PITHA 07/03
TTP/07-11
SFB/CPP-07-22
18 June 2007

Ultrasoft contribution to quarkonium production and annihilation

M. Beneke[a], Y. Kiyo[b], A.A. Penin[b,c,d]

[a] *Institut für Theoretische Physik E, RWTH Aachen, D–52056 Aachen, Germany*

[b] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D–76128 Karlsruhe, Germany*

[c] *Department of Physics, University Of Alberta Edmonton, AB T6G 2J1, Canada*

[d] *Institute for Nuclear Research, Russian Academy of Sciences, 119899 Moscow, Russia*

Abstract

We compute the third-order correction to electromagnetic S-wave quarkonium production and annihilation rates due to the emission and absorption of an ultrasoft gluon. Our result completes the analysis of the non-relativistic quarkonium bound-state dynamics in the next-to-next-to-next-to-leading order. The impact of the ultrasoft correction on the $\Upsilon(1S)$ leptonic width and the top quark-antiquark threshold production cross section is estimated.

PACS numbers: 12.38.Bx, 14.40.Gx, 14.65.Fy, 14.65.Ha

1 Introduction

The theoretical study of non-relativistic heavy quark-antiquark systems is among the earliest applications of perturbative quantum chromodynamics (QCD) [1]. Perturbation theory applies to the bound-state dynamics of bottomonium, at least within the sum rule approach [2], and top-antitop systems [3], since non-perturbative effects are under control [4,5]. This makes heavy quark-antiquark systems well suited to determine fundamental parameters of QCD, the strong coupling constant α_s and the heavy-quark masses m.

The binding energy of a quarkonium state and the value of its wave function at the origin – field-theoretically, the residues of two-point functions of local currents – are of primary phenomenological interest. The former determines the mass of the bound state, while the latter controls its production and annihilation rates. The quarkonium ground-state energy has been computed through $\mathcal{O}(m\alpha_s^5)$ including the third-order correction to the Coulomb approximation [6,7]. This result has been extended to the excited S-wave states [8,9]. For the wave function at the origin a complete result is only available including the second-order correction [10–12]. This correction is large even for top quarks, and for a reliable perturbative prediction the third-order approximation seems to be needed. This amounts to a difficult calculation, which can be broken into several well-defined pieces, some of which are already available, such as the double-logarithmically enhanced $\mathcal{O}(\alpha_s^3 \ln^2 \alpha_s)$ terms [13,14] and the single-logarithmic $\mathcal{O}(\alpha_s^3 \ln \alpha_s)$ terms [15,16]. The calculation of the most difficult non-logarithmic term has been started in [8,9], where the contribution to the wave function at the origin from the loop corrections to the colour-Coulomb potential have been evaluated. In this and the companion paper [17] the remaining contributions from the non-Coulomb potentials and due to the emission and absorption of an ultrasoft gluon by the quarkonium bound state are presented. The ultrasoft correction discussed below is of special interest, because it constitutes a qualitatively new effect, which shows up for the first time in the third order. No other such effects are expected in higher orders of the perturbative expansion. The complete third-order correction to the wave function at the origin can now be expressed in terms of a few yet unknown matching coefficients, which can be obtained by standard fixed-order loop calculations.

2 Ultrasoft correction to the wave function

2.1 Definitions

In non-relativistic bound states the quark velocity v is a small parameter. An expansion in v may be performed directly in the QCD Lagrangian by using the framework of effective field theory [18–20], or diagrammatically with the threshold expansion [21]. The relevant momentum regions are the hard region (energy k^0 and momentum $\boldsymbol{k}$ of order m), the soft region ($k^0, \boldsymbol{k} \sim mv$), the potential region ($k^0 \sim mv^2$, $\boldsymbol{k} \sim mv$), and the ultrasoft region ($k^0, \boldsymbol{k} \sim mv^2$). Integrating out the hard modes amounts to matching onto non-relativistic QCD (NRQCD) [19]. If one also integrates out the soft modes and potential gluons, one obtains the effective theory called potential NRQCD (PNRQCD), which contains potential heavy quarks and ultrasoft gluons as dynamical fields [20] (see also [22]). In this theory the leading colour-Coulomb potential is part of the unperturbed Lagrangian, so that the propagation of a colour-singlet quark-antiquark pair is described by the Green function of the Schrödinger equation

$$(H_0 - E)\, G_C^{(s)}(\boldsymbol{r}, \boldsymbol{r}'; E) = \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')\,, \tag{1}$$

with

$$H_0 = -\frac{\boldsymbol{\nabla}_{(r)}^2}{m} - \frac{\alpha_s C_F}{r}, \tag{2}$$

$r = |\boldsymbol{r}|$, m the heavy-quark pole mass, and $C_F = (N_c^2 - 1)/(2N_c)$, $N_c = 3$. The PNRQCD Lagrangian further contains interactions of quarks with the multipole-expanded ultrasoft gluon field and instantaneous, spatially non-local interactions ("potentials"), which can be treated as perturbations. This constitutes the basic framework for the perturbative analysis of quarkonium bound-state properties. The colour-singlet Coulomb Green function $G_C^{(s)}(\boldsymbol{r}, \boldsymbol{r}'; E)$ has an infinite number of bound-state poles with energies $E_n^{(0)} = -m(\alpha_s C_F)^2/(2n)^2$ and contains the information about the corresponding wave functions. In the quark-antiquark Fock state sector the perturbations to the energy levels and wave functions can be taken into account by replacing H_0 by the PNRQCD Hamiltonian H with the ultrasoft modes are excluded.

In this paper, however, we are interested in the leading ultrasoft effect. To connect the concept of a non-relativistic wave function at the origin to a physical quantity, we consider the two-point function of the electromagnetic heavy-quark current $j_\mu = \bar{Q}\gamma_\mu Q$ in full QCD,

$$\left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi(q^2) = i \int d^d x\, e^{iqx}\, \langle 0|T j_\mu(x) j_\nu(0)|0\rangle\,, \tag{3}$$

whose poles are related to electromagnetic production and annihilation rates of the corresponding quarkonium states. In PNRQCD j_μ is represented in terms of operators constructed from the non-relativistic quark and antiquark two-component Pauli spinor fields ψ and χ,

$$\boldsymbol{j} = c_v \psi^\dagger \boldsymbol{\sigma} \chi + \frac{d_v}{6m^2} \psi^\dagger \boldsymbol{\sigma} \boldsymbol{D}^2 \chi + \ldots\,. \tag{4}$$

The matching coefficients $c_v(\mu) = 1 + c_v^{(1)}(\mu)\alpha_s(\mu)/(4\pi) + \ldots$ and $d_v(\mu) = 1 + \mathcal{O}(\alpha_s)$ represent the contributions from the hard modes with μ a factorization scale that is also implicit in the renormalization convention for the operators on the right-hand side. (We use dimensional regularization with $d = 4 - 2\epsilon$ and the $\overline{\text{MS}}$ scheme.) We also introduce the PNRQCD two-point function

$$2(d-1)N_c\, G(E) = i \int d^d x\, e^{iEx^0}\, \langle 0|T[\psi^\dagger \sigma^i \chi](x)\, [\chi^\dagger \sigma^i \psi](0)|0\rangle\,, \tag{5}$$

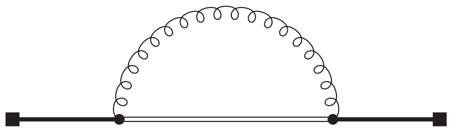

Figure 1: The ultrasoft correction as a PNRQCD Feynman diagram. The bold and the double lines stand for the singlet and octet Coulomb Green functions, respectively, the curly line represents the ultrasoft-gluon propagator, the black circles represent the chromoelectric dipole interaction $g_s \boldsymbol{r}\boldsymbol{E}$, and the squares correspond to the non-relativistic currents.

where $E = \sqrt{q^2} - 2m$. In leading order $G(E)$ coincides with (the correspondingly regularized) $G_C^{(s)}(0,0;E)$. Substituting the expansion (4) into (3) and using an equation-of-motion relation for the insertion of the derivative current in (4), we obtain

$$\Pi(q^2) = \frac{N_c c_v}{2m^2} \left[c_v - \frac{E}{m} \left(1 + \frac{d_v}{3} \right) + \ldots \right] G(E) \,, \tag{6}$$

which is valid up to the third order. $G(E)$ has Coulomb bound-state poles at energies $E_n \approx E_n^{(0)}$ with spin and orbital angular momentum $S = 1$ and $l = 0$, respectively, following from the form of the current, $\psi^\dagger \boldsymbol{\sigma} \chi$. Near the pole

$$G(0,0;E) \underset{E \to E_n}{\longrightarrow} \frac{|\psi_n(0)|^2}{E_n - E - i\varepsilon} \,, \tag{7}$$

which defines the "wave function at the origin". Beginning from the second-order correction, these wave-functions are factorization-scale dependent, but this dependence cancels in the residues of the poles of $\Pi(q^2)$, which determine the observable electromagnetic production and annihilation rates.

The third-order (NNNLO) corrections to $|\psi_n(0)|^2$ originate (i) from single insertions of the third-order potentials in the PNRQCD Lagrangian and multiple insertions of first- and second-order potentials, where the order is determined by the combined suppression in α_s and v relative to the leading-order Coulomb potential $\alpha_s/r \sim \alpha_s v$. This contribution has been computed in [8,9,17] and expressed in terms of the yet unknown three-loop correction to the Coulomb potential and four constants related to $\mathcal{O}(\epsilon)$ parts of loop corrections to other potentials; (ii) from the emission and absorption of an ultrasoft gluon (similar to the Lamb shift for energy levels), calculated below.

2.2 Calculation of the ultrasoft correction

The leading ultrasoft interactions in the PNRQCD Lagrangian are $g_s \psi^\dagger(x)(A^0(t,\mathbf{0}) - \boldsymbol{x}\boldsymbol{E}(t,\mathbf{0}))\psi(x)$ together with a similar term for the antiquark field. The contribution from the A^0 coupling cancels (or can be gauged away), leaving the chromoelectric dipole interaction, which results in a NNNLO correction [23–25]. The PNRQCD diagram representing this correction is shown in Fig. 1. The corresponding correction to $G(E)$ defined

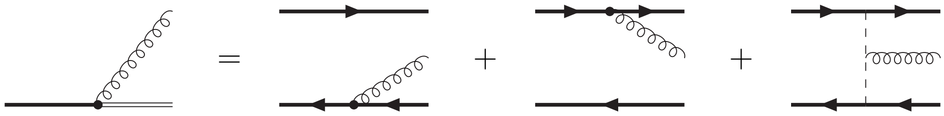

Figure 2: The NRQCD decomposition of the PNRQCD vertex. The dashed line corresponds to a potential Coulomb gluon. The black circles on the right-hand side correspond to the NRQCD vertices $g_s \boldsymbol{p}\boldsymbol{A}/m$. The bold arrows correspond to the potential quark and antiquark propagators with any number of the potential gluon exchanges.

in (5) reads

$$\delta^{us}G(E) = ig_s^2 C_F \int d^3\boldsymbol{r} d^3\boldsymbol{r}' \int \frac{d^4k}{(2\pi)^4} \left[\frac{k_0^2\, \boldsymbol{r}\boldsymbol{r}' - (\boldsymbol{r}\boldsymbol{k})(\boldsymbol{r}'\boldsymbol{k})}{k^2 + i\varepsilon} \right.$$
$$\left. \times\, G_C^{(s)}(0, \boldsymbol{r}; E) G_C^{(o)}(\boldsymbol{r}, \boldsymbol{r}'; E - k_0) G_C^{(s)}(\boldsymbol{r}', 0; E) \right] \tag{8}$$

with the understanding that one picks up only the pole at $k^0 = |\boldsymbol{k}| - i\epsilon$ in the gluon propagator. Here $G_C^{(o)}(\boldsymbol{r}, \boldsymbol{r}'; E)$ denotes the colour-octet Coulomb Green function corresponding to the Hamiltonian (2) with the octet potential $V_C^{(o)}(r) = (C_A/2 - C_F)\alpha_s/r$ $(C_A = N_c)$. Only the $l = 1$ partial wave of the octet Green function contributes to (8), and since the octet potential is repulsive, $G_C^{(8)}(\boldsymbol{r}, \boldsymbol{r}'; E)$ does not have bound-state poles.

Expression (8) cannot be used in practice, because the ultrasoft correction is divergent. Its definition requires specifying a regulator and subtractions, which must be chosen to be consistent with the calculation of the potential insertions; the potentials themselves; and the hard matching coefficients. In the following we provide the necessary definitions, deferring a detailed discussion of many interesting technical aspects of the calculation to a later publication that will provide results not only for $|\psi_n(0)|^2$ but also for the full correlation function $G(E)$. To apply dimensional regularization we transform (8) to momentum space. We also find it convenient to re-express the PNRQCD vertex in terms of the NRQCD vertices from which it is derived by equation-of-motion relations, see [25] and Fig. 2 for a graphical representation. The formulation of non-relativistic effective theory in dimensional regularization [6,12,26,27] is very convenient, because it allows one to combine bound-state calculations with loop calculations of matching coefficients, which are technically feasible only in dimensional regularization. Furthermore, when loop integrals are expanded in the sense of the threshold expansion [21], the matching of contributions from different regions is automatic.

The integral over the three-momentum $\boldsymbol{k}$ of the ultrasoft gluon is ultraviolet (UV) divergent. The divergence is related to the factorization of the ultrasoft scale from the other scales, and cancels when all pieces of the calculation are added. The UV-divergent part of the ultrasoft integral has the form of a single insertion of a third-order potential and of a one-loop correction to the coefficient d_v of the derivative current in (4). In fact, they cancel precisely infrared (IR) divergences in the calculation of these quantities

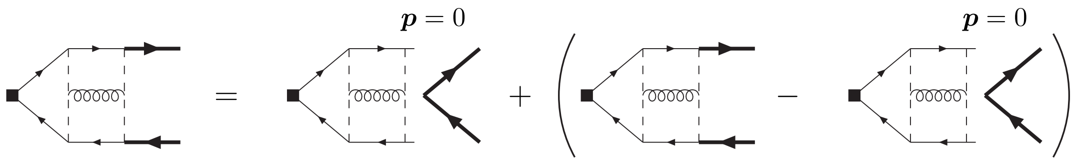

Figure 3: Example of a three-loop vertex correction with an ultrasoft exchange. The right-hand side illustrates the subtraction of the UV divergent part. The thin arrows correspond to free potential quark and antiquark propagators.

[6,28]. We therefore define the ultrasoft correction by adding counterterms that cancel these ultrasoft subdivergences,

$$\delta^{us}G(E) = \left[\tilde{\mu}^{2\epsilon}\right]^2 \int \frac{d^{d-1}\boldsymbol{\ell}}{(2\pi)^{d-1}} \frac{d^{d-1}\boldsymbol{\ell}'}{(2\pi)^{d-1}} \bigg\{ (-\delta d_v^{\rm div})\, \frac{\boldsymbol{\ell}^2 + \boldsymbol{\ell}'^2}{6m^2}\, \tilde{G}_C^{(s)}(\boldsymbol{\ell}, \boldsymbol{\ell}'; E)$$

$$+ \left[\tilde{\mu}^{2\epsilon}\right]^2 \int \frac{d^{d-1}\boldsymbol{p}}{(2\pi)^{d-1}} \frac{d^{d-1}\boldsymbol{p}'}{(2\pi)^{d-1}}\, \tilde{G}_C^{(s)}(\boldsymbol{\ell}, \boldsymbol{p}; E) \left[\delta U + \delta\tilde{V}_{c.t.}\right] \tilde{G}_C^{(s)}(\boldsymbol{p}', \boldsymbol{\ell}'; E) \bigg\}, \quad (9)$$

where δU follows from (8), and the counterterms read ($\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$)

$$\delta\tilde{V}_{c.t.} = \frac{\alpha_s C_F}{6\epsilon} \bigg[C_A^3 \frac{\alpha_s^3}{\boldsymbol{q}^2} + 4(C_A^2 + 2C_A C_F) \frac{\pi\alpha_s^2}{m|\boldsymbol{q}|}$$

$$+16 \left(C_F - \frac{C_A}{2} \right) \frac{\alpha_s}{m^2} + 16 C_A \frac{\alpha_s}{m^2} \frac{\boldsymbol{p}^2 + \boldsymbol{p}'^2}{2\boldsymbol{q}^2} \bigg], \quad (10)$$

$$\delta d_v^{\rm div} = -\frac{\alpha_s}{4\pi} \frac{16 C_F}{\epsilon}. \quad (11)$$

The counterterms added here are subtracted from the other parts of the calculation [17]. With $\tilde{\mu}^2 = e^{\gamma_E} \mu^2/(4\pi)$ subtracting poles in ϵ corresponds to the $\overline{\rm MS}$ subtraction.

After adding this counterterm the potential loops and Coulomb Green functions can be evaluated in three dimensions unless the potential loop integrations are divergent. Such divergences occur in potential vertex subgraphs up to three loops, and they correspond to IR divergences in the three-loop correction to c_v. (There are also over-all divergences in $G(E)$, but they are irrelevant to the calculation of the bound-state pole and residue.) We separate this vertex subdivergence by adding and subtracting the three-loop vertex subdiagram at zero external momentum $\boldsymbol{p}$, as shown in Fig. 3. The vertex UV divergence is logarithmic, and does not depend on the external momentum. Hence, it is isolated in first term on the right-hand side of the equation of Fig. 3, while the difference in the brackets is finite and can be computed in three dimensions. The first term factorizes into a three-loop diagram at $\boldsymbol{p} = 0$, which has to be computed in d dimensions, and the leading-order expression for $G(E)$. The divergent part cancels the contribution from c_v, and the remainder can again be evaluated in three dimensions. Thus, we do not need the $d-1$ dimensional Coulomb Green function, which is unknown.

n	1	2	3	4	5	6
δ_n^{us}	353.06	256.62	224.26	206.88	195.48	187.16

Table 1: Numerical result for the non-logarithmic part of the ultrasoft contribution, as defined in (13); n denotes the principal quantum number.

2.3 Result

We write the perturbative expansion for the wave function at the origin as

$$|\psi_n(0)|^2 = |\psi_n^C(0)|^2 \left(1 + \delta^{(1)}\psi_n + \delta^{(2)}\psi_n + \ldots\right) , \tag{12}$$

where the leading-order Coulomb wave function at the origin in three dimensions is given by $|\psi_n^C(0)|^2 = (m\alpha_s C_F)^3/(8\pi n^3)$, and $\delta^{(m)}\psi_n$ stands for the mth order correction. For the ultrasoft part of the third-order correction we obtain

$$\begin{aligned}
\delta^{us}\psi_n =\; & \frac{\alpha_s^3}{\pi}\Bigg\{\left[\frac{1}{4}C_A^2C_F + \frac{7}{12}C_AC_F^2 + \frac{1}{6}C_F^3\right]\frac{1}{\epsilon^2} + \left[\frac{1}{6}C_A^2C_F + \frac{1}{2}C_AC_F^2 + \frac{1}{3}C_F^3\right]\frac{1}{\epsilon}\ln\frac{\mu}{m} \\
&+\left[\left(\frac{5}{18} - \frac{\ln 2}{6}\right)C_A^2C_F + \left(\frac{25}{12} - \frac{5}{6}\ln 2\right)C_AC_F^2 + \left(\frac{19}{18} - \ln 2\right)C_F^3\right]\frac{1}{\epsilon} \\
&+\left[-2C_A^2C_F - \frac{16}{3}C_AC_F^2 - \frac{8}{3}C_F^3\right]\ln^2\alpha_s + \left[-\frac{5}{6}C_A^2C_F - \frac{11}{6}C_AC_F^2 - \frac{1}{3}C_F^3\right]\ln^2\frac{\mu}{m} \\
&+\left[\frac{8}{3}C_A^2C_F + \frac{20}{3}C_AC_F^2 + \frac{8}{3}C_F^3\right]\ln\alpha_s\ln\frac{\mu}{m} \\
&+\Bigg[C_A^3 + \left(\frac{52}{9} - \frac{8}{3}\ln 2 - 4H_n\right)C_A^2C_F + \left(6 - \frac{10}{3n^2} - \frac{4}{3}\ln 2 - \frac{32}{3}H_n\right)C_AC_F^2 \\
&\quad + \left(-\frac{52}{9} - \frac{4}{3n^2} + 8\ln 2 - \frac{16}{3}H_n\right)C_F^3\Bigg]\ln\alpha_s \\
&+\Bigg[-\frac{3}{4}C_A^3 + \left(-\frac{11}{3} + \frac{5}{3}\ln 2 + \frac{8}{3}H_n\right)C_A^2C_F + \left(-\frac{3}{2} + \frac{5}{3n^2} + \frac{1}{3}\ln 2 + \frac{20}{3}H_n\right)C_AC_F^2 \\
&\quad + \left(5 + \frac{2}{3n^2} - 6\ln 2 + \frac{8}{3}H_n\right)C_F^3\Bigg]\ln\frac{\mu}{m} + \delta_n^{us}\Bigg\} ,
\end{aligned} \tag{13}$$

where $H_n = \ln\frac{C_F}{n} - \frac{1}{n} + \sum_{k=1}^{n-1}\frac{1}{k}$. The most difficult part of the calculation is the non-logarithmic term δ_n^{us}, which we could compute only numerically. For the six lowest states its value is given in Table 1. Note that $|\psi_n^C(0)|^2$ in (12) is formally defined in d dimensions, but as explained above, we do not need the explicit d-dimensional expression, because the $1/\epsilon$ pole terms in (13) cancel with pole terms from $c_v^2\,|\psi_n^C(0)|^2$ contained in (6).

We verified that the logarithmic terms in (13) when combined with those from the third-order potentials insertions [17] agree with [15][1]. The sum of the $1/\epsilon$ poles in the complete third-order correction to $|\psi_n(0)|^2$ from (13) and [17] combined contains a term proportional to the single insertion of the first-order Coulomb potential and a term that must cancel against the infrared pole in twice the three-loop correction to c_v. This second term reads

$$\delta^{(3)}\psi_n^{\rm div} = \frac{\alpha_s^3}{\pi}\Bigg\{\left(\frac{1}{36}C_A^2C_F+\frac{5}{48}C_AC_F^2+\frac{5}{72}C_F^3\right)\left(\frac{1}{\epsilon^2}+\frac{6}{\epsilon}\ln\frac{\mu}{m}\right) - \left(\frac{1}{24}C_AC_F+\frac{1}{36}C_F^2\right)\frac{\beta_0}{\epsilon^2}+\Bigg[\left(\frac{4}{27}+\frac{\ln 2}{2}\right)C_A^2C_F+\left(\frac{113}{162}+\frac{\ln 2}{2}\right)C_AC_F^2 + \left(\frac{43}{72}-\ln 2\right)C_F^3-\frac{37}{216}C_AC_FT_Fn_f+\frac{1}{30}C_F^2T_F-\frac{25}{162}C_F^2T_Fn_f\Bigg]\frac{1}{\epsilon}\Bigg\}, \qquad (14)$$

where $\beta_0 = 11C_A/3 - 4T_Fn_f/3$ is the one-loop QCD beta-function, $T_F = 1/2$ and n_f is the number of light-quark flavors. The n_f-part of the three-loop coefficient $c_v^{(3)}$ is known [29], and we checked that the n_f-dependent pole parts cancel as required. Eq. (14) is consistent with the scale dependence of the hard matching coefficient given by Eqs. (8), (9) of [15], except for the rational part of the $C_AC_F^2$ term in $\gamma_v'^{(3)}$, which should be increased by 7/24 to agree with our result.[2] The difference is due to the fact that by using $[\sigma^i,\sigma^j] = i\epsilon^{ijk}\sigma^k$ the spin-algebra in [15] is not completely d-dimensional, hence the result for the scale-dependence of $c_v^{(3)}$ given there does not correspond to a calculation in conventional dimensional regularization.

3 Quarkonium phenomenology

We briefly discuss the size of the ultrasoft correction for the two most relevant cases, the leptonic decay of the $\Upsilon(1S)$ and the threshold production of top quark-antiquark pairs in e^+e^- annihilation. This discussion must necessarily be preliminary, since the ultrasoft correction alone is factorization scheme and scale dependent. For the quarkonium spin-triplet ground state, $n = 1$, we obtain from (13), omitting the $1/\epsilon$ poles,

$$\delta^{us}\psi_1 = \alpha_s^3\Big\{-18.71\ln^2\alpha_s+52.03\ln\alpha_s+112.38 + \Big[23.52\ln\alpha_s-30.98\Big]\ln\frac{\mu}{m}-6.55\ln^2\frac{\mu}{m}\Big\}. \qquad (15)$$

The scale of the coupling α_s is most naturally of order of the inverse Bohr radius $m\alpha_sC_F$ in two of the three powers of the overall factor α_s^3, and of order of the ultrasoft scale $m\alpha_s^2$

[1] In Eq. (7) of [15] for the excited states the term $4C_F^3(1-1/n^2)/3$ is missed and the $\beta_0\ln n$ term should be multiplied by three.

[2] In addition, as noted in [29], the term $-\frac{3}{2}\beta_0\gamma_v^{(2)}$ in Eq. (8) of [15] should read $+\beta_0\gamma_v^{(2)}$.

in the third. However, any other scale choice is formally equivalent at this order. In the following we evaluate α_s at $\mu_B = mC_F\alpha_s(\mu_B)$, wherever it appears. The scale μ in the $\ln(\mu/m)$ terms is related to scale-dependent potentials and hard matching coefficients. We vary μ/m between $\alpha_s C_F$ (corresponding to the scale μ_B) and 1 (hard scale).

$\Upsilon(1S)$ *leptonic width.* Up to a normalization factor the $\Upsilon(1S)$ leptonic decay width is given by the residue of $\Pi(q^2)$ at the ground state pole. The leading-order expression for the decay width follows from (6), resulting in $\Gamma_1^{\rm LO} = 4\pi N_c e_b^2\alpha^2|\psi_1^C(0)|^2/(3m_b^2)$, where e_q is the electric charge of the quark flavour q and α is the fine-structure constant. The non-perturbative contribution to the width is quite sizeable and out of control for higher resonances [30,31], hence we consider only the case $n=1$. Adopting $\alpha_s = 0.30$, which corresponds to $\mu_B \approx 2\,\text{GeV}$ for the $\Upsilon(1S)$, we obtain $\delta\Gamma_1/\Gamma_1 \approx 3.0$ from the non-logarithmic correction δ_1^{us} alone. Proceeding as described above for the logarithmic terms results in the estimate

$$\delta\Gamma_1 \approx \Big[0.61 - 1.93\Big]\,\Gamma_1^{\rm LO}. \tag{16}$$

It therefore appears that the large non-logarithmic term leads to a large enhancement of the width. Whether or not perturbation theory is out of control (as may be suggested by the upper limit of the given range) can be decided only after combining all third-order terms.

Top-quark production near threshold. For top quarks non-perturbative effects are negligible, but its decay width Γ_t smears out the Coulomb resonances below the threshold. The NNLO analysis of the cross section [32] shows that only the ground-state pole gives rise to a prominent resonance. Although the calculation of the normalized cross section $R = \sigma(e^+e^- \to t\bar{t}X)/\sigma(e^+e^- \to \mu^+\mu^-)$ requires the full Green function $G(E)$, the height of the resonance can be estimated from the wave function at the origin of the *would-be* toponium ground state. In the leading-order approximation $R_1^{\rm LO} \approx 6\pi N_c e_t^2|\psi_1^C(0)|^2/(m_t^2\Gamma_t)$. Adopting $\alpha_s = 0.14$, which corresponds to $\mu_B \approx 32.5\,\text{GeV}$, we obtain $\delta R_1/R_1 \approx 0.31$ from the non-logarithmic correction δ_1^{us} alone. Including an estimate of the logarithmic terms we find

$$\delta R_1 \approx \Big[(-0.17) - (+0.13)\Big]\,R_1^{\rm LO}. \tag{17}$$

Hence, despite the large quark mass, we may anticipate a sizeable third-order correction, unless there are cancellations.

4 Summary

We evaluated the third-order correction to electromagnetic quarkonium production and annihilation due to the emission and absorption of an ultrasoft gluon. Together with other contributions [8,9,13–17,29] already completed the problem of evaluating the total

$\mathcal{O}(\alpha_s^3)$ corrections is now reduced to the calculation of four $\mathcal{O}(\epsilon)$ terms in the NNNLO heavy quark-antiquark potential [17], the three-loop colour-singlet Coulomb potential, and the three-loop vector current matching coefficient in the $\overline{\rm MS}$ scheme in dimensional regularization. The previously unknown non-logarithmic ultrasoft contribution is large and significantly increases the production and annihilation rates. It might limit the accuracy of the perturbative analysis of the quarkonium even for top quarks. We should however emphasize that a definite conclusion can only be drawn once the full NNNLO result is available. In this respect the sizable negative third-order correction from the perturbation potentials [8,9,17] should be mentioned.

Acknowledgement

This work is supported by the DFG Sonderforschungsbereich/Transregio 9 "Computergestützte Theoretische Teilchenphysik". The work of A.A.P. is supported by BMBF Grant No. 05HT6VKA.

References

[1] T. Appelquist and H.D. Politzer, Phys. Rev. Lett. 34 (1975) 43.

[2] V.A. Novikov, L.B. Okun, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, and V.I. Zakharov, Phys. Rev. Lett. 38 (1977) 626; Erratum *ibid.* 38 (1977) 791, Erratum; Phys. Rep. C 41 (1978) 1.

[3] V.S. Fadin and V.A. Khoze, Pis'ma Zh. Eksp. Teor. Fiz. 46 (1987) 417 [JETP Lett. 46 (1987) 525].

[4] M.B. Voloshin, Nucl. Phys. B 154 (1979) 365; Yad. Fiz. 36 (1982) 247 [Sov. J. Nucl. Phys. 36 (1982) 143].

[5] H. Leutwyler, Phys. Lett. B 98 (1981) 447.

[6] B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Nucl. Phys. B 635 (2002) 357.

[7] A.A. Penin and M. Steinhauser, Phys. Lett. B 538 (2002) 335.

[8] A.A. Penin, V.A. Smirnov, and M. Steinhauser, Nucl. Phys. B 716 (2005) 303.

[9] M. Beneke, Y. Kiyo, and K. Schuller, Nucl. Phys. B 714 (2005) 67.

[10] K. Melnikov and A. Yelkhovsky, Phys. Rev. D 59 (1999) 114009.

[11] A.A. Penin and A.A. Pivovarov, Nucl. Phys. B 549 (1999) 21.

[12] M. Beneke, A. Signer, and V.A. Smirnov, Phys. Lett. B 454 (1999) 137.

[13] B.A. Kniehl and A.A. Penin, Nucl. Phys. B 577 (2000) 197.

[14] A.V. Manohar and I.W. Stewart, Phys. Rev. D 63 (2001) 054004.

[15] B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Phys. Rev. Lett. 90 (2003) 212001; Erratum *ibid.* 91 (2003) 139903.

[16] A.H. Hoang, Phys. Rev. D 69 (2004) 034009.

[17] M. Beneke, Y. Kiyo, and K. Schuller, arXiv:0705.4518 [hep-ph].

[18] W.E. Caswell and G.P. Lepage, Phys. Lett. B 167 (1986) 437.

[19] G.T. Bodwin, E. Braaten, and G.P. Lepage, Phys. Rev. D 51 (1995) 1125; Erratum *ibid.* 55 (1997) 5853.

[20] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. 64 (1998) 428.

[21] M. Beneke and V.A. Smirnov, Nucl. Phys. B 522 (1998) 321.

[22] M. Beneke, in: Proceedings of the 33rd Rencontres de Moriond: Electroweak Interactions and Unified Theories, Les Arcs, France, 14-21 Mar 1998 [hep-ph/9806429].

[23] B.A. Kniehl and A.A. Penin, Nucl. Phys. B 563 (1999) 200.

[24] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. B 566 (2000) 275.

[25] M. Beneke, in: Proceedings of the 8th International Symposium on Heavy Flavor Physics (Heavy Flavors 8), Southampton, England, 25-29 Jul 1999, [hep-ph/9911490].

[26] A. Pineda and J. Soto, Phys. Lett. B 420 (1998) 391; Phys. Rev. D 59 (1999) 016005.

[27] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A 59 (1999) 4316.

[28] M.E. Luke and M.J. Savage, Phys. Rev. D 57 (1998) 413.

[29] P. Marquard, J.H. Piclum, D. Seidel, and M. Steinhauser, Nucl. Phys. B 758 (2006) 144.

[30] S. Titard and F.J. Yndurain, Phys. Rev. D **51** (1995) 6348.

[31] A. Pineda, Nucl. Phys. B 494 (1997) 213.

[32] A.H. Hoang, M. Beneke, K. Melnikov, T. Nagano, A. Ota, A.A. Penin, A.A. Pivovarov, A. Signer, V.A. Smirnov, Y. Sumino, T. Teubner, O. Yakovlev, and A. Yelkhovsky, Eur. Phys. J. direct C 3 (2000) 1.