^{\rm mag} - 2\beta_0 k_1, \\
\tilde{\gamma}_2^{\rm mag} &= \gamma_2^{\rm mag} - 2\beta_1 k_1 + \beta_0(2k_1^2 - 4\gamma_0 k_1 - 4k_2), \\
\tilde{\gamma}_3^{\rm mag} &= \gamma_3^{\rm mag} - 2\beta_2 k_1 + \beta_1(2k_1^2 - 4\gamma_0 k_1 - 4k_2) - 12\beta_0^2 k_1 c_{cm}^{(1)} \\
&\quad + \beta_0\Big(3\gamma_0 k_1^2 - 2k_1^3 - 6\gamma_1 k_1 + 6k_1 k_2 - 6\gamma_0 k_2 - 6k_3\Big).
\end{aligned} \tag{53}
$$

The terms containing the β-function stem from the transformation of m_Q to $\bar{m}_Q$ and the coefficients k_i are defined through $m_Q/\bar{m}_Q(\bar{m}_Q) = 1 + k_1\alpha_s/\pi + k_2\alpha_s^2/\pi^2 + k_3\alpha_s^3/\pi^3 + \ldots$. They can be found in Ref. [40] (see also Refs. [41–43,16]).

In Fig. 2 the results for $C_{\rm mag}$ and $C_{\rm spin}$ are shown as a function of $\Lambda_{\rm QCD}/M^{RGI}$ for $n_l = 0$ (left) and $n_l = 4$ (right) where the LO, NLO and NNLO results are shown. We also added an estimation for the NNNLO result by assuming a vanishing four-loop anomalous dimension which is motivated by the smallness

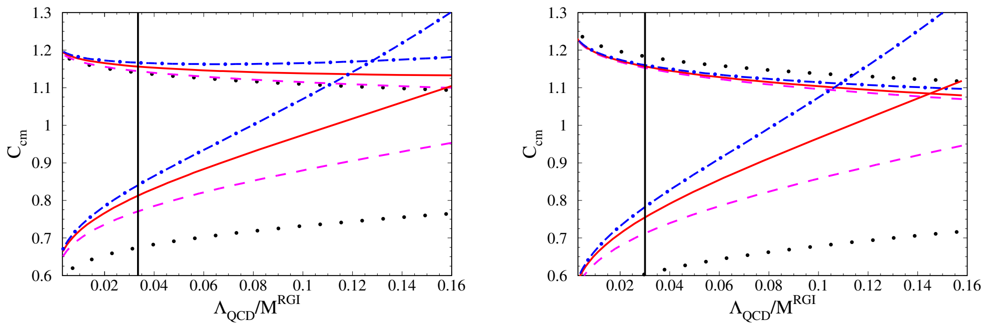

Fig. 2. C_{mag} and C_{spin} as a function of $\Lambda_{\mathrm{QCD}}/m_Q$ for $n_l = 0$ (left) and $n_l = 4$ (right). The upper group of lines corresponds to C_{spin} while the lower shows C_{mag}. Inside these groups the dotted, dashed and solid lines show the LO, NLO and NNLO, respectively. The estimation for the NNNLO result is shown by the dash-dotted line. The region relevant for the bottom quark is indicated by the vertical line.

of the higher order terms in Eq. (40). For the abscissa we choose $\Lambda_{\mathrm{QCD}} = 0.238$ which results from a lattice calculation for $n_l = 0$ [44] and vary the value for M^{RGI}.

For $\Lambda_{\mathrm{QCD}}/M^{RGI} \approx 0.03$, which is the range relevant for the bottom quark, one observes in the case of C_{mag} a relatively big shift when going from LO to NLO. However, the additional shifts after including the NNLO and the (estimated) NNNLO are smaller. The convergence improves significantly when going from C_{mag} to C_{spin}. In the case of the bottom quark the NLO corrections turn out to give a tiny contribution for $n_l = 0$, however, also the NNLO and NNNLO results are quite small. Going to smaller quark masses one observes moderate corrections for C_{spin} whereas for C_{mag} one has no convergence. We would like to mention that in the case $n_l = 4$ there is basically no change in the behaviour of C_{mag}. However, as far as C_{spin} is concerned one observes a moderate shift when including the NLO terms whereas the NNLO and NNNLO corrections are tiny.

6 Magnetic moment

The calculation of the anomalous magnetic moment of a heavy quark proceeds along the same lines as for the chromomagnetic moment. The only difference is that the external gluon is replaced by a photon. Our result reads

$$\frac{a_Q}{Q_Q} = \frac{\alpha_s^{(n_f)}(m_Q)}{2\pi} C_F$$

$$
\begin{aligned}
&+ \left(\frac{\alpha_s^{(n_f)}(m_Q)}{\pi}\right)^2 C_F\Big[\left(-\frac{31}{16} + \frac{5}{12}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta_3\right) C_F \\
&+ \left(\frac{317}{144} - \frac{1}{8}\pi^2 + \frac{1}{4}\pi^2 \ln 2 - \frac{3}{8}\zeta_3\right) C_A \\
&+ \left(-\frac{25}{36} n_l + \frac{119}{36} - \frac{1}{3}\pi^2\right) T_F\Big] \\
&+ \left(\frac{\alpha_s^{(n_f)}(m_Q)}{\pi}\right)^3 a_Q^{(3)} + \mathcal{O}\left(\alpha_s^4\right) . \qquad (54)
\end{aligned}
$$

Q_Q is the charge of the heavy quark in terms of the positron charge. Note that the strong coupling in Eq. (54) is defined for n_f active flavours and it is evaluated at the scale $\mu = m_Q$. The two-loop contribution was already computed in Ref. [45]. Recently, it has also been obtained by considering the on-shell limit in the calculation of the off-shell form factor [14]. We are in full agreement with Ref. [14] while we disagree with Ref. [45] by an overall factor of four in the coefficient of $(\alpha_s/\pi)^2$.

Our new three-loop term is given by

$$
\begin{aligned}
a_Q^{(3)} = &\Big(\frac{20}{3}a_4 + \frac{5}{18}\ln^4 2 - \frac{5}{18}\pi^2\ln^2 2 - \frac{22}{3}\pi^2\ln 2 - \frac{235}{24}\zeta_5 + \frac{103}{72}\pi^2\zeta_3 \\
&\quad - \frac{139}{2160}\pi^4 + \frac{241}{24}\zeta_3 + \frac{23}{6}\pi^2 - \frac{101}{64}\Big) C_F^3 \\
&- \Big(\frac{20}{3}a_4 + \frac{5}{18}\ln^4 2 - \frac{49}{36}\pi^2\ln^2 2 + \frac{31}{12}\pi^2\ln 2 - \frac{185}{24}\zeta_5 + \frac{5}{12}\pi^2\zeta_3 \\
&\quad + \frac{35}{432}\pi^4 + \frac{113}{48}\zeta_3 - \frac{1505}{432}\pi^2 + \frac{955}{72}\Big) C_F^2 C_A \\
&+ \Big(\frac{5}{3}a_4 + \frac{5}{72}\ln^4 2 - \frac{11}{18}\pi^2\ln^2 2 + \frac{25}{8}\pi^2\ln 2 - \frac{65}{32}\zeta_5 + \frac{29}{288}\pi^2\zeta_3 \\
&\quad + \frac{103}{2880}\pi^4 - \frac{1}{2}\zeta_3 - \frac{463}{216}\pi^2 + \frac{31231}{2592}\Big) C_F C_A^2 \\
&- \Big(\frac{8}{3}a_4 + \frac{1}{9}\ln^4 2 + \frac{2}{9}\pi^2\ln^2 2 - \frac{5}{3}\pi^2\ln 2 - \frac{11}{216}\pi^4 + 3\zeta_3 \\
&\quad + \frac{79}{54}\pi^2 - \frac{125}{32}\Big) C_F^2 T_F n_l \\
&+ \Big(\frac{4}{3}a_4 + \frac{1}{18}\ln^4 2 + \frac{1}{9}\pi^2\ln^2 2 - \frac{5}{6}\pi^2\ln 2 - \frac{11}{432}\pi^4 + \frac{19}{12}\zeta_3 \\
&\quad + \frac{77}{216}\pi^2 - \frac{2411}{324}\Big) C_F C_A T_F n_l \\
&+ \Big(\frac{32}{3}a_4 + \frac{4}{9}\ln^4 2 - \frac{4}{9}\pi^2\ln^2 2 - \frac{16}{9}\pi^2\ln 2 + \frac{4}{135}\pi^4 - \frac{263}{72}\zeta_3 \\
&\quad + \frac{11}{162}\pi^2 + \frac{7703}{864}\Big) C_F^2 T_F
\end{aligned}
$$

$$
\begin{aligned}
&- \Big(\frac{20}{3}a_4 + \frac{5}{18}\ln^4 2 - \frac{5}{18}\pi^2\ln^2 2 - \frac{32}{9}\pi^2\ln 2 + \frac{25}{24}\zeta_5 - \frac{1}{8}\pi^2\zeta_3 \\
&\quad + \frac{143}{2160}\pi^4 + \frac{241}{36}\zeta_3 + \frac{1375}{648}\pi^2 - \frac{2117}{162}\Big) C_F C_A T_F \\
&+ \Big[\frac{1}{27}\pi^2(n_l+1) + \frac{317}{324}n_l - \frac{61}{162}\Big] C_F T_F^2 n_l \\
&+ \Big(\frac{8}{3}\zeta_3 - \frac{4}{135}\pi^2 - \frac{943}{324}\Big) C_F T_F^2 + \frac{d_F^{abc}d_F^{abc}}{N_F} X_{\rm sin}\,,
\end{aligned} \tag{55}
$$

where d_F^{abc} is the symmetrized trace of three generators in the fundamental representation (for SU(N_c), $d_F^{abc}d_F^{abc} = (N_c^2-1)(N_c^2-4)/(16N_c)$). $X_{\rm sin}$ denotes the contribution from singlet diagrams (cf. Fig. 1(g)). It is given by

$$
\begin{aligned}
X_{\rm sin} = {}& 16a_4 + \frac{2}{3}\ln^4 2 - \frac{2}{3}\pi^2\ln^2 2 - 24\pi^2\ln 2 + \frac{5}{6}\zeta_5 - \frac{5}{18}\pi^2\zeta_3 - \frac{41}{540}\pi^4 \\
& - \frac{4}{3}\zeta_3 + \frac{931}{54}\pi^2 + \frac{5}{9}\,,
\end{aligned} \tag{56}
$$

where we only include the contribution from diagrams with closed heavy-quark loops. In principle there are contributions from diagrams with massless quarks as well. However, within perturbation theory they are divergent. Their evaluation either requires non-perturbative methods or phenomenological models describing the interaction of light mesons in intermediate states. [5] This is in analogy to the light-by-light contribution to the anomalous magnetic moment of the muon [46], which contains logarithms of the electron mass.

It is possible to obtain the known three-loop result for the electron anomalous magnetic moment from the expressions in Eqs. (54)–(56) by setting $C_F = T_F = 1$, $C_A = 0$, $N_F = 1$, $d_F^{abc}d_F^{abc} = 1$ and $n_l = 0$. This result was first obtained in analytical form in Ref. [20] and confirmed in Ref. [15].

Let us evaluate the quark magnetic moment numerically for charm, bottom and top quarks. Inserting the numerical values for the coefficients, we find

$$
\begin{aligned}
\frac{a_Q}{Q_Q} = {}& 0.2122\,\alpha_s^{(n_f)}(m_Q) + (0.8417 - 0.0469\,n_l)\left[\alpha_s^{(n_f)}(m_Q)\right]^2 \\
& + \left(4.5763 - 0.5856\,n_l + 0.0145\,n_l^2\right)\left[\alpha_s^{(n_f)}(m_Q)\right]^3 + \mathcal{O}\left(\alpha_s^4\right)\,.
\end{aligned} \tag{57}
$$

Using $\alpha_s^{(4)}(m_c) = 0.3378$, $\alpha_s^{(5)}(m_b) = 0.2169$ and $\alpha_s^{(6)}(m_t) = 0.1075$ we obtain

$$
a_c = \quad 0.0478 + 0.0533 + 0.0758 = \quad 0.1770\,,
$$

[5] In principle it is possible to calculate the contributions from massive charm and bottom quarks to a_b and a_t, respectively. However, since the integrals involved contain two mass scales, this is beyond the scope of this work.

$$
\begin{aligned}
a_b &= -0.0153 - 0.0103 - 0.0084 = -0.0340\,, \\
a_t &= \quad 0.0152 + 0.0047 + 0.0017 = \quad 0.0215\,.
\end{aligned}
\tag{58}
$$

The pattern is very similar to the chromomagnetic moment of the heavy quark: no convergence is observed for the charm quark, the corrections for the bottom quark are large and amount to more than 50% of the one-loop contribution, and in the top quark case, we find that our new term gives a 10% contribution which provides quite some confidence that the uncertainty of the final prediction for a_t is small.

As already mentioned in the Introduction the LEP1 bound of Ref. [13] for the bottom quark — $a_b/Q_b < 0.045$ (68%C.L.) — was found to be saturated by the two-loop correction. It is therefore interesting to see what happens if we include our three-loop term. For this purpose, we have to evaluate a_b for $\mu = m_Z$. We find

$$
a_b(m_Z) = -0.0083 - 0.0066 - 0.0056 = -0.0206\,. \tag{59}
$$

Since the three-loop correction is almost as large as the two-loop one this overshoots the bound by about 25%. In this context we want to mention again that the contributions from closed light-quark loops could not be included in our calculation. These contributions might decrease the three-loop correction. In any case, a more precise measurement of a_b would certainly be interesting.

7 Conclusion

Our main results are the three-loop anomalous dimension of the HQET chromomagnetic-interaction operator (13) and the three-loop coefficient of this operator in the HQET Lagrangian (14). They can be used for the investigation of various effects of the heavy-quark spin symmetry violation (e.g., mass splittings) using continuum or lattice techniques. Furthermore, we have obtained the three-loop anomalous magnetic moments of heavy quarks (54). This result does not include the light-quark light-by-light contribution which cannot be calculated perturbatively.

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