FOUR-LOOP TADPOLES AND PRECISION CALCULATIONS IN PERTURBATIVE QCD

JOHANN H. KÜHN*

Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany

Recent results on four-loop tadpoles amplitudes and their implications for precision calculations are presented.

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1. Introduction

During the first and partly even the second decade after QCD had been proposed as the theory of strong interactions, a number of tests were suggested to probe the qualitative features of this novel theory. In recent years the emphasis has shifted to different topics: on one hand the precise determination of the fundamental parameters of the theory, in particular the strong coupling constant α_s and the quark masses, on the other hand to the prediction of quantities that enter observables of interest, with the top quark contribution to the ρ parameter or lepton spectra and rates for B-meson decays as characterisic examples. This second application then allows to determine the parameters of the electroweak theory, e.g. the elements of the CKM matrix or, indirectly, the mass of the Higgs bosons, or to even find indications for a theory beyond the Standard Model. Most of these investigations are based on the perturbative treatment of QCD, although interesting and successful cross checks and predictions have been made within lattice gauge theory.

As a consequence of the relatively large value of the strong coupling, and to match the increasing experimental precision, the inclusion of quantum corrections is a must for any meaningful evaluation of the forementioned fundamental parameters. The evaluation of Feynman amplitudes, corresponding to diagrams with several external legs, with several mass- and energy-scales and more than one loop is difficult, often impossible. In practice, however, many physical problems exhibit a drastic hierarchy of scales, with the quark masses $m_t^2 \gg m_b^2 \gg m_c^2 \gg m_s^2 \gg m_{u,d}^2$ as most prominent examples. Frequently only one of them can be considered as non-vanishing, the effects of the other ones can be considered through appropriate expansions. Furthermore, the characteristic energy scale may often be assumed to be significantly smaller or larger than the quark mass, an assumption leading to further drastic simplifications. In the high energy limit quark masses are neglected or again taken into account through expansions. In the low energy limit, in contrast, the external momentum flow may be ignored, and one arrives at vacuum, also denoted tadpole diagrams. As long, as only one quark mass m_Q is kept non-vanishing, the scale of the corresponding amplitude can be factored out and the calculation is reduced to the evaluation of a dimensionless integral.

In many cases the approximation of small external momentum, $q^2 \ll m_Q^2$, can be ameliorated by considering a Taylor series in q^2/m_Q^2. The Taylor coefficients can again be expressed by tadpole diagrams, and often an excellent approximation to the complete result is constructed[1] from the low- and the high-energy behaviour. For the correlator of two vector currents, the vacuum polarization, this has been acchieved in three-loop approximation about ten years ago and similar results are available for axial vector and scalar correlators. Important applications along these lines were the evaluation of the R-ratio (with $R \equiv \sigma(e^+e^- \to had)/\sigma_{pt}$) for the production of massive quarks, the Higgs boson decay rate into massive quarks, the coupling of the Higgs boson to gluons and other observables of interest[2]. The Taylor coefficients of the vacuum polarization are closely related to moments of the R-ratio for the production of heavy quarks, and this correspondence has lead to a highly precise evaluation of the charm and bottom quark mass[3]. In view of the importance and the success of this approach a vigorous research program has been initiated during the last years towards the evaluation of tadpole amplitudes in four-loop approximation. A variety of results with important physical applications has been presented during the past months which will be described below.

2. Calculational Techniques

From the technical side all these calculations are based on the the traditional Integration-by-Parts method in combination with the Laporta algorithm[4], which allows to reduce the huge number of Feynman integrals algebraically to a few master integrals through an automated FORM3 program[5]. The master integrals are then evaluated either analytically or numerically with high precision. Difference equations[6] and, alternatively, numerical methods based on a Padé approximation for three-loop propagator integrals[7] lead to a precision of thirty or seven digits, respectively.(For a detailled discussion and a list of references see[8].)

In this context it is useful to select a set of master integrals whose coefficient functions are finite in the limit $\epsilon \to 0$. This choice of an "epsilon-finite basis" [8] gives important insights into the structure of the master integrals. In particular it leads directly to analytic expressions for those terms which cancel algebraically through renormalization and is particularly suited for the numerical evaluation through the formentioned Padé method. Let us now describe the results in more detail.

3. Heavy quark decoupling and matching conditions

According to the Appelquist-Carazzone theorem, effects due to heavy particles should eventually "decouple" from the low-energy physics. However, a peculiarity of mass-independent renormalization schemes is that the decoupling theorem does not hold in its naive form in theories renormalized in this framework: In general potentially large mass logarithms appear, when one calculates a physical observable.

The well-known way to bring these logarithms under control is to construct an effective field theory by "integrating out" the heavy quark field h. For QCD with $n_l = n_f - 1$ light and one heavy quark h the coupling constants in both theories, the full n_f-flavor QCD and the effective n_l-flavor one, $\alpha_s^{(n_f)}$ and $\alpha_s^{(n_l)}$ are then related by the so-called matching condition

$$\alpha_s^{(n_l)}(\mu) = \zeta_g^2(\mu, \alpha_s^{(n_f)}(\mu), m)\ \alpha_s^{(n_f)}(\mu). \quad (1)$$

Here $m = m(\mu)$ is the (running) mass of the heavy quark and ζ_g the decoupling function is now available in four-loop approximation[9,10].

The bare decoupling constants can all be expressed in terms of massive Feynman integrals without any external momenta[11], namely the gluon and ghost vacuum polarization functions and the ghost-gluon vertex function, all at vanishing external momenta. Inclusion of the additional term leads to a minute shift of $\alpha_s(M_Z)$ by 0.001 when running from m_τ to M_Z and a slight reduction of the theory error.

4. Coupling of the Higgs boson to gluons

With the SM Higgs boson mass below 200 GeV, its ggH coupling is essentially generated by the top quark only and becomes independent of the top quark mass M_t in the limit $M_H \ll 2M_t$. Then the situation may be greatly simplified[11] by constructing a heavy-top-quark effective Lagrangian, which is a linear combination of certain dimension-four operators acting in QCD with five quark flavors and an effective coupling C_1, directly related to the logarithmic derivative of ζ_g w.r.t. m. Now, armed with the newly computed four-loop term in ζ_g, the old result[11] for C_1 (earlier obtained indirectly) was checked. In fact, one could even construct C_1 at five loops in terms of the known four-loop QCD decoupling function, the quark anomalous dimension γ_m and the β-function and the only yet unknown parameter: the n_f-dependent piece of the five-loop contribution to the β-function.

5. Taylor coefficients of the vacuum polarization in $\mathcal{O}(\alpha_s^3)$

The vacuum polarization function $\Pi(q^2)$ can be linked to moments $\mathcal{M}_n$ with the help of dispersion relations:

$$\mathcal{M}_n^{\rm exp} = \int ds \frac{R(s)}{s^{n+1}} = \frac{12\pi^2}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2)\big|_{q^2=0}, \qquad (2)$$

$\mathcal{M}_n^{\rm exp}$ depends on the experimental measurable R-ratio, the derivatives of the vacuum polarization function can be calculated in perturbative QCD. This method can be used to determine charm- and bottom-quark masses. Taking into account moments up to three-loop order[1], precise results for both quark masses have been obtained[3]. In view of the present and foreseeable precision the four-loop contributions are of phenomenological importance and have been evaluated[12,13] for the two lowest terms of the expansion. The uncertainties from the variation of the renormalization scale are thus significantly reduced. For the lowest moment, however, the experimental uncertainties are still dominant.

A preliminary analysis[14] based on the lowest three moments which includes the latest experimental information and an estimate of the still unknown four-loop terms of $\mathcal{M}_2$ and $\mathcal{M}_3$ leads to

$$\overline{m}_c(3\text{GeV}) = 1007 \pm 16\text{MeV}, \qquad (3)$$

$$\overline{m}_b(10\text{GeV}) = 3630 \pm 35\text{MeV}. \qquad (4)$$

and thus to a significant reduction of the error.

6. Four-loop QCD corrections to the ρ parameter

A group of dominant radiative corrections can be absorbed in the shift of the ρ parameter from its lowest order value $\rho_{\rm Born} = 1$. The result for the one-loop approximation $\delta\rho = 3\,x_t = 3\,G_F m_t^2/(8\sqrt{2}\,\pi^2)$, hence quadratic in m_t, was first obtained by Veltman and used to establish a limit on the mass splitting within one fermion doublet. In order to make full use of the present experimental precision, this one-loop calculation was improved by two-loop and even three-loop QCD corrections. Also important are two-loop and three-loop electroweak effects proportional to x_t^2 and x_t^3, respectively, and the three-loop mixed corrections[15] of order $\alpha_s x_t^2$.

For fixed pole mass of the top quark, the three-loop result leads to a shift of about 10 MeV in the mass of the W-boson. (This applies both to the pure QCD corrections[16] and the mixed QCD-electroweak ones[15].) Conversely, the corresponding shift of the top quark pole mass amounts to 1.5 GeV. These values are comparable to the experimental precision anticipated for top- and W-mass measurements at the International Linear Collider. An improvement of the theoretical accuracy seems, therefore, desirable. The W-

self-energy receives contributions from the correlator of the "non-diagonal" t-b-current only. Contributions to the Z self-energy originate only from the "diagonal" axial current correlator induced by top quark loops. The nonvanishing parts of $\Pi_T^Z(0)$ are conveniently decomposed into non-singlet and singlet pieces characterized by Feynman diagrams where the external current couples to the same and to two different closed fermion lines, respectively. In three-loop approximation the singlet-piece is larger than the non-singlet piece by nearly a factor twenty. This has motivated the authors of[17] to evaluate, in a first step, the four-loop singlet piece.

To complete the evaluation of the four-loop QCD corrections to the ρ parameter, $\Pi_T^W(0)$ and the non-singlet parts of $\Pi_T^Z(0)$ are required. The result for the shift in the ρ parameter can be cast into the following form[18,19]:

$$\delta\rho^{\overline{\rm MS}} = 3x_t \sum_{i=0}^{3} \left(\frac{\alpha_s}{\pi}\right)^i \delta\rho_i^{\overline{\rm MS}} \tag{5}$$

For m_t defined in the $\overline{\rm MS}$-scheme one finds[8,19]

$$\begin{aligned}\delta\rho_3^{\overline{\rm MS}} &= \delta\rho_3^{\overline{\rm MS}}(\text{singlet}) + \delta\rho_3^{\overline{\rm MS}}(\text{non-singlet}) \\ &= -3.2866 + 1.6067 = -1.6799.\end{aligned} \tag{6}$$

If expressed in terms of the pole mass, a major shift originates from the large correction in the pole-$\overline{\rm MS}$ relation: $\delta\rho_3^{pole} = -93.1501$. For fixed top mass, this corresponds to shift of around 2 MeV in the W-boson mass. The result based on the three-loop calculation is thus stabilized.

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