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Relation between the pole and the minimally subtracted mass in dimensional regularization and dimensional reduction to three-loop order

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Abstract

We compute the relation between the pole quark mass and the minimally subtracted quark mass in the framework of QCD applying dimensional reduction as a regularization scheme. Special emphasis is put on the evanescent couplings and the renormalization of the ε-scalar mass. As a by-product we obtain the three-loop on-shell renormalization constants $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ in dimensional regularization and thus provide the first independent check of the analytical results computed several years ago.

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1 Introduction

In quantum chromodynamics (QCD), like in any other renormalizeable quantum field theory, it is crucial to specify the precise meaning of the parameters appearing in the underlying Lagrangian — in particular when higher order quantum corrections are considered. The canonical choice for the coupling constant of QCD, α_s, is the so-called modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme [1] which has the advantage that the beta function, ruling the scale dependence of the coupling, is mass-independent. On the other hand, for a heavy quark besides the $\overline{\mathrm{MS}}$ scheme also other definitions are important — first and foremost the pole mass. Whereas the former definition is appropriate for those processes where the relevant energy scales are much larger than the quark mass the pole mass is the relevant definition for threshold processes. Thus it is important to have precise conversion formulae at hand in order to convert one definition into the other.

By far the most loop calculations performed within QCD are based on dimensional regularization (`DREG`) in order to handle the infinities which occur in intermediate steps. It is well known, however, that it is not convenient to apply `DREG` to supersymmetric theories since it introduces a mismatch between the numbers of fermionic and bosonic degrees of freedom in super-multiplets. In order to circumvent this problem and, at the same time, take over as many advantages as possible from `DREG` the regularization scheme dimensional reduction (`DRED`) has been invented (see, e.g., Refs. [2,3]). Indeed, the application of `DRED` to supersymmetric theories leads to a relatively small price one has to pay at the technical level. An elegant way is to introduce an additional scalar particle (the so-called ε scalar) at the level of the Lagrangian and to proceed for the practical calculation of the Feynman diagrams as in `DREG`. The situation becomes more complicated in case the symmetry between fermions and bosons in the underlying theory is distorted, e.g., after some heavy squarks have been integrated out. In such situations couplings involving the ε scalar, which are called evanescent couplings, renormalize differently from the gauge couplings and therefore one has to allow for new couplings in the theory. The same is true if `DRED` is applied to QCD: in addition to α_s four new couplings have to be introduced, each of which has their own beta function governing both the running and the renormalization. In this paper we take over the notation from [4] and denote them by α_e (the coupling between ε scalars and quarks) and η_i ($i = 1, 2, 3$), which describe three different four-ε vertices.

The one-loop relation between the $\overline{\texttt{MS}}$ and pole mass has been considered long ago in Ref. [5]. At the beginning of the nineties the mass relation to two-loop order [6] was one of the first applications of two-loop on-shell integrals. In a subsequent paper also the two-loop result for the on-shell wave function counterterm has been obtained [7]. The three-loop mass relation has been computed for the first time in Ref. [8,9] where the off-shell fermion propagator has been considered for small and large external momenta. The on-shell quantities have been obtained with the help of a conformal mapping and Padé approximation. The numerical results of Ref. [8,9] have later been confirmed in Ref. [10] by an analytical on-shell calculation. The three-loop result for Z_2^{OS} has been obtained in

Ref. [11]. In this paper we provide the first independent check of the analytical results for $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ in the DREG scheme.

The renormalization scheme based on DRED together with modified minimal subtraction is called $\overline{\rm DR}$. As far as the relation between the $\overline{\rm DR}$ and the pole mass is concerned one can find the one-loop result in Ref. [12]. The two-loop calculation [13] has been performed in DRED, identifying, however, the evanescent coupling α_e with α_s. In this paper we will provide the more general result and furthermore extend the relation to three-loop order.

The paper is organized as follows: In Section 2 we outline the general strategy, provide technical details and derive the result for the on-shell mass and wave function counterterm in the case of dimensional regularization. The peculiarities of dimensional reduction are explained in Section 3 where all relevant counterterms are listed. Finally, Section 4 contains the main result of our paper: the relation between the pole mass and the minimally subtracted mass in the framework of dimensional reduction up to three-loop order.

2 On-shell counterterms for mass and wave function in dimensional regularization

In order to compute the counterterms for the quark mass and wave function one has to put certain requirements on the pole and the residual of the quark propagator. More precisely, in the on-shell scheme we demand that the quark two-point function has a zero at the position of the on-shell mass and that the residual of the propagator is $-i$. Thus, the renormalized quark propagator is given by

$$S_F(q) = \frac{-iZ_2^{\rm OS}}{\not{q} - m_{q,0} + \Sigma(q, M_q)} \tag{1}$$

$$\stackrel{q^2\to M_q^2}{\longrightarrow} \frac{-i}{\not{q} - M_q}\,, \tag{2}$$

where the renormalization constants are defined as

$$m_{q,0} = Z_m^{\rm OS}\, M_q\,, \tag{3}$$

$$\psi_0 = \sqrt{Z_2^{\rm OS}}\,\psi\,. \tag{4}$$

ψ is the quark field with mass m_q, M_q is the on-shell mass and bare quantities are denoted by a subscript 0. Σ denotes the quark self-energy contributions which can be decomposed as

$$\Sigma(q, m_q) = m_q\,\Sigma_1(q^2, m_q) + (\not{q} - m_q)\,\Sigma_2(q^2, m_q)\,. \tag{5}$$

The calculation outlined in Ref. [6] for the evaluation of $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ reduces all occurring Feynman diagrams to the evaluation of on-shell integrals at the bare mass scale. In particular, it avoids the introduction of explicit counterterm diagrams. At three-loop order we find it more convenient to follow the more direct approach described in Ref. [10, 11], which requires the calculation of diagrams with mass counterterm insertion. Following the latter reference, we expand Σ around $q^2 = M_q^2$

$$\begin{aligned}
\Sigma(q, M_q) &\approx M_q\,\Sigma_1(M_q^2, M_q) + (\not{q} - M_q)\,\Sigma_2(M_q^2, M_q) \\
&\quad + M_q \frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2}(q^2 - M_q^2) + \ldots \\
&\approx M_q\,\Sigma_1(M_q^2, M_q) \\
&\quad + (\not{q} - M_q)\left(2M_q^2\frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)\right) + \ldots . \qquad (6)
\end{aligned}$$

Inserting Eq. (6) into Eq. (1) and comparing to Eq. (2) we find the following formulae for the renormalization constants

$$Z_m^{\rm OS} = 1 + \Sigma_1(M_q^2, M_q)\,, \qquad (7)$$

$$\left(Z_2^{\rm OS}\right)^{-1} = 1 + 2M_q^2\frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)\,. \qquad (8)$$

If we consider the external momentum of the quarks to be $q = Q(1 + t)$ with $Q^2 = M_q^2$, the self-energy can be written as

$$\Sigma(q, M_q) = M_q\Sigma_1(q^2, M_q) + (\not{Q} - M_q)\Sigma_2(q^2, M_q) + t\not{Q}\,\Sigma_2(q^2, M_q)\,. \qquad (9)$$

Let us now consider the quantity ${\rm Tr}\{\frac{\not{Q}+M_q}{4M_q^2}\Sigma\}$ and expand to first order in t

$$\begin{aligned}
{\rm Tr}\left\{\frac{\not{Q} + M_q}{4M_q^2}\Sigma(q, M_q)\right\} &= \Sigma_1(q^2, M_q) + t\Sigma_2(q^2, M_q) \\
&= \Sigma_1(M_q^2, M_q) + \left(2M_q^2\frac{\partial}{\partial q^2}\Sigma_1(q^2, M_q)\Big|_{q^2=M_q^2} + \Sigma_2(M_q^2, M_q)\right)t \\
&\quad + \mathcal{O}\left(t^2\right)\,. \qquad (10)
\end{aligned}$$

Thus, to obtain $Z_m^{\rm OS}$ one only needs to calculate Σ_1 for $q^2 = M_q^2$. To calculate $Z_2^{\rm OS}$, one has to compute the first derivative of the self-energy diagrams. The mass renormalization is taken into account iteratively by calculating one- and two-loop diagrams with zero-momentum insertions.

Sample diagrams contributing to the quark propagator are shown in Fig. 1. All occurring Feynman integrals can be mapped onto $J_+^{(3)}$ and $L_+^{(3)}$ as given in Eq. (4) of Ref. [14] and to seven more similar ones which will be listed in Ref. [15].

All Feynman diagrams are generated with `QGRAF` [16] and the various topologies are identified with the help of `q2e` and `exp` [17, 18]. In a next step the reduction of the various functions to so-called master integrals has to be achieved. For this step we use the so-called

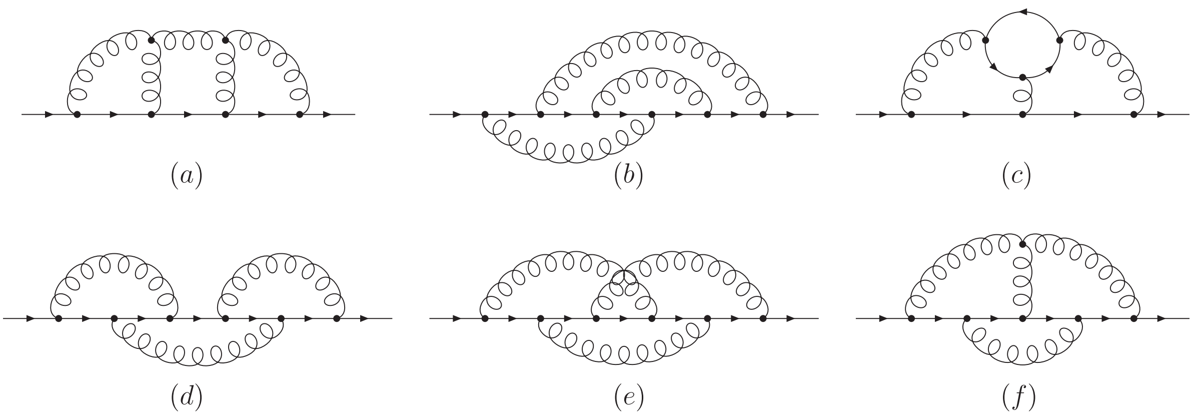

Figure 1: Sample three-loop diagrams. Solid lines denote massive quarks with mass m_q and curly lines denote gluons. In the closed fermion loops all quark flavours have to be considered.

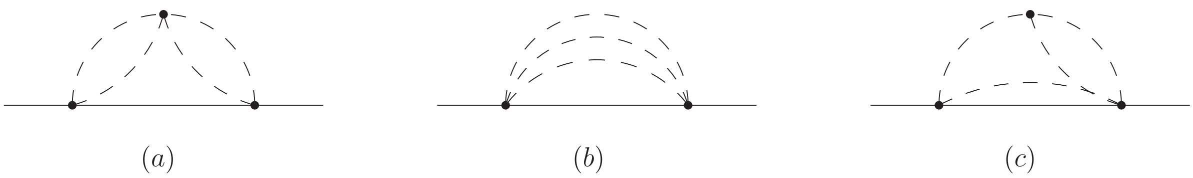

Figure 2: Three-loop master integrals. Solid lines denote massive and dashed lines are massless scalar propagators.

Laporta method [19,20] which reduces the three-loop integrals to 19 master integrals. We use the implementation of Laporta's algorithm in the program `Crusher` [21]. It is written in `C++` and uses `GiNaC` [22] for simple manipulations like taking derivatives of polynomial quantities. In the practical implementation of the Laporta algorithm one of the most time-consuming operations is the simplification of the coefficients appearing in front of the individual integrals. This task is performed with the help of `Fermat` [23] where a special interface has been used (see Ref. [24]). The main features of the implementation are the automated generation of the integration-by-parts (IBP) identities [25] and a complete symmetrization of the diagrams.

The results for the master integrals can be found in Ref. [11]. We have checked the results numerically with the Mellin-Barnes method [26,27] and the program `MB` [28]. We did, however, find some differences to Ref. [11]: In addition to the 18 master integrals given in that reference, we found the integral depicted in Fig. 2(a) where the result can be found in Eq. (A.2) of Ref. [14]. Furthermore, we already pointed out in Ref. [14], that the result for Fig 2(b) (denoted by I_{16} in Ref. [11]) is wrong. As it turns out, the result given in [11] corresponds to the integral depicted in Fig. 2(c). Since there is a one-to-one correspondence between the two integrals with regard to the IBP relations, it does not matter which is chosen to be the master integral.

Most of the master integrals of Ref. [11] were already calculated in Ref. [19]. However, it turns out that there are differences in the order $\mathcal{O}(\epsilon)$ terms of the integrals I_2–I_7 of these references. The expressions given in Ref. [11] agree with our numerical results.

The results for the renormalization constants can be cast into the following form

$$\begin{aligned} Z_i^{\rm OS} &= 1+\frac{\alpha_s(\mu)}{\pi}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon}\delta Z_i^{(1)}+\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon}\delta Z_i^{(2)} \\ &\quad +\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\epsilon}\delta Z_i^{(3)}+\mathcal{O}\left(\alpha_s^4\right)\,, \end{aligned} \tag{11}$$

with $i\in\{m,2\}$. It is convenient to further decompose the three-loop contribution in terms of the different colour factors

$$\begin{aligned} \delta Z_i^{(3)} &= C_F^3\,Z_i^{FFF}+C_F^2C_A\,Z_i^{FFA}+C_FC_A^2\,Z_i^{FAA} \\ &\quad +C_FT_Fn_l\left(C_F\,Z_i^{FFL}+C_A\,Z_i^{FAL}+T_Fn_l\,Z_i^{FLL}+T_Fn_h\,Z_i^{FHL}\right) \\ &\quad +C_FT_Fn_h\left(C_F\,Z_i^{FFH}+C_A\,Z_i^{FAH}+T_Fn_h\,Z_i^{FHH}\right)\,, \end{aligned} \tag{12}$$

where $C_F=(N_c^2-1)/(2N_c)$ and $C_A=N_c$ are the eigenvalues of the quadratic Casimir operators of the fundamental and adjoint representation of $SU(N_c)$, respectively. In the case of QCD we have $N_c=3$. $T_F=1/2$ is the index of the fundamental representation and $n_f=n_l+n_h$ is the number of quark flavours. n_l and n_h are the number of light and heavy quark flavours, respectively. The former are considered to be massless, while the latter have mass m_q. Although we have $n_h=1$ in our applications, we keep a generic label which is useful when tracing the origin of the individual contributions. $\alpha_s(\mu)$ is the strong coupling constant defined in the $\overline{\mathtt{MS}}$ scheme with n_f active flavours.

The one- and two-loop contributions to the mass renormalization constant are

$$\begin{aligned} \delta Z_m^{(1)} &= -C_F\left[\frac{3}{4\epsilon}+1+\frac{3}{4}L_\mu+\left(2+\frac{1}{16}\pi^2+L_\mu+\frac{3}{8}L_\mu^2\right)\epsilon\right. \\ &\quad \left.+\left(4+\frac{1}{12}\pi^2-\frac{1}{4}\zeta_3+\left(2+\frac{1}{16}\pi^2\right)L_\mu+\frac{1}{2}L_\mu^2+\frac{1}{8}L_\mu^3\right)\epsilon^2\right]\,, \end{aligned} \tag{13}$$

and

$$
\begin{aligned}
\delta Z_m^{(2)} &= \left[\frac{9}{32\epsilon^2} + \left(\frac{45}{64} + \frac{9}{16}L_\mu\right)\frac{1}{\epsilon} + \frac{199}{128} - \frac{17}{64}\pi^2 + \frac{1}{2}\pi^2\ln 2 - \frac{3}{4}\zeta_3 + \frac{45}{32}L_\mu + \frac{9}{16}L_\mu^2\right. \\
&\quad + \left(\frac{677}{256} - \frac{205}{128}\pi^2 + 3\pi^2\ln 2 - \pi^2\ln^2 2 - \frac{135}{16}\zeta_3 + \frac{7}{40}\pi^4 - \frac{1}{2}\ln^4 2 - 12a_4\right. \\
&\quad \left.\left. + \left(\frac{199}{64} - \frac{17}{32}\pi^2 + \pi^2\ln 2 - \frac{3}{2}\zeta_3\right)L_\mu + \frac{45}{32}L_\mu^2 + \frac{3}{8}L_\mu^3\right)\epsilon\right] C_F^2 \\
&\quad + \left[\frac{11}{32\epsilon^2} - \frac{97}{192\epsilon} - \frac{1111}{384} + \frac{1}{12}\pi^2 - \frac{1}{4}\pi^2\ln 2 + \frac{3}{8}\zeta_3 - \frac{185}{96}L_\mu - \frac{11}{32}L_\mu^2\right. \\
&\quad + \left(-\frac{8581}{768} + \frac{271}{1152}\pi^2 - \frac{3}{2}\pi^2\ln 2 + \frac{1}{2}\pi^2\ln^2 2 + \frac{13}{4}\zeta_3 - \frac{7}{80}\pi^4 + \frac{1}{4}\ln^4 2 + 6a_4\right. \\
&\quad \left.\left. + \left(-\frac{1463}{192} + \frac{7}{64}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta_3\right)L_\mu - \frac{229}{96}L_\mu^2 - \frac{11}{32}L_\mu^3\right)\epsilon\right] C_A C_F \\
&\quad + \left[-\frac{1}{8\epsilon^2} + \frac{5}{48\epsilon} + \frac{71}{96} + \frac{1}{12}\pi^2 + \frac{13}{24}L_\mu + \frac{1}{8}L_\mu^2\right. \\
&\quad \left. + \left(\frac{581}{192} + \frac{97}{288}\pi^2 + \zeta_3 + \left(\frac{103}{48} + \frac{3}{16}\pi^2\right)L_\mu + \frac{17}{24}L_\mu^2 + \frac{1}{8}L_\mu^3\right)\epsilon\right] C_F T_F n_l \\
&\quad + \left[-\frac{1}{8\epsilon^2} + \frac{5}{48\epsilon} + \frac{143}{96} - \frac{1}{6}\pi^2 + \frac{13}{24}L_\mu + \frac{1}{8}L_\mu^2 + \left(\frac{1133}{192} - \frac{227}{288}\pi^2\right.\right. \\
&\quad \left.\left. + \pi^2\ln 2 - \frac{7}{2}\zeta_3 + \left(\frac{175}{48} - \frac{5}{16}\pi^2\right)L_\mu + \frac{17}{24}L_\mu^2 + \frac{1}{8}L_\mu^3\right)\epsilon\right] C_F T_F n_h\,, \qquad (14)
\end{aligned}
$$

where $L_\mu = \ln(\mu^2/M_q^2)$. ζ_n is Riemann's zeta function with integer argument n and $a_4 = \mathrm{Li}_4(1/2)$. We give the one- and two-loop results to order $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon)$, respectively. In general these terms are necessary for three-loop calculations.

The individual three-loop terms read

$$
\begin{aligned}
Z_m^{FFF} &= -\frac{9}{128\epsilon^3} - \left(\frac{63}{256} + \frac{27}{128}L_\mu\right)\frac{1}{\epsilon^2} + \left(-\frac{457}{512} + \frac{111}{512}\pi^2 - \frac{3}{8}\pi^2\ln 2\right. \\
&\quad \left. + \frac{9}{16}\zeta_3 - \frac{189}{256}L_\mu - \frac{81}{256}L_\mu^2\right)\frac{1}{\epsilon} - \frac{14225}{3072} - \frac{6037}{3072}\pi^2 + 5\pi^2\ln 2 \\
&\quad + \frac{5}{4}\pi^2\ln^2 2 + \frac{153}{128}\zeta_3 - \frac{73}{480}\pi^4 - \frac{1}{16}\pi^2\zeta_3 + \frac{5}{8}\zeta_5 - \frac{1}{8}\ln^4 2 - 3a_4 \\
&\quad + \left(-\frac{1371}{512} + \frac{333}{512}\pi^2 - \frac{9}{8}\pi^2\ln 2 + \frac{27}{16}\zeta_3\right)L_\mu - \frac{567}{512}L_\mu^2 - \frac{81}{256}L_\mu^3\,, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FFA} \;=\;& -\frac{33}{128\epsilon^3} + \left(\frac{49}{768} - \frac{33}{128}L_\mu\right)\frac{1}{\epsilon^2} + \left(\frac{3311}{1536} - \frac{43}{512}\pi^2 + \frac{3}{16}\pi^2\ln 2\right.\\
&\left. -\frac{9}{32}\zeta_3 + \frac{379}{256}L_\mu + \frac{33}{256}L_\mu^2\right)\frac{1}{\epsilon} + \frac{100247}{9216} + \frac{6545}{9216}\pi^2 + \frac{25}{36}\pi^2\ln 2\\
& -\frac{89}{72}\pi^2\ln^2 2 - \frac{3995}{384}\zeta_3 + \frac{1867}{8640}\pi^4 - \frac{19}{16}\pi^2\zeta_3 + \frac{45}{16}\zeta_5 - \frac{35}{144}\ln^4 2 - \frac{35}{6}a_4\\
& + \left(\frac{14311}{1536} - \frac{1135}{1536}\pi^2 + \frac{71}{48}\pi^2\ln 2 - \frac{71}{32}\zeta_3\right)L_\mu + \frac{1797}{512}L_\mu^2 + \frac{121}{256}L_\mu^3\,, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FAA} \;=\;& -\frac{121}{576\epsilon^3} + \frac{1679}{3456\epsilon^2} - \frac{11413}{20736\epsilon} - \frac{1322545}{124416} - \frac{1955}{3456}\pi^2 - \frac{115}{72}\pi^2\ln 2\\
& +\frac{11}{36}\pi^2\ln^2 2 + \frac{1343}{288}\zeta_3 - \frac{179}{3456}\pi^4 + \frac{51}{64}\pi^2\zeta_3 - \frac{65}{32}\zeta_5 + \frac{11}{72}\ln^4 2 + \frac{11}{3}a_4\\
& + \left(-\frac{13243}{1728} + \frac{11}{72}\pi^2 - \frac{11}{24}\pi^2\ln 2 + \frac{11}{16}\zeta_3\right)L_\mu - \frac{2341}{1152}L_\mu^2 - \frac{121}{576}L_\mu^3\,, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FFL} \;=\;& \frac{3}{32\epsilon^3} + \left(-\frac{5}{192} + \frac{3}{32}L_\mu\right)\frac{1}{\epsilon^2} - \left(\frac{65}{384} + \frac{7}{128}\pi^2 + \frac{1}{4}\zeta_3 + \frac{23}{64}L_\mu + \frac{3}{64}L_\mu^2\right)\frac{1}{\epsilon}\\
& +\frac{575}{2304} + \frac{1091}{2304}\pi^2 - \frac{11}{9}\pi^2\ln 2 + \frac{2}{9}\pi^2\ln^2 2 + \frac{145}{96}\zeta_3 - \frac{119}{2160}\pi^4 + \frac{1}{9}\ln^4 2\\
& +\frac{8}{3}a_4 - \left(\frac{497}{384} - \frac{5}{384}\pi^2 + \frac{1}{3}\pi^2\ln 2 + \frac{1}{4}\zeta_3\right)L_\mu - \frac{117}{128}L_\mu^2 - \frac{11}{64}L_\mu^3\,, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FAL} \;=\;& \frac{11}{72\epsilon^3} - \frac{121}{432\epsilon^2} + \left(\frac{139}{1296} + \frac{1}{4}\zeta_3\right)\frac{1}{\epsilon} + \frac{70763}{15552} + \frac{175}{432}\pi^2 + \frac{11}{18}\pi^2\ln 2\\
& -\frac{1}{9}\pi^2\ln^2 2 + \frac{89}{144}\zeta_3 + \frac{19}{2160}\pi^4 - \frac{1}{18}\ln^4 2 - \frac{4}{3}a_4\\
& + \left(\frac{869}{216} + \frac{7}{72}\pi^2 + \frac{1}{6}\pi^2\ln 2 + \frac{1}{2}\zeta_3\right)L_\mu + \frac{373}{288}L_\mu^2 + \frac{11}{72}L_\mu^3\,, \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FLL} \;=\;& -\frac{1}{36\epsilon^3} + \frac{5}{216\epsilon^2} + \frac{35}{1296\epsilon} - \frac{2353}{7776} - \frac{13}{108}\pi^2 - \frac{7}{18}\zeta_3\\
& - \left(\frac{89}{216} + \frac{1}{18}\pi^2\right)L_\mu - \frac{13}{72}L_\mu^2 - \frac{1}{36}L_\mu^3\,, \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
Z_m^{FHL} \;=\;& -\frac{1}{18\epsilon^3} + \frac{5}{108\epsilon^2} + \frac{35}{648\epsilon} - \frac{5917}{3888} + \frac{13}{108}\pi^2 + \frac{2}{9}\zeta_3\\
& - \left(\frac{143}{108} - \frac{1}{18}\pi^2\right)L_\mu - \frac{13}{36}L_\mu^2 - \frac{1}{18}L_\mu^3\,, \qquad (21)
\end{aligned}
$$

$$Z_m^{FFH} = \frac{3}{32\epsilon^3} + \left(-\frac{5}{192} + \frac{3}{32}L_\mu\right)\frac{1}{\epsilon^2} - \left(\frac{281}{384} - \frac{17}{128}\pi^2 + \frac{1}{4}\zeta_3 + \frac{23}{64}L_\mu + \frac{3}{64}L_\mu^2\right)\frac{1}{\epsilon}$$
$$-\frac{5257}{2304} - \frac{1327}{6912}\pi^2 + \frac{5}{36}\pi^2\ln 2 - \frac{1}{9}\pi^2\ln^2 2 + \frac{37}{96}\zeta_3 + \frac{91}{2160}\pi^4 + \frac{1}{9}\ln^4 2$$
$$+\frac{8}{3}a_4 - \left(\frac{1145}{384} - \frac{221}{384}\pi^2 + \frac{1}{3}\pi^2\ln 2 + \frac{1}{4}\zeta_3\right)L_\mu - \frac{117}{128}L_\mu^2 - \frac{11}{64}L_\mu^3\,, \tag{22}$$

$$Z_m^{FAH} = \frac{11}{72\epsilon^3} - \frac{121}{432\epsilon^2} + \left(\frac{139}{1296} + \frac{1}{4}\zeta_3\right)\frac{1}{\epsilon} + \frac{144959}{15552} - \frac{449}{144}\pi^2 + \frac{32}{9}\pi^2\ln 2$$
$$+\frac{1}{18}\pi^2\ln^2 2 - \frac{109}{144}\zeta_3 - \frac{43}{1080}\pi^4 + \frac{1}{8}\pi^2\zeta_3 - \frac{5}{8}\zeta_5 - \frac{1}{18}\ln^4 2 - \frac{4}{3}a_4$$
$$+\left(\frac{583}{108} - \frac{13}{36}\pi^2 + \frac{1}{6}\pi^2\ln 2 + \frac{1}{2}\zeta_3\right)L_\mu + \frac{373}{288}L_\mu^2 + \frac{11}{72}L_\mu^3\,, \tag{23}$$

$$Z_m^{FHH} = -\frac{1}{36\epsilon^3} + \frac{5}{216\epsilon^2} + \frac{35}{1296\epsilon} - \frac{9481}{7776} + \frac{4}{135}\pi^2 + \frac{11}{18}\zeta_3$$
$$-\left(\frac{197}{216} - \frac{1}{9}\pi^2\right)L_\mu - \frac{13}{72}L_\mu^2 - \frac{1}{36}L_\mu^3\,. \tag{24}$$

For the wave function renormalization constant, we have at the one-loop level $\delta Z_2^{(1)} = \delta Z_m^{(1)}$. The two-loop contribution to the wave function renormalization constant reads

$$\delta Z_2^{(2)} = \left[\frac{9}{32\epsilon^2} + \left(\frac{51}{64} + \frac{9}{16}L_\mu\right)\frac{1}{\epsilon} + \frac{433}{128} - \frac{49}{64}\pi^2 + \pi^2\ln 2 - \frac{3}{2}\zeta_3 + \frac{51}{32}L_\mu + \frac{9}{16}L_\mu^2\right.$$
$$+\left(\frac{211}{256} - \frac{339}{128}\pi^2 + \frac{23}{4}\pi^2\ln 2 - 2\pi^2\ln^2 2 - \frac{297}{16}\zeta_3 + \frac{7}{20}\pi^4 - \ln^4 2 - 24a_4\right.$$
$$\left.\left.+\left(\frac{433}{64} - \frac{49}{32}\pi^2 + 2\pi^2\ln 2 - 3\zeta_3\right)L_\mu + \frac{51}{32}L_\mu^2 + \frac{3}{8}L_\mu^3\right)\epsilon\right]C_F^2$$
$$+\left[\frac{11}{32\epsilon^2} - \frac{127}{192\epsilon} - \frac{1705}{384} + \frac{5}{16}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta_3 - \frac{215}{96}L_\mu - \frac{11}{32}L_\mu^2\right.$$
$$+\left(-\frac{9907}{768} + \frac{769}{1152}\pi^2 - \frac{23}{8}\pi^2\ln 2 + \pi^2\ln^2 2 + \frac{129}{16}\zeta_3 - \frac{7}{40}\pi^4 + \frac{1}{2}\ln^4 2 + 12a_4\right.$$
$$\left.\left.+\left(-\frac{2057}{192} + \frac{109}{192}\pi^2 - \pi^2\ln 2 + \frac{3}{2}\zeta_3\right)L_\mu - \frac{259}{96}L_\mu^2 - \frac{11}{32}L_\mu^3\right)\epsilon\right]C_AC_F$$
$$+\left[-\frac{1}{8\epsilon^2} + \frac{11}{48\epsilon} + \frac{113}{96} + \frac{1}{12}\pi^2 + \frac{19}{24}L_\mu + \frac{1}{8}L_\mu^2\right.$$
$$\left.+\left(\frac{851}{192} + \frac{127}{288}\pi^2 + \zeta_3 + \left(\frac{145}{48} + \frac{3}{16}\pi^2\right)L_\mu + \frac{23}{24}L_\mu^2 + \frac{1}{8}L_\mu^3\right)\epsilon\right]C_FT_Fn_l$$
$$+\left[\left(\frac{1}{16} + \frac{1}{4}L_\mu\right)\frac{1}{\epsilon} + \frac{947}{288} - \frac{5}{16}\pi^2 + \frac{11}{24}L_\mu + \frac{3}{8}L_\mu^2 + \left(\frac{17971}{1728} - \frac{445}{288}\pi^2\right.\right.$$
$$\left.\left.+2\pi^2\ln 2 - \frac{85}{12}\zeta_3 + \left(\frac{1043}{144} - \frac{29}{48}\pi^2\right)L_\mu + \frac{5}{8}L_\mu^2 + \frac{7}{24}L_\mu^3\right)\epsilon\right]C_FT_Fn_h\,. \tag{25}$$

Again, we give the result to order $\mathcal{O}(\epsilon)$, which is necessary for three-loop calculations. The individual three-loop terms are given by

$$
\begin{aligned}
Z_2^{FFF} &= -\frac{9}{128\epsilon^3} - \left(\frac{81}{256} + \frac{27}{128}L_\mu\right)\frac{1}{\epsilon^2} + \left(-\frac{1039}{512} + \frac{303}{512}\pi^2 - \frac{3}{4}\pi^2\ln 2\right. \\
&\quad \left. + \frac{9}{8}\zeta_3 - \frac{243}{256}L_\mu - \frac{81}{256}L_\mu^2\right)\frac{1}{\epsilon} - \frac{10823}{3072} - \frac{58321}{9216}\pi^2 + \frac{685}{48}\pi^2\ln 2 \\
&\quad + 3\pi^2\ln^2 2 - \frac{739}{128}\zeta_3 - \frac{41}{120}\pi^4 + \frac{1}{8}\pi^2\zeta_3 - \frac{5}{16}\zeta_5 - \frac{5}{12}\ln^4 2 - 10a_4 \\
&\quad + \left(-\frac{3117}{512} + \frac{909}{512}\pi^2 - \frac{9}{4}\pi^2\ln 2 + \frac{27}{8}\zeta_3\right)L_\mu - \frac{729}{512}L_\mu^2 - \frac{81}{256}L_\mu^3\,, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FFA} &= -\frac{33}{128\epsilon^3} + \left(\frac{95}{768} - \frac{33}{128}L_\mu\right)\frac{1}{\epsilon^2} + \left(\frac{1787}{512} - \frac{131}{512}\pi^2 + \frac{3}{8}\pi^2\ln 2\right. \\
&\quad \left. - \frac{5}{8}\zeta_3 + \frac{469}{256}L_\mu + \frac{33}{256}L_\mu^2\right)\frac{1}{\epsilon} + \frac{136945}{9216} + \frac{29695}{9216}\pi^2 - \frac{755}{288}\pi^2\ln 2 \\
&\quad - \frac{235}{72}\pi^2\ln^2 2 - \frac{6913}{384}\zeta_3 + \frac{1793}{3456}\pi^4 - \frac{45}{16}\pi^2\zeta_3 + \frac{145}{16}\zeta_5 - \frac{55}{144}\ln^4 2 - \frac{55}{6}a_4 \\
&\quad + \left(\frac{25609}{1536} - \frac{3335}{1536}\pi^2 + \frac{71}{24}\pi^2\ln 2 - \frac{37}{8}\zeta_3\right)L_\mu + \frac{2155}{512}L_\mu^2 + \frac{121}{256}L_\mu^3\,, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FAA} &= -\frac{121}{576\epsilon^3} + \frac{2009}{3456\epsilon^2} - \left[\frac{12793}{20736} + \frac{3}{128}\zeta_3 + \frac{1}{1080}\pi^4 + \left(\frac{1}{768} + \frac{3}{256}\zeta_3\right.\right. \\
&\quad \left.\left. - \frac{1}{4320}\pi^4\right)\xi\right]\frac{1}{\epsilon} - \frac{1654711}{124416} - \frac{4339}{3456}\pi^2 - \frac{325}{144}\pi^2\ln 2 + \frac{127}{144}\pi^2\ln^2 2 \\
&\quad + \frac{5857}{576}\zeta_3 - \frac{3419}{23040}\pi^4 + \frac{127}{72}\pi^2\zeta_3 - \frac{37}{6}\zeta_5 + \frac{85}{288}\ln^4 2 + \frac{85}{12}a_4 \\
&\quad + \left(-\frac{13}{768} - \frac{1}{256}\pi^2 - \frac{13}{256}\zeta_3 + \frac{17}{27648}\pi^4 + \frac{1}{144}\pi^2\zeta_3 + \frac{7}{384}\zeta_5\right)\xi \\
&\quad + \left[-\frac{36977}{3456} + \frac{55}{96}\pi^2 - \frac{11}{12}\pi^2\ln 2 + \frac{167}{128}\zeta_3 - \frac{1}{360}\pi^4\right. \\
&\quad \left. + \left(-\frac{1}{256} - \frac{9}{256}\zeta_3 + \frac{1}{1440}\pi^4\right)\xi\right]L_\mu - \frac{2671}{1152}L_\mu^2 - \frac{121}{576}L_\mu^3\,, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FFL} &= \frac{3}{32\epsilon^3} + \left(-\frac{19}{192} + \frac{3}{32}L_\mu\right)\frac{1}{\epsilon^2} - \left(\frac{235}{384} + \frac{7}{128}\pi^2 + \frac{1}{4}\zeta_3 + \frac{41}{64}L_\mu + \frac{3}{64}L_\mu^2\right)\frac{1}{\epsilon} \\
&\quad - \frac{3083}{2304} + \frac{2845}{2304}\pi^2 - \frac{47}{18}\pi^2\ln 2 + \frac{4}{9}\pi^2\ln^2 2 + \frac{473}{96}\zeta_3 - \frac{229}{2160}\pi^4 + \frac{2}{9}\ln^4 2 \\
&\quad + \frac{16}{3}a_4 + \left(-\frac{1475}{384} + \frac{133}{384}\pi^2 - \frac{2}{3}\pi^2\ln 2 + \frac{1}{4}\zeta_3\right)L_\mu - \frac{179}{128}L_\mu^2 - \frac{11}{64}L_\mu^3\,, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FAL} &= \frac{11}{72\epsilon^3} - \frac{169}{432\epsilon^2} + \left(\frac{313}{1296} + \frac{1}{4}\zeta_3\right)\frac{1}{\epsilon} + \frac{111791}{15552} + \frac{13}{48}\pi^2 + \frac{47}{36}\pi^2 \ln 2 \\
&\quad - \frac{2}{9}\pi^2 \ln^2 2 - \frac{35}{72}\zeta_3 + \frac{19}{1080}\pi^4 - \frac{1}{9}\ln^4 2 - \frac{8}{3}a_4 \\
&\quad + \left(\frac{169}{27} - \frac{1}{18}\pi^2 + \frac{1}{3}\pi^2 \ln 2 + \frac{1}{4}\zeta_3\right) L_\mu + \frac{469}{288} L_\mu^2 + \frac{11}{72} L_\mu^3 \,, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FLL} &= -\frac{1}{36\epsilon^3} + \frac{11}{216\epsilon^2} + \frac{5}{1296\epsilon} - \frac{5767}{7776} - \frac{19}{108}\pi^2 - \frac{7}{18}\zeta_3 \\
&\quad - \left(\frac{167}{216} + \frac{1}{18}\pi^2\right) L_\mu - \frac{19}{72} L_\mu^2 - \frac{1}{36} L_\mu^3 \,, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FHL} &= \left(\frac{1}{36} + \frac{1}{12} L_\mu\right)\frac{1}{\epsilon^2} + \left(-\frac{5}{216} + \frac{1}{144}\pi^2 - \frac{1}{9} L_\mu + \frac{1}{24} L_\mu^2\right)\frac{1}{\epsilon} \\
&\quad - \frac{4721}{1296} + \frac{19}{54}\pi^2 - \frac{1}{36}\zeta_3 + \left(-\frac{329}{108} + \frac{25}{144}\pi^2\right) L_\mu - \frac{7}{12} L_\mu^2 - \frac{5}{72} L_\mu^3 \,, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FFH} &= -\left(\frac{7}{192} + \frac{3}{16} L_\mu\right)\frac{1}{\epsilon^2} - \left(\frac{707}{384} - \frac{15}{64}\pi^2 + \frac{29}{64} L_\mu + \frac{15}{32} L_\mu^2\right)\frac{1}{\epsilon} \\
&\quad - \frac{76897}{6912} - \frac{11551}{20736}\pi^2 + \frac{7}{18}\pi^2 \ln 2 - \frac{1}{2}\pi^2 \ln^2 2 + \frac{1763}{288}\zeta_3 + \frac{31}{720}\pi^4 + \frac{1}{2}\ln^4 2 \\
&\quad + 12 a_4 + \left(-\frac{2891}{384} + \frac{233}{192}\pi^2 - \frac{2}{3}\pi^2 \ln 2 + \zeta_3\right) L_\mu - \frac{143}{128} L_\mu^2 - \frac{19}{32} L_\mu^3 \,, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FAH} &= \frac{1-\xi}{192\epsilon^3} - \left[\frac{7}{72} - \frac{1}{64}\xi + \left(\frac{41}{192} + \frac{1}{64}\xi\right) L_\mu\right]\frac{1}{\epsilon^2} + \left[\frac{13}{216} - \frac{41}{2304}\pi^2\right. \\
&\quad \left. - \left(\frac{35}{576} + \frac{1}{768}\pi^2\right)\xi + \left(\frac{83}{144} + \frac{3}{64}\xi\right) L_\mu - \left(\frac{35}{384} + \frac{3}{128}\xi\right) L_\mu^2\right]\frac{1}{\epsilon} \\
&\quad + \frac{49901}{2592} - \frac{36019}{5184}\pi^2 + \frac{80}{9}\pi^2 \ln 2 + \frac{1}{3}\pi^2 \ln^2 2 - \frac{77}{16}\zeta_3 - \frac{17}{360}\pi^4 + \frac{11}{48}\pi^2\zeta_3 \\
&\quad - \frac{15}{16}\zeta_5 - \frac{1}{3}\ln^4 2 - 8 a_4 + \left(\frac{407}{1728} + \frac{1}{256}\pi^2 - \frac{7}{192}\zeta_3\right)\xi \\
&\quad + \left[\frac{4141}{432} - \frac{641}{768}\pi^2 + \frac{1}{3}\pi^2 \ln 2 - \frac{1}{2}\zeta_3 - \left(\frac{35}{192} + \frac{1}{256}\pi^2\right)\xi\right] L_\mu \\
&\quad + \left(\frac{35}{16} + \frac{9}{128}\xi\right) L_\mu^2 + \left(\frac{247}{1152} - \frac{3}{128}\xi\right) L_\mu^3 \,, \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
Z_2^{FHH} &= \frac{1}{72\epsilon^2} - \left(\frac{5}{432} + \frac{1}{12} L_\mu^2\right)\frac{1}{\epsilon} - \frac{8425}{2592} + \frac{2}{45}\pi^2 + \frac{7}{3}\zeta_3 \\
&\quad - \left(\frac{481}{216} - \frac{5}{24}\pi^2\right) L_\mu - \frac{11}{72} L_\mu^2 - \frac{1}{6} L_\mu^3 \,. \qquad (35)
\end{aligned}
$$

Starting from the three-loop level, the wave function renormalization constant depends on the gauge parameter, ξ. The parameter in the above equations is defined through the gluon propagator as

$$D^{ab}_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} - \xi\,\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\,, \tag{36}$$

where a and b are colour indices.

We want to mention that our results for $Z_m^{\rm OS}$ and $Z_2^{\rm OS}$ agree with the literature [10, 11]. Whereas in [10, 11] they are expressed in terms of the bare coupling we decided to use the renormalized α_s as an expansion parameter which to our opinion is more convenient in practical applications.

The genuine three-loop integrals which appear in the `DREG` calculation are the same for `DRED`. The main complication is the more involved renormalization which is discussed in more detail in the next Section.

3 Renormalization in `DRED`

Let us in this Section collect the `DRED` counterterms needed for our calculation. In addition to the strong coupling α_s also the evanescent coupling[1] α_e has to be renormalized to two-loop order. The evanescent couplings[2] η_1, η_2 and η_3 appear for the first time at three-loop order and thus no renormalization is necessary. Both for the heavy quark mass, m_q, and the ε-scalar mass, m_ε, two-loop counterterms are necessary. Whereas the couplings are renormalized using minimal subtraction the masses are renormalized on-shell. The corresponding counterterms are defined through

$$\begin{aligned}
\alpha_s^{0,\overline{\rm DR}} &= \mu^{2\epsilon}\left(Z_s^{\overline{\rm DR}}\right)^2\alpha_s^{\overline{\rm DR}}\,, & \alpha_e^0 &= \mu^{2\epsilon}\left(Z_e\right)^2\alpha_e\,,\\
m_q^{0,\overline{\rm DR}} &= M_q Z_m^{{\rm OS},\overline{\rm DR}}\,, & \left(m_\varepsilon^0\right)^2 &= m_\varepsilon^2 Z_{m_\varepsilon}^{\rm OS}\,.
\end{aligned} \tag{37}$$

We attach an additional index "$\overline{\rm DR}$" to the quark mass renormalization constant in order to remind that it relates the pole mass to the bare mass in the `DRED` scheme[3] (in this context see also Ref. [30]).

Recently the quantities Z_e and $Z_s^{\overline{\rm DR}}$ have been computed to three- and four-loop order [4, 30], respectively. The results have been presented in terms of the corresponding β functions. For completeness we present in the following the two-loop results for the

[1] We refer to [4, 29] for a precise definition of the evanescent couplings.

[2] For the $SU(3)$ gauge group, which we exclusively consider in this paper, there are three independent such couplings [29].

[3] In principle such an index would also be necessary in Section 2. However, since the $\overline{\rm MS}$ scheme in connection with `DREG` constitutes the standard framework we refrain from introducing an additional index there.

renormalization constants

$$
\begin{aligned}
Z_s^{\overline{\rm DR}} &= 1+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{1}{\epsilon}\left(-\frac{11}{24}C_A+\frac{1}{6}T_Fn_f\right)+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{121}{384}C_A^2-\frac{11}{48}C_AT_Fn_f\right.\right.\\
&\quad\left.\left.+\frac{1}{24}n_f^2T_F^2\right)+\frac{1}{\epsilon}\left(-\frac{17}{96}C_A^2+\frac{5}{48}C_AT_Fn_f+\frac{1}{16}C_FT_Fn_f\right)\right], \qquad (38)\\
Z_e &= 1+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{1}{\epsilon}\left(-\frac{3}{4}C_F\right)+\frac{\alpha_e}{\pi}\frac{1}{\epsilon}\left(-\frac{1}{4}C_A+\frac{1}{2}C_F+\frac{1}{4}T_Fn_f\right)\\
&\quad+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{11}{32}C_AC_F+\frac{9}{32}C_F^2-\frac{1}{8}C_FT_Fn_f\right)+\frac{1}{\epsilon}\left(\frac{7}{256}C_A^2-\frac{55}{192}C_AC_F\right.\right.\\
&\quad\left.\left.-\frac{3}{64}C_F^2-\frac{1}{32}C_AT_Fn_f+\frac{5}{48}C_FT_Fn_f\right)\right]+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{\alpha_e}{\pi}\left[\frac{1}{\epsilon^2}\left(\frac{3}{8}C_AC_F-\frac{3}{4}C_F^2\right.\right.\\
&\quad\left.\left.-\frac{3}{8}C_FT_Fn_f\right)+\frac{1}{\epsilon}\left(\frac{3}{32}C_A^2-\frac{5}{8}C_AC_F+\frac{11}{16}C_F^2+\frac{5}{32}C_FT_Fn_f\right)\right]\\
&\quad+\left(\frac{\alpha_e}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{3}{32}C_A^2-\frac{3}{8}C_AC_F+\frac{3}{8}C_F^2-\frac{3}{16}C_AT_Fn_f+\frac{3}{8}C_FT_Fn_f+\frac{3}{32}T_F^2n_f^2\right)\right.\\
&\quad\left.+\frac{1}{\epsilon}\left(-\frac{3}{32}C_A^2+\frac{5}{16}C_AC_F-\frac{1}{4}C_F^2+\frac{3}{32}C_AT_Fn_f-\frac{3}{16}C_FT_Fn_f\right)\right]\\
&\quad+\frac{\alpha_e}{\pi}\frac{1}{\epsilon}\left(\frac{\eta_1}{\pi}\frac{9}{32}-\frac{\eta_2}{\pi}\frac{5}{16}-\frac{\eta_3}{\pi}\frac{3}{16}\right)-\left(\frac{\eta_1}{\pi}\right)^2\frac{1}{\epsilon}\frac{27}{256}+\left(\frac{\eta_2}{\pi}\right)^2\frac{1}{\epsilon}\frac{15}{16}+\frac{\eta_1}{\pi}\frac{\eta_3}{\pi}\frac{1}{\epsilon}\frac{9}{64}\\
&\quad-\left(\frac{\eta_3}{\pi}\right)^2\frac{1}{\epsilon}\frac{21}{128}\,. \qquad (39)
\end{aligned}
$$

Note that we set $N_c = 3$ in all terms containing the couplings η_i since our implementation of these couplings is only valid for $SU(3)$. We would also like to point out that the analytical form of $Z_s^{\overline{\rm DR}}$ is identical to the corresponding result in the $\overline{\rm MS}$ scheme. This has been shown by an explicit calculation in Ref. [3]. The one-loop result for Z_e can be found in Ref. [29].

The one-loop corrections to $Z_m^{\rm OS,\overline{DR}}$ can be found in Ref. [12] and the two-loop terms have been computed in Ref. [13]. For our calculation we also need the $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon)$ parts of the one- and two-loop terms, respectively.

In Section 4 we want to present the finite result obtained by considering the ratio $Z_m^{\rm OS,\overline{DR}}/Z_m^{\overline{\rm DR}}$. The quantity $Z_m^{\overline{\rm DR}}$ has been computed in Ref. [30] to three and in Ref. [4] even to four-loop order. Whereas in [4, 30] only the anomalous dimensions are given we want to present the explicit result for the renormalization constant

$$
\begin{aligned}
Z_m^{\overline{\rm DR}} \;=\;& 1+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{1}{\epsilon}\left(-\frac{3}{4}C_F\right)+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{11}{32}C_AC_F+\frac{9}{32}C_F^2-\frac{1}{8}C_FT_Fn_f\right)\right.\\
&\left.+\frac{1}{\epsilon}\left(-\frac{91}{192}C_AC_F-\frac{3}{64}C_F^2+\frac{5}{48}C_FT_Fn_f\right)\right]+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\frac{\alpha_e}{\pi}\left(\frac{3}{16}\frac{1}{\epsilon}C_F^2\right)\\
&+\left(\frac{\alpha_e}{\pi}\right)^2\frac{1}{\epsilon}\left(\frac{1}{16}C_AC_F-\frac{1}{8}C_F^2-\frac{1}{16}C_FT_Fn_f\right)+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^3\left[\frac{1}{\epsilon^3}\left(-\frac{121}{576}C_A^2C_F\right.\right.\\
&\left.-\frac{33}{128}C_AC_F^2-\frac{9}{128}C_F^3+\frac{11}{72}C_AC_FT_Fn_f+\frac{3}{32}C_F^2T_Fn_f-\frac{1}{36}C_FT_F^2n_f^2\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{1613}{3456}C_A^2C_F+\frac{295}{768}C_AC_F^2+\frac{9}{256}C_F^3-\frac{59}{216}C_AC_FT_Fn_f-\frac{29}{192}C_F^2T_Fn_f\right.\\
&\left.+\frac{5}{216}C_FT_F^2n_f^2\right)+\frac{1}{\epsilon}\left(-\frac{10255}{20736}C_A^2C_F+\frac{133}{768}C_AC_F^2-\frac{43}{128}C_F^3+\left(\frac{281}{2592}\right.\right.\\
&\left.\left.\left.+\frac{1}{4}\zeta_3\right)C_AC_FT_Fn_f+\left(\frac{23}{96}-\frac{1}{4}\zeta_3\right)C_F^2T_Fn_f+\frac{35}{1296}C_FT_F^2n_f^2\right)\right]\\
&+\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^2\frac{\alpha_e}{\pi}\left[\frac{1}{\epsilon^2}\left(-\frac{11}{192}C_AC_F^2-\frac{15}{64}C_F^3+\frac{1}{48}C_F^2T_Fn_f\right)+\frac{1}{\epsilon}\left(\frac{5}{256}C_A^2C_F\right.\right.\\
&\left.\left.+\frac{7}{32}C_AC_F^2+\frac{9}{64}C_F^3-\frac{3}{32}C_F^2T_Fn_f\right)\right]+\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\left(\frac{\alpha_e}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(-\frac{9}{64}C_AC_F^2+\frac{9}{32}C_F^3\right.\right.\\
&\left.+\frac{9}{64}C_F^2T_Fn_f\right)+\frac{1}{\epsilon}\left(-\frac{1}{64}C_A^2C_F+\frac{7}{32}C_AC_F^2-\frac{3}{8}C_F^3-\frac{1}{64}C_AC_FT_Fn_f\right.\\
&\left.\left.-\frac{1}{8}C_F^2T_Fn_f\right)\right]+\left(\frac{\alpha_e}{\pi}\right)^3\left[\frac{1}{\epsilon^2}\left(-\frac{1}{48}C_A^2C_F+\frac{1}{12}C_AC_F^2-\frac{1}{12}C_F^3\right.\right.\\
&\left.+\frac{1}{24}C_AC_FT_Fn_f-\frac{1}{12}C_F^2T_Fn_f-\frac{1}{48}C_FT_F^2n_f^2\right)+\frac{1}{\epsilon}\left(\frac{1}{32}C_A^2C_F-\frac{1}{8}C_AC_F^2+\frac{1}{8}C_F^3\right.\\
&\left.\left.-\frac{1}{24}C_AC_FT_Fn_f+\frac{5}{48}C_F^2T_Fn_f+\frac{1}{96}C_FT_F^2n_f^2\right)\right]-\frac{1}{8}\left(\frac{\alpha_e}{\pi}\right)^2\frac{\eta_1}{\pi}\frac{1}{\epsilon}\\
&+\frac{5}{36}\left(\frac{\alpha_e}{\pi}\right)^2\frac{\eta_2}{\pi}\frac{1}{\epsilon}+\frac{1}{12}\left(\frac{\alpha_e}{\pi}\right)^2\frac{\eta_3}{\pi}\frac{1}{\epsilon}+\frac{3}{64}\frac{\alpha_e}{\pi}\left(\frac{\eta_1}{\pi}\right)^2\frac{1}{\epsilon}-\frac{5}{12}\frac{\alpha_e}{\pi}\left(\frac{\eta_2}{\pi}\right)^2\frac{1}{\epsilon}\\
&+\frac{7}{96}\frac{\alpha_e}{\pi}\left(\frac{\eta_3}{\pi}\right)^2\frac{1}{\epsilon}-\frac{1}{16}\frac{\alpha_e}{\pi}\frac{\eta_1}{\pi}\frac{\eta_3}{\pi}\frac{1}{\epsilon}\,. \qquad (40)
\end{aligned}
$$

In order to achieve the finite result for the relation between the pole and the $\overline{\texttt{DR}}$ quark mass it is necessary to fix a renormalization scheme also for the mass of the ε scalar, m_ε. Although, there is in general no tree-level term in the Lagrangian, there are loop induced contributions to m_ε which require the introduction of corresponding counterterms. The relevant Feynman diagrams contributing to the ε-scalar propagator show quadratic

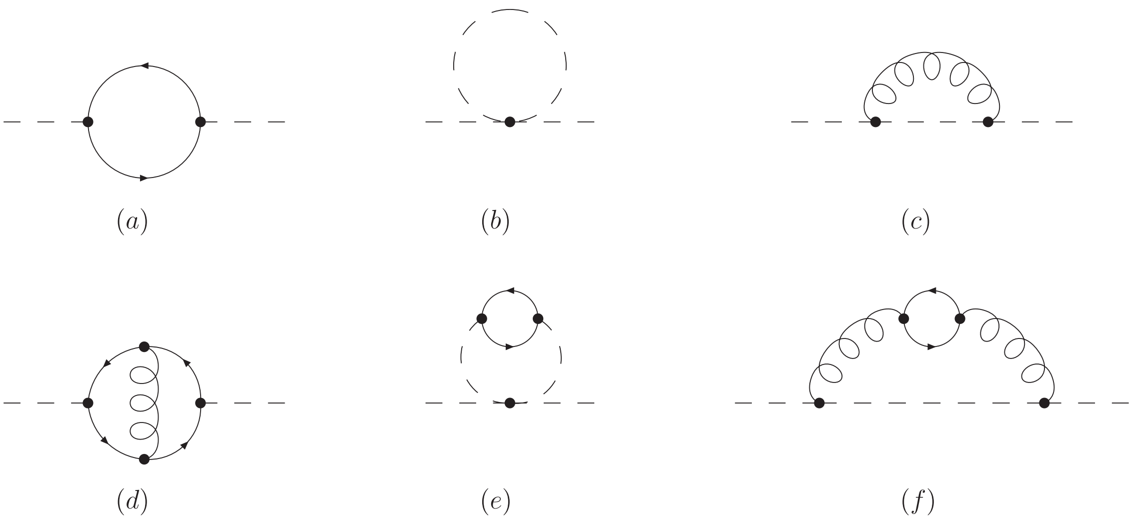

Figure 3: One- and two-loop Feynman diagrams contributing to the ε-scalar propagator. Dashed lines denote ε scalars, curly lines denote gluons and solid lines denote massive quarks with mass m_q.

divergences and therefore, one needs to consider only contributions from massive particles. Thus, in our case, only diagrams involving a massive quark have to be taken into account. Some sample diagrams are shown in Fig. 3.

It is common practice to renormalize m_ε on-shell and require that the renormalized mass is zero to each order in perturbation theory [31]. This scheme is known as the $\overline{\texttt{DR}}'$ scheme [31] and offers the advantage that the ε-scalar mass completely decouples from the physical observables. For supersymmetric theories the $\overline{\texttt{DR}}$ and $\overline{\texttt{DR}}'$ renormalization schemes are the same, while for theories with broken supersymmetry the latter one is most convenient. At one-loop order there is only one relevant diagram (cf. Fig. 3(a)) which has to be evaluated for vanishing external momentum. A closer look to the two-loop diagrams shows that they develop infra-red divergences in the limit $m_\varepsilon \to 0$ (cf., e.g., Fig. 3(e)). They can be regulated by introducing a small but non-vanishing mass for the ε scalars. After the subsequent application of an asymptotic expansion [32] in the limit $q^2 = m_\varepsilon^2 \ll M_q^2$ the infra-red divergences manifest themselves as $\ln(m_\varepsilon)$ terms. Furthermore, one-loop diagrams like the ones in Fig. 3(b) and (c) do not vanish anymore and have to be taken into account as well. Although they are proportional to m_ε^2, after renormalization they induce two-loop contributions which are proportional to M_q^2, partly multiplied by $\ln(m_\varepsilon)$ terms. It is interesting to note that in the sum of the genuine two-loop diagrams and the counterterm contributions the limit $m_\varepsilon \to 0$ can be taken which demonstrates the infra-red finiteness of the on-shell mass of the ε scalar.

Taking the infra-red finiteness for granted, it is also possible to choose $q^2 = m_\varepsilon^2 = 0$ from the very beginning. Then the individual diagrams are infra-red divergent, however, the sum is not. We have performed the calculation both ways and checked that the final result is the same. It is given by

$$
\begin{aligned}
\frac{M_q^2}{m_\varepsilon^2} Z_{m_\varepsilon}^{\rm OS} &= 1 - \frac{\alpha_e}{\pi} n_h T_F \left[\frac{2}{\epsilon} + 2 + 2L_\mu + \epsilon \left(2 + \frac{1}{6}\pi^2 + 2L_\mu + L_\mu^2 \right) \right] \\
& - \left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi} \right)^2 n_h T_F \left(\frac{3}{4}\frac{1}{\epsilon} + \frac{1}{4} + \frac{3}{2} L_\mu \right) C_A + \frac{\alpha_s^{\overline{\rm DR}}}{\pi} \frac{\alpha_e}{\pi} n_h T_F \left\{ \frac{1}{\epsilon^2} \left(\frac{3}{8} C_A + \frac{3}{2} C_F \right) \right. \\
& + \frac{1}{\epsilon} \left[\frac{7}{8} C_A + \frac{3}{2} C_F + \left(\frac{3}{4} C_A + \frac{3}{2} C_F \right) L_\mu \right] + \left(\frac{15}{8} + \frac{1}{16}\pi^2 \right) C_A \\
& \left. + \left(\frac{3}{2} + \frac{1}{8}\pi^2 \right) C_F + \left(\frac{7}{4} C_A + \frac{3}{2} C_F \right) L_\mu + \left(\frac{3}{4} C_A + \frac{3}{4} C_F \right) L_\mu^2 \right\} \\
& + \left(\frac{\alpha_e}{\pi} \right)^2 n_h T_F \left\{ \frac{1}{\epsilon^2} \left(\frac{1}{4} C_A - \frac{1}{2} C_F - \frac{1}{2} T_F n_f \right) + \frac{1}{\epsilon} \left[\frac{1}{2} C_F \right. \right. \\
& \left. - \frac{1}{2} \left(1 + L_\mu \right) T_F n_f \right] - \frac{1}{2} C_A + \frac{5}{2} C_F - \left(\frac{1}{2} + \frac{1}{24}\pi^2 \right) T_F n_f \\
& \left. - \left(\frac{1}{2} C_A - 2 C_F + \frac{1}{2} T_F n_f \right) L_\mu - \left(\frac{1}{4} C_A - \frac{1}{2} C_F + \frac{1}{4} T_F n_f \right) L_\mu^2 \right\} \\
& + \frac{\alpha_e}{\pi} \frac{\eta_1}{\pi} n_h \left[\frac{3}{16} \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\frac{3}{16} + \frac{3}{8} L_\mu \right) + \frac{3}{16} + \frac{1}{32}\pi^2 + \frac{3}{8} L_\mu + \frac{3}{8} L_\mu^2 \right] \\
& - \frac{\alpha_e}{\pi} \frac{\eta_2}{\pi} n_h \left[\frac{5}{4} \frac{1}{\epsilon^2} + \left(5 + \frac{5}{2} L_\mu \right) \frac{1}{\epsilon} + \frac{25}{2} + \frac{5}{24}\pi^2 + 10 L_\mu + \frac{5}{2} L_\mu^2 \right] \\
& - \frac{\alpha_e}{\pi} \frac{\eta_3}{\pi} n_h \left[\frac{7}{16} \frac{1}{\epsilon^2} + \left(\frac{7}{16} + \frac{7}{8} L_\mu \right) \frac{1}{\epsilon} + \frac{7}{16} + \frac{7}{96}\pi^2 + \frac{7}{8} L_\mu + \frac{7}{8} L_\mu^2 \right] ,
\end{aligned}
\tag{41}
$$

where the constants are defined after Eq. (14). The overall factor n_h in front of the one- and two-loop corrections shows that the renormalization of m_ε only influences those terms which contain a closed heavy quark loop.

4 $Z_m^{\rm OS}$ in dimensional reduction

The approach to extract the on-shell mass counterterm in `DRED` can be taken over from `DREG` as described in Section 2, i.e. one considers the inverse quark propagator and requires that it has a zero at the position of the pole. Again, the counterterm diagrams are generated order-by-order in a generic way.

The major complication as compared to the calculation in `DREG` is the appearance of the evanescent couplings and the ε scalars. In particular, there are three different four-ε vertices. This leads to many more Feynman diagrams which have to be considered. Whereas in the case of `DREG` about 130 diagrams contribute there are more than 1100 in

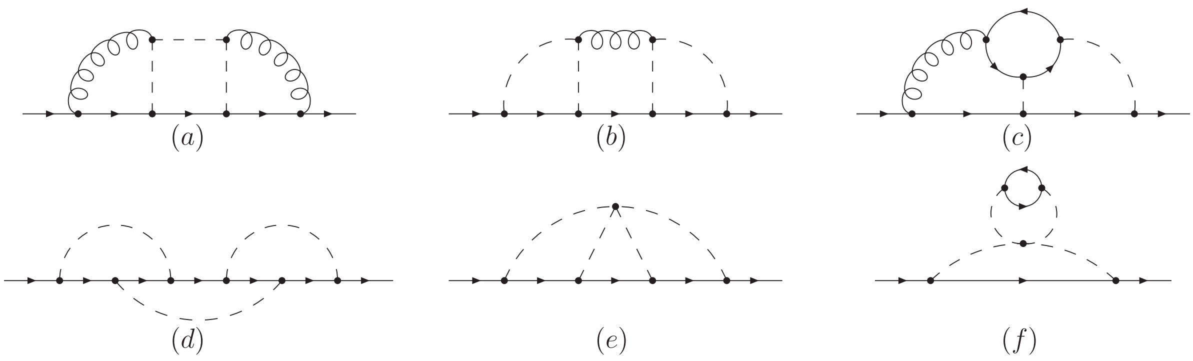

Figure 4: Sample three-loop diagrams contributing to the quark propagator which have to be considered additionally in case dimensional reduction is used for the regularization. Solid lines denote massive quarks with mass m_q, curly lines denote gluons and the ε scalars are represented by dashed lines. In the closed fermion loops all quark flavours have to be considered.

the case of DRED. Typical Feynman diagrams are shown in Fig. 1 and Fig. 4. The more involved renormalization has already been discussed in Section 3.

We avoid to present the result for the divergent quantity $Z_m^{\mathrm{OS},\overline{\mathrm{DR}}}$ but show the result for the ratio to the $\overline{\mathrm{DR}}$ quantity which is finite. We cast our result in the following form

$$z_m^{\mathrm{OS},\overline{\mathrm{DR}}} = \frac{Z_m^{\mathrm{OS},\overline{\mathrm{DR}}}}{Z_m^{\overline{\mathrm{DR}}}} = \frac{m_q^{\overline{\mathrm{DR}}}}{M_q^{\mathrm{OS}}} = 1 + \delta^{(1)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} + \delta^{(2)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} + \delta^{(3)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} . \quad (42)$$

Since our implementation of the couplings η_i is only valid for $SU(3)$ we furthermore set $N_c = 3$. Our one-, two- and three-loop results read:

$$\delta^{(1)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} = -\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\left(\frac{4}{3} + L_\mu\right) - \frac{1}{3}\frac{\alpha_e}{\pi} , \quad (43)$$

$$\begin{aligned}\delta^{(2)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} = {} & \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^2 \Bigg[-\frac{3143}{288} - \frac{2}{9}\pi^2 - \frac{1}{9}\pi^2 \ln 2 + \frac{1}{6}\zeta_3 - \frac{151}{24} L_\mu - \frac{7}{8} L_\mu^2 \\ & + \left(\frac{71}{144} + \frac{1}{18}\pi^2 + \frac{13}{36} L_\mu + \frac{1}{12} L_\mu^2\right) n_l + \left(\frac{143}{144} - \frac{1}{9}\pi^2 + \frac{13}{36} L_\mu \right. \\ & \left. + \frac{1}{12} L_\mu^2\right) n_h \Bigg] + \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\frac{\alpha_e}{\pi}\left(-\frac{3}{8} + \frac{1}{3} L_\mu\right) + \left(\frac{\alpha_e}{\pi}\right)^2 \left(\frac{1}{6} + \frac{1}{48} n_f\right) , \quad (44)\end{aligned}$$

$$
\begin{aligned}
\delta^{(3)} z_m^{\mathrm{OS},\overline{\mathrm{DR}}} &= \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^3 \Bigg\{ -\frac{1160387}{10368} - \frac{24707}{2592}\pi^2 - \frac{38}{9}\pi^2 \ln 2 + \frac{7}{27}\pi^2 \ln^2 2 + \frac{67}{72}\zeta_3 \\
&+ \frac{341}{2592}\pi^4 + \frac{1331}{432}\pi^2\zeta_3 - \frac{1705}{216}\zeta_5 + \frac{19}{54}\ln^4 2 + \frac{76}{9}a_4 - \bigg(\frac{20089}{288} + \pi^2 \\
&+ \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta_3\bigg) L_\mu - \frac{1475}{96} L_\mu^2 - \frac{21}{16} L_\mu^3 + \bigg[\frac{42235}{3888} + \frac{923}{648}\pi^2 + \frac{11}{81}\pi^2 \ln 2 \\
&- \frac{2}{81}\pi^2 \ln^2 2 + \frac{707}{216}\zeta_3 - \frac{61}{1944}\pi^4 - \frac{1}{81}\ln^4 2 - \frac{8}{27}a_4 + \bigg(\frac{3463}{432} + \frac{35}{108}\pi^2 \\
&+ \frac{1}{27}\pi^2 \ln 2 + \frac{7}{9}\zeta_3\bigg) L_\mu + \frac{35}{16} L_\mu^2 + \frac{2}{9} L_\mu^3\bigg] n_l - \bigg[\frac{2353}{23328} + \frac{13}{324}\pi^2 + \frac{7}{54}\zeta_3 \\
&+ \left(\frac{89}{648} + \frac{1}{54}\pi^2\right) L_\mu + \frac{13}{216} L_\mu^2 + \frac{1}{108} L_\mu^3\bigg] n_l^2 - \bigg[\frac{5917}{11664} - \frac{13}{324}\pi^2 - \frac{2}{27}\zeta_3 \\
&+ \left(\frac{143}{324} - \frac{1}{54}\pi^2\right) L_\mu + \frac{13}{108} L_\mu^2 + \frac{1}{54} L_\mu^3\bigg] n_l n_h + \bigg[\frac{77065}{3888} - \frac{13375}{1944}\pi^2 \\
&+ \frac{640}{81}\pi^2 \ln 2 + \frac{1}{81}\pi^2 \ln^2 2 - \frac{751}{216}\zeta_3 - \frac{41}{972}\pi^4 + \frac{1}{4}\pi^2\zeta_3 - \frac{5}{4}\zeta_5 - \frac{1}{81}\ln^4 2 \\
&- \frac{8}{27}a_4 + \left(\frac{4435}{432} - \frac{23}{54}\pi^2 + \frac{1}{27}\pi^2 \ln 2 + \frac{7}{9}\zeta_3\right) L_\mu + \frac{35}{16} L_\mu^2 + \frac{2}{9} L_\mu^3\bigg] n_h \\
&- \bigg[\frac{9481}{23328} - \frac{4}{405}\pi^2 - \frac{11}{54}\zeta_3 + \left(\frac{197}{648} - \frac{1}{27}\pi^2\right) L_\mu + \frac{13}{216} L_\mu^2 + \frac{1}{108} L_\mu^3\bigg] n_h^2 \Bigg\} \\
&+ \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^2 \frac{\alpha_e}{\pi} \bigg\{ \frac{41105}{20736} + \frac{2}{27}\pi^2 + \frac{1}{27}\pi^2 \ln 2 + \frac{7}{24}\zeta_3 + \frac{35}{12} L_\mu + \frac{7}{24} L_\mu^2 \\
&- \left[\frac{27}{64} + \frac{1}{54}\pi^2 + \frac{11}{54} L_\mu + \frac{1}{36} L_\mu^2\right] n_l - \left[\frac{113}{192} - \frac{1}{27}\pi^2 + \frac{11}{54} L_\mu + \frac{1}{36} L_\mu^2\right] n_h \bigg\} \\
&+ \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi} \left(\frac{\alpha_e}{\pi}\right)^2 \left\{ \frac{1397}{2592} - \frac{5}{36}\zeta_3 - \frac{1}{6} L_\mu + \left[\frac{55}{1728} + \frac{5}{36}\zeta_3 - \frac{1}{48} L_\mu\right] n_f \right\} \\
&+ \left(\frac{\alpha_e}{\pi}\right)^3 \left\{ -\frac{7}{144} - \frac{5}{216}\zeta_3 - \frac{31}{576} n_f + \frac{5}{576} n_f^2 \right\} - \frac{5}{24}\left(\frac{\alpha_e}{\pi}\right)^2 \frac{\eta_2}{\pi} \\
&- \frac{9}{256}\frac{\alpha_e}{\pi}\left(\frac{\eta_1}{\pi}\right)^2 + \frac{15}{16}\frac{\alpha_e}{\pi}\left(\frac{\eta_2}{\pi}\right)^2 - \frac{7}{128}\frac{\alpha_e}{\pi}\left(\frac{\eta_3}{\pi}\right)^2 + \frac{3}{64}\frac{\alpha_e}{\pi}\frac{\eta_1}{\pi}\frac{\eta_3}{\pi} \,. \qquad (45)
\end{aligned}
$$

For $\alpha_e = \alpha_s$ the two-loop result agrees with [13], while the three-loop one is new.

There is a very strong cross check of the results in Eqs. (43), (44) and (45) for the limit $n_h = 0$. The starting point is the relation between the $\overline{\mathtt{MS}}$ and the on-shell mass in `DREG` which can easily be obtained from the results of Section 2 and the $\overline{\mathtt{MS}}$ counterpart of Eq. (40) (see, e.g., Ref. [33]). In this relation, which depends on $\alpha_s^{\overline{\mathrm{MS}}}$, both $m^{\overline{\mathrm{MS}}}$ and $\alpha_s^{\overline{\mathrm{MS}}}$ are replaced by their `DRED` counterparts using Eqs. (4.2) and (4.3) of Ref. [4]. In this way we could verify the results for $z_m^{\overline{\mathrm{DR}},\mathrm{OS}}|_{n_h=0}$. This provides a strong consistency check both on the results presented in this paper but also on the approach used in [4] for the extraction of the conversion formulae between the $\overline{\texttt{MS}}$ and $\overline{\texttt{DR}}$ quantities.

In order to get an impression of the numerical size of the corrections, both in `DREG` and `DRED`, let us consider the relation between the minimally subtracted and the pole mass for the case of the bottom and top quark. As input we use $\alpha_s^{(5),\overline{\rm MS}}(M_Z) = 0.1189$ [34], $M_b = 4.800$ GeV and $M_t = 171.4$ GeV [35].

In the framework of `DREG` it is straightforward to convert $\alpha_s^{(5),\overline{\rm MS}}(M_Z)$ to $\alpha_s^{(5),\overline{\rm MS}}(M_b)$ and $\alpha_s^{(6),\overline{\rm MS}}(M_t)$ using four-loop accuracy[4] leading to

$$\begin{aligned}
\text{bottom}: \quad & m_b^{\overline{\rm MS}}(M_b) = 3.953 \text{ GeV} = 4.800(1 - 0.0929 - 0.0493 - 0.0342) \text{ GeV} \\
& \qquad = (4.800 - 0.445 - 0.236 - 0.164) \text{ GeV}\,, \qquad (46) \\
\text{top}: \quad & m_t^{\overline{\rm MS}}(M_t) = 161.1 \text{ GeV} = 171.4(1 - 0.0461 - 0.0109 - 0.0033) \text{ GeV} \\
& \qquad = (171.4 - 7.9 - 1.9 - 0.6) \text{ GeV}\,, \qquad (47)
\end{aligned}$$

with $\alpha_s^{(5),\overline{\rm MS}}(M_b) = 0.2188$ and $\alpha_s^{(6),\overline{\rm MS}}(M_t) = 0.1085$. The numbers given in the round brackets of the above equations indicate the contributions from the tree-level, one-, two- and three-loop conversion relation.

Within `DRED` the numerical analysis gets more involved since four more couplings appear whose values are needed for $\mu = M_b$ and $\mu = M_t$. As mentioned in the Introduction, `DRED` is an appropriate scheme for supersymmetric theories where in the strong sector only one coupling constant is present — like in usual QCD using `DREG`. Since all supersymmetric particles are heavier than the electroweak scale it is necessary to match the full theory to the Standard Model at some properly chosen scale $\mu^{\rm dec}$. At this step the additional couplings appear.

For the computation of $\alpha_e^{(5)}(M_b)$ and $\eta_i^{(5)}(M_b)$ we use $\mu^{\rm dec} = M_Z$, evaluate in a first step $\alpha_s^{(5),\overline{\rm DR}}(M_Z)$ using the three-loop relation given in Ref. [4] and require

$$\begin{aligned}
&\alpha_s^{(5),\overline{\rm DR}}(M_Z) = \alpha_e^{(5)}(M_Z) = \eta_1^{(5)}(M_Z)\,, \\
&\eta_2^{(5)}(M_Z) = \eta_3^{(5)}(M_Z) = 0\,. \qquad (48)
\end{aligned}$$

In a second step the renormalization group functions, which are known to four ($\alpha_s^{\overline{\rm DR}}$), three ($\alpha_e$) and one-loop order ($\eta_i$) [4,30] are used to obtain

$$\begin{aligned}
\alpha_s^{(5),\overline{\rm DR}}(M_b) &= 0.2241\,, \\
\alpha_e^{(5)}(M_b) &= 0.1721\,, \\
\eta_1^{(5)}(M_b) &= 0.2152\,, \\
\eta_2^{(5)}(M_b) &= -0.01798\,, \\
\eta_3^{(5)}(M_b) &= -0.005777\,. \qquad (49)
\end{aligned}$$

[4] We use `RunDec` [36] for the running and decoupling of α_s in the $\overline{\rm MS}$ scheme.

In the case of the top quark we choose $\mu^{\rm dec} = M_t$. It is necessary to know the couplings for six active flavours and thus we evaluate in a first step $\alpha_s^{(6),\overline{\rm MS}}(M_Z) = 0.1178$ and pose the analogue requirements as in Eq. (48) with "(5)" replaced by "(6)" and M_Z by M_t. This leads to

$$
\begin{aligned}
&\alpha_s^{(6),\overline{\rm DR}}(M_t) = \alpha_e^{(6)}(M_t) = \eta_1^{(6)}(M_t) = 0.1096\,, \\
&\eta_2^{(6)}(M_t) = \eta_3^{(6)}(M_t) = 0\,. \qquad (50)
\end{aligned}
$$

Finally, we can insert the results of Eqs. (49) and (50) into Eq. (42) and obtain

$$
\begin{aligned}
\text{bottom}: \quad m_b^{\overline{\rm DR}}(M_b) = 3.859 \text{ GeV} &= 4.800(1 - 0.1134 - 0.0495 - 0.033) \text{ GeV} \\
&= (4.800 - 0.544 - 0.238 - 0.159) \text{ GeV}\,, \qquad (51) \\
\text{top}: \quad m_t^{\overline{\rm DR}}(M_t) = 159.0 \text{ GeV} &= 171.4(1 - 0.0581 - 0.0105 - 0.0030) \text{ GeV} \\
&= (171.4 - 10.0 - 1.8 - 0.5) \text{ GeV}\,. \qquad (52)
\end{aligned}
$$

The perturbative expansion shows a similar behaviour as for the $\overline{\rm MS}$–on-shell mass relation: the three-loop terms amount to 159 MeV and 500 MeV, respectively, and are thus far above the current uncertainty for the bottom quark mass [37] and much larger than the expected uncertainty for the top quark mass [38].

5 Conclusions

The main result of this paper is the relation between the pole and $\overline{\texttt{DR}}$ quark mass to three-loop order in QCD where dimensional reduction has been used as a regularization scheme. The conversion formula has been obtained in analytical form where all occurring integrals have been reduced to a small set of master integrals with the help of the Laporta algorithm. Due to the occurrence of evanescent couplings when using `DRED` within QCD it is more advantageous to use in this case dimensional regularization. However, the latter cannot be used in supersymmetric models. Thus, our result constitutes an important preparation for similar calculations in supersymmetric extensions of the Standard Model.

As a by-product we have obtained the corresponding relation and the on-shell wave function renormalization using dimensional regularization. This constitutes the first check of the analytical results obtained about seven years ago. Furthermore, we can confirm the observation of Ref. [11] that $Z_2^{\rm OS}$ depends on the gauge fixing parameter starting from three-loop order.

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