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Standard and ϵ-finite Master Integrals for the ρ-Parameter

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Abstract

We have constructed an ϵ-finite basis of master integrals for all new types of one-scale tadpoles which appear in the calculation of the four-loop QCD corrections to the electroweak ρ-parameter. Using transformation rules from the ϵ-finite basis to the standard "minimal-number-of-lines" basis, we obtain as a by-product analytical expressions for few leading terms of the ϵ-expansion of all members of the standard basis. The new master integrals have been computed with the help of the Padé method and by use of difference equations independently.

1 Introduction

In many multi-loop calculations the standard method to calculate physical observables is to use the traditional integration-by-parts (IBP) method in combination with Laporta's algorithm [1, 2] in order to reduce all appearing integrals to a small set of master integrals. Once this reduction is completed one is left with the calculation of the master integrals, addressed in this work.
One physical quantity which has recently been evaluated applying these methods is the contribution from top- and bottom-quarks to the ρ-parameter at four-loop order in perturbative QCD [3]. A subsequent independent evaluation of the same quantity has been performed in ref. [4]. This completes the partial result from the so-called singlet term already determined in ref. [5]. For the ρ-parameter, the result can be expressed in terms of 63 four-loop tadpole master integrals. A subset of 13 master integrals was already required in earlier four-loop calculations, like the determination of the matching condition for the strong coupling constant α_s at a heavy quark threshold in the modified minimal subtraction scheme [6, 7] and the evaluation of the two lowest terms of the Taylor expansion of the vacuum polarization function [8, 9]. This subset of master integrals has been evaluated in ref. [10] with high precision using the method of difference equations [1, 2, 11–13]. *All* results relevant for the four-loop calculation of ref. [10] have been confirmed with a completely different method (see below) in ref. [14]. Some of these master integrals have also been found in refs. [5, 13, 15–21].
A different method, based on the idea of the ϵ-finite basis, has been suggested in ref. [14]. In this approach one avoids so-called spurious poles in the coefficient functions, which multiply the master integrals. These poles may arise in general while solving the linear system of IBP equations, when a division by $\epsilon = (4-d)/2$ occurs. Master integrals which have a spurious pole as coefficient need to be evaluated deeper in the ϵ-expansion. The calculation of each additional order in this expansion of a master integral is in general increasingly tedious. In contrast, the approach of the ϵ-finite basis exploits the freedom in the choice of master integrals in order to select a basis of master integrals in such a way that the coefficients, being functions in the space time dimension d, are finite in the limit $\epsilon \to 0$. In ref. [14] the ϵ-finite basis, shown in fig. 1, has been constructed for the subset of 13 master integrals discussed above.

All of them have been calculated semi-analytically. For the problem of the four-loop QCD corrections to the ρ-parameter an ϵ-finite basis with more elements is required which is constructed in the following. The corresponding master integrals are particularly suited for the evaluation by a semi-numerical method based on Padé approximations [14, 22, 23]. Most of the results are checked independently by the method of difference equations and have been used for the

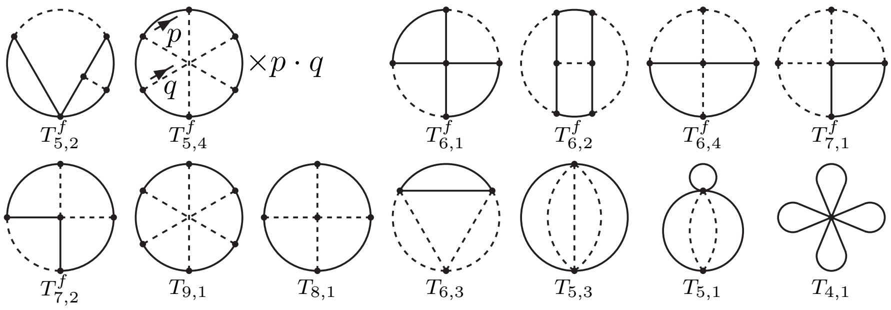


Figure 1: Diagrams of the subset of 13 master integrals of the ϵ-finite basis addressed in ref. [14]. Note that the last four diagrams are known completely analytically and are not considered in the construction of the ϵ-finite basis.

evaluation of the ρ-parameter in ref. [3]. They are in full agreement with those recently published in [4].

This paper is structured as follows. In the next section we present the results for the master integrals in the ϵ-finite basis, calculated by the Padé method. Then, in section 3, we give results for the master integrals in the standard basis evaluated by difference equations. In this case we also include the analytical information obtained in section 2. Our summary and conclusions are given in section 4.

2 The master integrals in the ϵ-finite basis

For the construction of the ϵ-finite basis we follow the algorithm described in reference [14]. In the evaluation of the ρ-parameter we have again a vast choice in defining the ϵ-finite basis with elements suitable for the Padé method. We exclude 14 diagrams from the construction of the ϵ-finite basis, which are known to high orders in ϵ analytically or even completely analytically. Four of them are depicted in fig. 1 ($T_{4,1}, T_{5,1}, T_{5,3}, T_{6,3}$), the remaining ones are listed in appendix A.

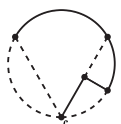
The application of the method [14, 22, 23] to the actual calculation of the ϵ-finite integrals is straightforward. The only adjustment necessary arises from the appearance of massless cuts inherent in many of the relevant diagrams. In addition to the function used to subtract the high energy logarithms, a second function was introduced to subtract the logarithms appearing in the calculation of diagrams with a massless cut in the low energy limit. This approach was also verified by recalculating the diagrams presented in fig. 1 (e.g. $T^f_{6,4}$); this time performing the cut in such a way that a self energy diagram with a massless cut arises. Full agreement between the two approaches was observed. Following this procedure we again calculate all divergent parts of the diagrams analytically, and the constant part in the limit $\epsilon \to 0$ numerically. All the digits of our numerical results in eqs. (3)-(20) are valid digits. The analytical information of the expansion then allows to study the cancellation of all divergences in the physical problem during the renormalization procedure analytically. Just as in the problem of the diagonal current, some diagrams are simultaneously members of the standard and the ϵ-finite basis. These integrals are discussed in section 3. Now we give the results for the integrals which are different in the two bases, i.e. the replaced integrals due to the existence of a spurious pole. As in ref. [14] the name of the ϵ-finite master integral is deduced by using the name of the original master integral, which has been replaced, and the additional superscript letter "f". The loop integrals are defined as follows,

$$J = \int [dk_1][dk_2][dk_3][dk_4] \frac{1}{D_1 D_2 \dots D_{N_d}}\,, \tag{1}$$

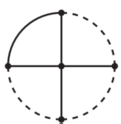
with N_d denominators $D_j = (p_j^2 - m_j^2)$. The momenta p_j are defined by the diagrams shown below as a linear combination of the four loop momenta k_i. For briefness we set the single mass scale m of the integrals in the following $m = 1$, thus $m_j \in \{0, 1\}$. Dashed(solid) lines in all the diagrams denote massless(massive) propagators. The integration measure is given by $[dk_j] = e^{\epsilon\,\gamma_E}/(i\,\pi^{d/2})\, d^d k_j$, where $\gamma_E = 0.577216\dots$ is the Euler's constant. In this conventions the one-loop tadpole reads:

$$T_1 = -e^{\epsilon\,\gamma_E}\,\Gamma(1-d/2) = \frac{1}{\epsilon} + 1 + \epsilon\left(1 + \frac{\pi^2}{12}\right) + \epsilon^2\left(1 + \frac{\pi^2}{12} - \frac{\zeta_3}{3}\right) + \mathcal{O}(\epsilon^3)\,. \tag{2}$$

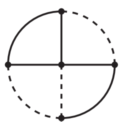
The final expressions for the remaining master integrals in the ϵ-finite basis read:


$T^f_{5,8}$

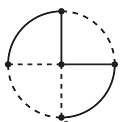
$$= \frac{3\,\zeta_3}{2\,\epsilon^2} - \frac{1}{\epsilon}\Big(\frac{91}{360}\,\pi^4 - \frac{15}{2}\,\zeta_3 - \frac{4}{3}\,\log^4(2) + \frac{4}{3}\,\pi^2\,\log^2(2) - 32\,\mathrm{Li}_4\left(\tfrac{1}{2}\right)\Big) + 43.4562683 + \mathcal{O}(\epsilon)\,, \tag{3}$$


$T^f_{6,14}$

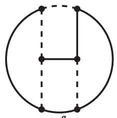
$$= \frac{5\,\zeta_5}{\epsilon} - 23.2862341 + \mathcal{O}(\epsilon)\,, \tag{4}$$


$T^f_{7,15}$

$$= \frac{5\,\zeta_5}{\epsilon} - 24.8172810 + \mathcal{O}(\epsilon)\,, \tag{5}$$

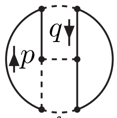

$T^f_{7,6}$

$$= \frac{5\,\zeta_5}{\epsilon} - 18.9284641 + \mathcal{O}(\epsilon)\,, \tag{6}$$



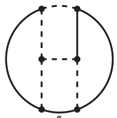
$T^f_{6,9}$

$$= -3.2095107 + \mathcal{O}(\epsilon)\,, \tag{7}$$

$p \cdot q \times$ 

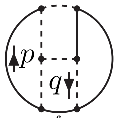
$T^f_{6,12}$

$$= \frac{1}{2\,\epsilon}\,(5\,\zeta_5 - 3\,\zeta_3) - 3.67204 + \mathcal{O}(\epsilon)\,, \tag{8}$$



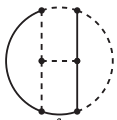
$T^f_{6,13}$

$$= -5.1732238 + \mathcal{O}(\epsilon)\,, \tag{9}$$

$p \cdot q \times$ 

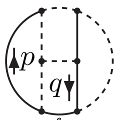
$T^f_{6,18}$

$$= \frac{1}{2\,\epsilon}\,(5\,\zeta_5 - 3\,\zeta_3) - 1.589373 + \mathcal{O}(\epsilon)\,, \tag{10}$$



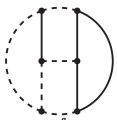
$T^f_{7,11}$

$$= -5.126697 + \mathcal{O}(\epsilon)\,, \tag{11}$$

$p \cdot q \times$ 

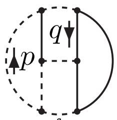
$T^f_{7,14}$

$$= \frac{1}{2\,\epsilon}\,(5\,\zeta_5 - 3\,\zeta_3) + 0.0114944 + \mathcal{O}(\epsilon)\,, \tag{12}$$



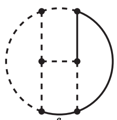
$T^f_{5,9}$

$$= -6.8481671 + \mathcal{O}(\epsilon)\,, \tag{13}$$

$p \cdot q \times$ 

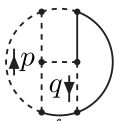
$T^f_{5,10}$

$$= \frac{1}{2\,\epsilon}\,(5\,\zeta_5 - 3\,\zeta_3) - 3.272794 + \mathcal{O}(\epsilon)\,, \tag{14}$$



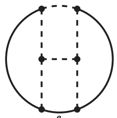
$T^f_{6,16}$

$$= -9.5955369 + \mathcal{O}(\epsilon)\,, \tag{15}$$

$p \cdot q \times$ 

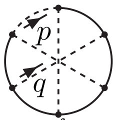
$T^f_{6,19}$

$$= \frac{1}{2\,\epsilon}\,(5\,\zeta_5 - 3\,\zeta_3) - 0.80223 + \mathcal{O}(\epsilon)\,, \tag{16}$$



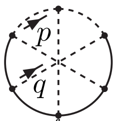
$T^f_{6,15}$

$$= -8.816973 + \mathcal{O}(\epsilon)\,, \tag{17}$$

$p \cdot q \,\times$ 

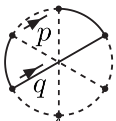
$T^f_{9,4}$

$$= -\frac{5\,\zeta_5}{4\,\epsilon} + 6.2801043 + \mathcal{O}(\epsilon)\,, \tag{18}$$

$p \cdot q \,\times$ 

$T^f_{6,10}$

$$= -\frac{5\,\zeta_5}{4\,\epsilon} + 5.68333946 + \mathcal{O}(\epsilon)\,, \tag{19}$$

$p \cdot q \,\times$ 

$T^f_{6,17}$

$$= -\frac{5\,\zeta_5}{4\,\epsilon} - 4.872849 + \mathcal{O}(\epsilon)\,, \tag{20}$$

with the Riemann zeta function ζ_n and the poly-logarithm function $\mathrm{Li}_n(z)$ being defined by:

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \text{and} \quad \mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}\,. \tag{21}$$

3 The master integrals in the standard basis

The results from the previous section can also be translated into analytical results for the standard "minimal-number-of-lines" basis for a few leading terms of the ϵ-expansion by using the IBP-relations between both bases. Since the order ϵ^0 of the ϵ-finite master integrals in section 2 is known numerically, one can also derive one additional order numerically through the IBP-relations for the members of the standard basis from the ϵ-finite one. On the other hand one can perform a formal expansion of the standard master integrals:

$$T_{i,j} = \sum_{k=n_{\min}}^{\infty} \epsilon^k\, T_{i,j}^{(k)} \tag{22}$$

and insert it into the IBP-relations, which express the ϵ-finite master integrals of section 2 in terms of the standard ones. Here we denote each of the master integrals according to the following rule: after a capital letter "T" we write the number of lines (index i) in the given diagram. The second number (j) enumerates the different topologies with the same number of lines. If a particular order k of the ϵ-expansion of the master integral $T_{i,j}$ is considered, it is denoted by an additional last index (k). After performing the ϵ-expansion of the IBP-relations one compares then the different orders in ϵ and obtains thus a linear system of equations. Its solution gives additional relations among particular orders of different master integrals. They are given in appendix B.

Another very convenient approach for evaluating master integrals with eight or less number of lines is given by the numerical solution of difference equations [1, 2, 11–13]. This method has been applied directly to the diagrams of the standard basis providing an independent check of the results in the previous section and leading to much higher numerical accuracy. The idea of this method is described in detail in ref. [1], we restrict ourself on a brief sketch. In this approach one of the massive propagators is raised to a symbolic power x

$$J(x) = \int [dk_1][dk_2][dk_3][dk_4] \frac{1}{D_1^x D_2 \dots D_{N_d}} , \tag{23}$$

and the IBP-method is used to construct difference equations for the master integrals by solving a linear system of equations. The appearance of two variables, the space-time dimension d and the power x, complicates the solution of the system of IBP-identities. It is solved with the help of a modified version of the setup described in ref. [24] using `Form`[25–27] and `Fermat`[28]. In the case of the ρ-parameter also master integrals with increased powers of a propagator appear. Then the increased power is chosen in such a way that it is carried by the propagator D_1, which carries in the above approach the symbolical power x. In this case the integral is obtained as a by-product of the solution of the integral without extra power. The difference equation for $J(x)$ can be written in the form

$$\sum_{j=0}^{R_h} p_j(x) J(x-j) = \sum_{n=1}^{N_k} \sum_{j=0}^{R_i^{(n)}} q_j^{(n)}(x) K^{(n)}(x-j) , \tag{24}$$

where p_j and $q_j^{(n)}$ are polynomials in x and d. $K^{(n)}(x)$ are sub-topologies of $J(x)$, i. e. topologies with at least one propagator less, which also satisfy difference equations. At this point the solutions for the sub-topologies are assumed to be already known.

The solution is split into several parts determined by using a factorial series

$$J_i(x) = \mu_i^x \sum_{s=0}^{\infty} a_s^{(i)} \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_i)} \tag{25}$$

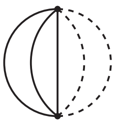
as an ansatz. From the difference equation one gets the values of μ_i and K_i and a recursion formula for the coefficients $a_s^{(i)}$. This leads to a particular solution $J_p(x)$ of the inhomogeneous equation and to one or more homogeneous solutions $J_h^{(k)}(x)$. The full solution is given by

$$J(x) = J_p(x) + \sum_k \eta_k(d) J_h^{(k)}(x) \tag{26}$$

with functions $\eta_k(d)$, fixed by boundary conditions. The $\eta_k(d)$ can be determined from the low energy expansion of the self-energy diagram which one obtains by cutting the line that carries the power x in the diagram $J(x)$. To obtain a numerical result, the coefficients $a_s^{(i)}$ are calculated as an expansion in $\epsilon = (4-d)/2$ up to a pre-defined depth ϵ_{max}. Summing the series to a specified $s = s_{max}$ and estimating the remainder gives the numerical result.

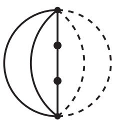
This method provides a possibility to calculate the ϵ-expansions of master integrals with high numerical precision up to high orders of ϵ.

Integrals that we did not calculate by difference equations have been constructed from the ϵ-finite basis. The lowest order, which is only known numerically, can also be obtained through the IBP-relations from the ϵ-finite basis, calculated by the Padé approach, but with less precision. The precision obtained with the Padé approach is marked in the results with an underlined (_) digit. For 11 integrals ($T_{7,10}, T_{7,11}, T_{7,12}, T_{7,13}, T_{8,3}, T_{8,5}, T_{9,3}, T_{9,4}, T_{9,5}, T_{9,6}, T_{9,7}$) more precise numerical results have in the meantime been obtained in ref. [4]. They are appended in the following to our results and separated by a vertical bar (|). All given digits of our numerical results are valid digits. The results for the master integrals read:



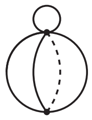
$T_{5,8}$

$$
\begin{aligned}
&= \frac{1}{4\,\epsilon^4} + \frac{1}{\epsilon^3} + \frac{1}{12\,\epsilon^2}\left(\frac{97}{4} + \pi^2\right) + \frac{1}{3\,\epsilon}\left(\frac{833}{96} + \pi^2 - \zeta_3\right) \\
&+ \frac{26509}{1728} + \frac{97}{144}\,\pi^2 + \frac{\pi^4}{12} - \frac{4}{3}\,\zeta_3 - \frac{11}{2}\,\sqrt{3}\,s_2 + 4\,s_2^2 \\
&+ 80.895\underline{5}061678534024741104898217973159065185099065309\,\epsilon \\
&+ 1085.2836587072798385770251574638225630474844919094 7\,\epsilon^2 \\
&+ 4545.3038841344258039136014565753388532852325754477 4\,\epsilon^3 \\
&+ 35998.993832632655631332960614101730730253696962197 1\,\epsilon^4 \\
&+ 134897.30755205360470411847742587239300739773511352 0\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{27}
$$



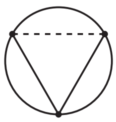
$T_{5,9}$

$$
\begin{aligned}
&= -\frac{1}{4\,\epsilon^3} - \frac{43}{48\,\epsilon^2} - \frac{1}{4\,\epsilon}\left(\frac{51}{8} + \frac{\pi^2}{3}\right) \\
&+ \frac{161}{192} - \frac{43}{144}\,\pi^2 + \frac{\zeta_3}{3} - \frac{3}{2}\,\sqrt{3}\,s_2 \\
&+ 7.3892\underline{4}322701052496152773911803203314613623175263798\,\epsilon \\
&+ 101.91151081956229384618574711657726556591568884032 7\,\epsilon^2 \\
&+ 413.97147492372213242294121414435108634991908634611 1\,\epsilon^3 \\
&+ 2899.9938588823884522572419700526882190692126453802 8\,\epsilon^4 \\
&+ 6906.3332640066949561571764315111712598668993904841 7\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{28}
$$



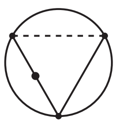
$T_{5,10}$

$$
\begin{aligned}
&= \frac{1}{\epsilon^4} + \frac{19}{4\,\epsilon^3} + \frac{1}{\epsilon^2}\,\left(\frac{103}{8} + \frac{\pi^2}{3}\right) + \frac{1}{\epsilon}\,\left(\frac{341}{16} + \frac{19}{12}\,\pi^2 + 3\,\sqrt{3}\,s_2 - \frac{4}{3}\,\zeta_3\right)\\
&+ 56.280301\underline{6}207811654954905101988728760185938023940072\\
&+ 0.628615916518815190241344693553761974378363047641 83\,\epsilon\\
&- 713.454611137588809603443079328921106975591183151813\,\epsilon^2\\
&- 3252.499009048080289940992784552141807091903793301 22\,\epsilon^3\\
&- 14732.774860454907766842451896380569392371710243490 9\,\epsilon^4\\
&- 48411.369188043121463269455599097226590258284539842 0\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{29}
$$



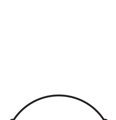
$T_{6,9}$

$$
\begin{aligned}
&= \frac{1}{\epsilon^4} + \frac{25}{4\,\epsilon^3} + \frac{1}{3\,\epsilon^2}\,\left(\frac{257}{4} + \pi^2\right) + \frac{1}{\epsilon}\,\left(\frac{433}{12} + \frac{25}{12}\,\pi^2 + 6\,\sqrt{3}\,s_2 - \frac{10}{3}\,\zeta_3\right)\\
&+ 13.591650\underline{4}31438131078488915440408858928007185810 1489\\
&- 595.072765215957129399869660561653897265114979088125\,\epsilon\\
&- 6875.841137441240810147207301867665126550506930910 46\,\epsilon^2\\
&- 31308.184930459252070388975190806777387805967436177 7\,\epsilon^3\\
&- 192167.397958807277872297783881991122229270292735722\,\epsilon^4\\
&- 729545.659023744499039120484965719141090831740292666\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{30}
$$



$T_{6,10}$

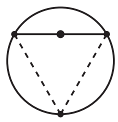
$$
\begin{aligned}
&= \frac{5}{12\,\epsilon^4} + \frac{7}{4\,\epsilon^3} + \frac{1}{12\,\epsilon^2}\,\left(43 + \frac{5}{3}\,\pi^2\right)\\
&- \frac{1}{\epsilon}\,\left(\frac{13}{4} - \frac{7}{12}\,\pi^2 - 2\,\sqrt{3}\,s_2 + \frac{19}{18}\,\zeta_3\right)\\
&- 37.457464\underline{9}3129682605507825546331033437467035360 06083\\
&- 296.071923695141317174869536814657610506616897735608\,\epsilon\\
&- 2022.200106529030007095218583041763273692820451036 55\,\epsilon^2\\
&- 8244.016338272020344231811722110354212125826825341 10\,\epsilon^3\\
&- 46235.439634860670200290541279212728709649063777242 9\,\epsilon^4\\
&- 162640.942801514076139130843600849619080175249655387\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{31}
$$



$T_{6,11}$

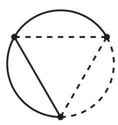
$$
\begin{aligned}
&= \frac{7}{12\,\epsilon^4} + \frac{43}{12\,\epsilon^3} + \frac{1}{4\,\epsilon^2}\,\left(47 + \frac{7}{9}\,\pi^2\right)\\
&+ \frac{1}{\epsilon}\,\left(\frac{187}{12} + \frac{43}{36}\,\pi^2 + 6\,\sqrt{3}\,s_2 + \frac{7}{18}\,\zeta_3\right)\\
&- 51.2252\underline{2}212801189152418819912274965922080241128506972
\end{aligned}
$$

$$
\begin{aligned}
&- 477.9474609951946192772855621918974532030740255333346\,\epsilon \\
&- 6127.934332491282993501512228770044793436571944255540\,\epsilon^2 \\
&- 23752.74646204112761680848816068935407828882935788 10\,\epsilon^3 \\
&- 161921.4001951448958665663782642147209892227864161 51\,\epsilon^4 \\
&- 555837.6820689275490274911346937524332839696017260 90\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{32}
$$



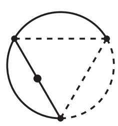
$T_{6,12}$

$$
\begin{aligned}
&= \frac{1}{3\,\epsilon^4} + \frac{4}{3\,\epsilon^3} + \frac{1}{3\,\epsilon^2}\,\left(7 + \frac{\pi^2}{3}\right) \\
&+ \frac{1}{\epsilon}\,\left(-\frac{16}{3} + \frac{4}{9}\,\pi^2 + 4\,\sqrt{3}\,s_2 - \frac{4}{9}\,\zeta_3\right) \\
&- 57.585\underline{1}192392905729834630220283359182548958203213894 \\
&- 226.9066886386370871112117327604881822531175510564 18\,\epsilon \\
&- 2357.428848232581062733641238088441827062289972624 73\,\epsilon^2 \\
&- 6885.098360977711276347127904732211834613340039696 82\,\epsilon^3 \\
&- 51888.44463619730787079356776034804035048270644703 40\,\epsilon^4 \\
&- 139706.6496151791445943491020970424353811071741243 16\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{33}
$$



$T_{6,13}$

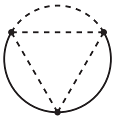
$$
\begin{aligned}
&= \frac{1}{3\,\epsilon^4} + \frac{23}{12\,\epsilon^3} + \frac{1}{3\,\epsilon^2}\,\left(\frac{65}{4} + \frac{\pi^2}{3}\right) \\
&+ \frac{1}{\epsilon}\,\left(\frac{13}{12} + \frac{5}{9}\,\pi^2 + 3\,\sqrt{3}\,s_2 + \frac{20}{9}\,\zeta_3\right) \\
&- 59.198\underline{7}779967727811953138849540094767971219526012487 \\
&- 549.0427267950283240607628342536157266781169813547 86\,\epsilon \\
&- 4293.526172966570450214137607441026022265212649191 04\,\epsilon^2 \\
&- 20846.82974176864614764975013356475087551987021690 76\,\epsilon^3 \\
&- 110899.1740770305717954004091240842162489151135321 75\,\epsilon^4 \\
&- 470220.6567701511797928554609839719959623155220216 62\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{34}
$$



$T_{6,14}$

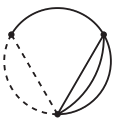
$$
\begin{aligned}
&= \frac{1}{4\,\epsilon^4} + \frac{11}{12\,\epsilon^3} + \frac{1}{12\,\epsilon^2}\,\left(13 + \pi^2\right) \\
&+ \frac{1}{\epsilon}\,\left(-\frac{89}{12} + \frac{11}{36}\,\pi^2 + 2\,\sqrt{3}\,s_2 + \frac{17}{6}\,\zeta_3\right) \\
&- 62.926\underline{3}9532675207901828584026900091433864756452442 29 \\
&- 258.6929411391298008706565131718407791226565113793 67\,\epsilon \\
&- 2245.883289004246009121283748554760880785826901496 58\,\epsilon^2
\end{aligned}
$$

$$
\begin{aligned}
&- 7643.284417608632445333687904824415143807330149508 51\,\epsilon^3 \\
&- 48359.266517640802325161975102835625167624930775189 2\,\epsilon^4 \\
&- 156259.77328206506161626460799016705627691384465346 2\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{35}
$$



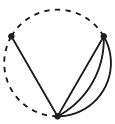
$T_{6,15}$

$$
\begin{aligned}
&= \frac{1}{12\,\epsilon^4} + \frac{5}{12\,\epsilon^3} + \frac{1}{12\,\epsilon^2}\left(7 + \frac{\pi^2}{3}\right) - \frac{1}{6\,\epsilon}\left(\frac{67}{2} + \frac{\pi^2}{6} - \frac{37}{3}\,\zeta_3\right) \\
&- \frac{235}{4} - \frac{65}{36}\,\pi^2 - \frac{7}{40}\,\pi^4 + \frac{293}{18}\,\zeta_3 \\
&+ \frac{\epsilon}{2}\left(-\frac{1497}{2} - \frac{601}{18}\,\pi^2 - \frac{511}{180}\,\pi^4 + \frac{1555}{9}\,\zeta_3 + \frac{181}{27}\,\pi^2\,\zeta_3 + \frac{1813}{15}\,\zeta_5\right) \\
&- 3088.5\underline{5}3855409970210226125854528856995516343078007 97\,\epsilon^2 \\
&- 14498.109291874519354681662299222882222179448413067 4\,\epsilon^3 \\
&- 75204.133631489144762717873912142014253375518679273 5\,\epsilon^4 \\
&- 315844.16876826479665546307129394451265015976106737 9\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{36}
$$



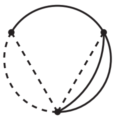
$T_{6,16}$

$$
\begin{aligned}
&= \frac{7}{8\,\epsilon^4} + \frac{85}{16\,\epsilon^3} + \frac{1}{8\,\epsilon^2}\left(\frac{601}{4} + \frac{13}{3}\,\pi^2\right) + \frac{1}{\epsilon}\left(\frac{2747}{64} + \frac{155}{48}\,\pi^2 - \frac{8}{3}\,\zeta_3\right) \\
&+ \frac{2329}{128} + \frac{1091}{96}\,\pi^2 + \frac{\pi^4}{4} + \frac{43}{2}\,\zeta_3 + \frac{17}{2}\,\sqrt{3}\,s_2 + 2\,s_2^2 \\
&+ 6.4118\underline{7}24837253080630619146816327061097014836699200 8\,\epsilon \\
&- 1028.331288333875338344586673526226806343239152573 41\,\epsilon^2 \\
&- 16249.877755264195774749291779233471688052225484124 1\,\epsilon^3 \\
&- 65696.898973387503919698614093333766728209454048510 6\,\epsilon^4 \\
&- 430175.28109482482144156875158518920623772193068161 7\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{37}
$$



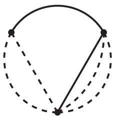
$T_{6,17}$

$$
\begin{aligned}
&= \frac{5}{8\,\epsilon^4} + \frac{33}{8\,\epsilon^3} + \frac{1}{24\,\epsilon^2}\left(\frac{775}{2} + 11\,\pi^2\right) \\
&+ \frac{1}{\epsilon}\left(\frac{1389}{32} + \frac{23}{8}\,\pi^2 + 3\,\sqrt{3}\,s_2 - \frac{7}{3}\,\zeta_3\right) \\
&+ 147.807\underline{8}7035341945700602002788127775202079600611232 3 \\
&+ 467.918626785158952385003318864377551570797031294221\,\epsilon \\
&- 1988.539793801800446261028948924608881763251588792 71\,\epsilon^2 \\
&- 4066.0500495929160439993639312969140638382499365038 1\,\epsilon^3 \\
&- 84210.538667468960167279985288401769862468535263735 7\,\epsilon^4 \\
&- 187546.77189281473365783574449006758336389247133655 3\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{38}
$$



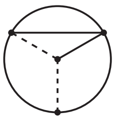
$T_{6,18}$

$$
\begin{aligned}
&= \frac{1}{2\,\epsilon^4} + \frac{47}{16\,\epsilon^3} + \frac{1}{\epsilon^2}\,\left(\frac{317}{32} + \frac{\pi^2}{3}\right) + \frac{1}{\epsilon}\,\left(\frac{1315}{64} + \frac{31}{16}\,\pi^2 + 3\,\sqrt{3}\,s_2 - \frac{5}{3}\,\zeta_3\right)\\
&+ 83.135\underline{8}88200080780846532273482259297546210098646 5128\\
&+ 27.174091904545384840866187400570635467263966987 4261\,\epsilon\\
&- 1554.564950466625606304544850420407571810793945898 06\,\epsilon^2\\
&- 9710.654665542054776998495100145365357868345538116 46\,\epsilon^3\\
&- 59651.10297545411348029771724103090149675162280785 74\,\epsilon^4\\
&- 263782.7127603203350623398311189790314015246849032 80\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{39}
$$



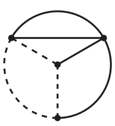
$T_{6,19}$

$$
\begin{aligned}
&= \frac{5}{24\,\epsilon^4} + \frac{55}{48\,\epsilon^3} + \frac{1}{24\,\epsilon^2}\,\left(\frac{331}{4} + \frac{11}{3}\,\pi^2\right)\\
&+ \frac{1}{3\,\epsilon}\,\left(\frac{977}{64} + \frac{121}{48}\,\pi^2 + \frac{17}{3}\,\zeta_3\right) - \frac{1831}{128} + \frac{661\,\pi^2}{288} - \frac{11\,\pi^4}{120} + \frac{187}{18}\,\zeta_3\\
&- \epsilon\,\left(\frac{42485}{256} - \frac{335}{576}\,\pi^2 + \frac{121}{240}\,\pi^4 - \frac{1327}{36}\,\zeta_3 - \frac{77}{27}\,\pi^2\,\zeta_3 - \frac{172}{3}\,\zeta_5\right)\\
&- 1308.40\underline{0}4597549072358809428979475258313272928922540 0\,\epsilon^2\\
&- 6297.444100073721695254809203611638516863563801775 51\,\epsilon^3\\
&- 39381.32918463861391307026227540061650332306177657 64\,\epsilon^4\\
&- 156879.0790319171980944411083222541830974370763907 86\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{40}
$$



$T_{7,5}$

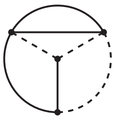
$$
\begin{aligned}
&= \frac{1}{8\,\epsilon^4} + \frac{5}{4\,\epsilon^3} + \frac{1}{\epsilon^2}\,\left(\frac{65}{8} + \frac{\pi^2}{8} + \zeta_3\right)\\
&+ \frac{1}{\epsilon}\,\left(\frac{175}{4} + \frac{3}{4}\,\pi^2 + \frac{\pi^4}{60} - 6\,\sqrt{3}\,s_2 + \frac{10}{3}\,\zeta_3\right)\\
&+ 188.248\underline{4}938415207225124816764363888710719498263689 57\\
&+ 965.7462952542046954227857365808493667626212531481 87\,\epsilon\\
&+ 2625.626185456401796637930657407416319152281080245 70\,\epsilon^2\\
&+ 17532.05859815281533447078440000892486871673719849 49\,\epsilon^3\\
&+ 36105.67080011986386614910474542233629251672163771 88\,\epsilon^4\\
&+ 300793.6333362216379702529282592313193921705652325 62\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{41}
$$



$T_{7,6}$

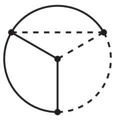
$$
\begin{aligned}
&= \frac{1}{12\,\epsilon^4} + \frac{5}{6\,\epsilon^3} + \frac{1}{\epsilon^2}\,\left(\frac{65}{12} + \frac{7}{36}\,\pi^2 + \zeta_3\right)\\
&+ \frac{1}{3\,\epsilon}\,\left(\frac{175}{2} + \frac{23}{6}\,\pi^2 + \frac{\pi^4}{20} - \frac{13}{3}\,\zeta_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{567}{4}+\frac{191}{36}\,\pi^2+\frac{19}{36}\,\pi^4-\frac{367}{9}\,\zeta_3-\frac{1}{3}\pi^2\,\zeta_3-\frac{89}{3}\,\zeta_5+8\,\sqrt{3}\,s_2+4\,s_2^2\\
&+915.\underline{9}967427155624078215822328798370385453624693238 20\,\epsilon\\
&+2767.39379170194861041563852820827715025382193870924\,\epsilon^2\\
&+16538.67925059045805662392896015192713282101269389 73\,\epsilon^3\\
&+40456.09097133940954100133510087645992074927777022 47\,\epsilon^4\\
&+280703.97757787624072770249448854200076354735643160 5\,\epsilon^5+\mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{42}
$$



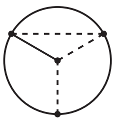
$T_{7,7}$

$$
\begin{aligned}
&=\frac{1}{8\,\epsilon^4}+\frac{13}{12\,\epsilon^3}+\frac{1}{\epsilon^2}\,\left(\frac{143}{24}+\frac{\pi^2}{8}+\zeta_3\right)+\frac{1}{3\,\epsilon}\,\left(\frac{317}{4}+\frac{43}{12}\,\pi^2+\frac{\pi^4}{20}+19\,\zeta_3\right)\\
&+158.265\underline{8}949856147547656090936644509280816216852541 72\\
&+1040.51973958438238064321813618885698154945163332335\,\epsilon\\
&+2220.52649830986620272733311300227293842057928920038\,\epsilon^2\\
&+19009.38193679337810544859063801972743982451002370 12\,\epsilon^3\\
&+29872.28905548133616884820860538428353768929239730 47\,\epsilon^4\\
&+325486.61852564220571884842895839387794606451875328 1\,\epsilon^5+\mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{43}
$$



$T_{7,8}$

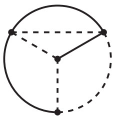
$$
\begin{aligned}
&=\frac{1}{6\,\epsilon^4}+\frac{3}{2\,\epsilon^3}+\frac{1}{\epsilon^2}\,\left(\frac{26}{3}+\frac{\pi^2}{18}+\frac{\zeta_3}{2}\right)\\
&+\frac{1}{\epsilon}\,\left(41+\frac{\pi^2}{2}+\frac{\pi^4}{120}-\frac{7}{18}\,\zeta_3\right)\\
&+192.14\underline{3}9499851897927831333572589872626147270561225 21\\
&+895.58323270744196015137233006351500233752280353468 2\,\epsilon\\
&+3015.50609158685899428979534043459900694651565328232\,\epsilon^2\\
&+15431.71415650542061656299748662377650169092755062 81\,\epsilon^3\\
&+45950.38515771104559998246982001838395837283237323 33\,\epsilon^4\\
&+256376.55367165790411186099995313036410369499797179 9\,\epsilon^5+\mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{44}
$$



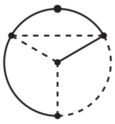
$T_{7,9}$

$$
\begin{aligned}
&=\frac{1}{8\,\epsilon^4}+\frac{5}{4\,\epsilon^3}+\frac{1}{2\,\epsilon^2}\,\left(\frac{65}{4}+\frac{\pi^2}{4}+\zeta_3\right)\\
&+\frac{1}{\epsilon}\,\left(\frac{175}{4}+\frac{3}{4}\,\pi^2+\frac{\pi^4}{120}-6\,\sqrt{3}\,s_2-\frac{\zeta_3}{6}\right)\\
&+210.851\underline{2}3475830598461209011260543498573296118935 9568\\
&+790.91035286867799946838195547029378186280989570793 2\,\epsilon\\
&+3471.90742617992105936864664247656869003732285384889\,\epsilon^2
\end{aligned}
$$

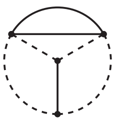
$$
\begin{aligned}
&+ 13388.2008481237206516463529137761462371387930576955\,\epsilon^3 \\
&+ 54666.0539850126748720361226803316191993587586541377\,\epsilon^4 \\
&+ 219636.9108467716176611227337164104390397575972251 40\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned}
\tag{45}
$$


$T_{7,10}$

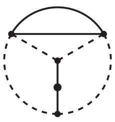
$$
\begin{aligned}
&= \frac{1}{12\,\epsilon^4} + \frac{5}{6\,\epsilon^3} + \frac{1}{2\,\epsilon^2}\left(\frac{65}{6} + \frac{7}{18}\,\pi^2 + \zeta_3\right) \\
&+ \frac{1}{6\,\epsilon}\left(175 + \frac{23\,\pi^2}{3} + \frac{\pi^4}{20} - \frac{89}{3}\,\zeta_3\right) \\
&+ 194.097392\underline{6}55331|631239559043545 \\
&+ 770.17891682309|6887339876558212\,\epsilon \\
&+ 3399.9948243974|1716780023304896\,\epsilon^2 \\
&|+ 13282.1053505362611944372969993\,\epsilon^3 \\
&|+ 54753.4641354130075585240962863\,\epsilon^4 + \mathcal{O}(\epsilon^5)\,,
\end{aligned}
\tag{46}
$$


$T_{7,11}$

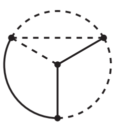
$$
\begin{aligned}
&= \frac{\zeta_3}{2\,\epsilon^2} + \frac{1}{\epsilon}\left(\frac{\pi^4}{120} + \frac{3}{2}\,\zeta_3\right) \\
&- 4.5950107\underline{7}9222|629663368197802 \\
&+ 78.94876834312|825216261582947\,\epsilon \\
&- 340.518381398|9136598834071036\,\epsilon^2 \\
&+ 1861.12230255|2258601194625502\,\epsilon^3 \\
&|- 8297.0012966735720711956780 72\,\epsilon^4 + \mathcal{O}(\epsilon^5)\,,
\end{aligned}
\tag{47}
$$


$T_{7,12}$

$$
\begin{aligned}
&= \frac{1}{12\,\epsilon^4} + \frac{2}{3\,\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{13}{4} + \frac{7}{36}\,\pi^2 + \zeta_3\right) \\
&+ \frac{1}{3\,\epsilon}\left(\frac{71}{2} + \frac{14}{3}\,\pi^2 + \frac{\pi^4}{20} + \frac{68}{3}\,\zeta_3\right) \\
&+ 154.60463\underline{1}|5446170184994708554 34 \\
&|+ 894.807380418881266203622406485\,\epsilon \\
&|+ 2607.13652650580177242514674361\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,,
\end{aligned}
\tag{48}
$$

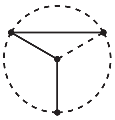

$T_{7,13}$

$$
\begin{aligned}
&= \frac{1}{12\,\epsilon^4} + \frac{1}{3\,\epsilon^3} + \frac{7}{12\,\epsilon^2}\left(1 + \frac{1}{3}\,\pi^2\right) - \frac{1}{3\,\epsilon}\left(\frac{7}{2} - \frac{7}{3}\,\pi^2 - \frac{5}{3}\,\zeta_3\right) \\
&+ 10.00336626|15671539152667896043 \\
&|+ 121.389726250269120381127113931\,\epsilon \\
&|- 338.169587547191526007167931728\,\epsilon^2 + \mathcal{O}(\epsilon^3)\,,
\end{aligned}
\tag{49}
$$


$T_{7,14}$

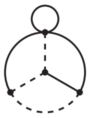
$$
= \frac{1}{6\,\epsilon^4} + \frac{3}{2\,\epsilon^3} + \frac{1}{3\,\epsilon^2}\left(26 + \frac{\pi^2}{6}\right) + \frac{1}{\epsilon}\left(41 + \frac{\pi^2}{2} - \frac{35}{9}\,\zeta_3\right)
$$

$$
\begin{aligned}
&+ \frac{1039}{6} + \frac{55}{18}\,\pi^2 + \frac{37}{120}\,\pi^4 - \frac{94}{3}\,\zeta_3 + 2\,\sqrt{3}\,s_2 + 10\,s_2^2 \\
&+ 736.116\underline{8}322998660413786288752695122012275688154107 47\,\epsilon \\
&+ 3742.4598433116742266248254477738044749184738761775 6\,\epsilon^2 \\
&+ 11749.430216934050608343876190539983485201106796461 2\,\epsilon^3 \\
&+ 62286.583600193037587315791231469767222497285037081 5\,\epsilon^4 \\
&+ 184394.06758712300035468502780917450873697303937893 3\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{50}
$$



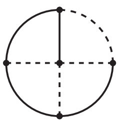
$T_{7,15}$

$$
\begin{aligned}
&= \frac{1}{8\,\epsilon^4} + \frac{13}{12\,\epsilon^3} + \frac{1}{2\,\epsilon^2}\left(\frac{143}{12} + \frac{\pi^2}{4} + \zeta_3\right) + \frac{1}{6\,\epsilon}\left(\frac{317}{2} + \frac{43}{6}\,\pi^2 + \frac{\pi^4}{20} + 17\,\zeta_3\right) \\
&+ \frac{2455}{24} + \frac{545}{72}\,\pi^2 + \frac{\pi^4}{90} + \frac{169}{18}\,\zeta_3 - \frac{\pi^2\,\zeta_3}{6} - \frac{59}{6}\,\zeta_5 + 2\,\sqrt{3}\,s_2 - 8\,s_2^2 \\
&+ 895.\underline{2}672230345077923365459347469719030900741771785 63\,\epsilon \\
&+ 2851.2448774512992022605638916897715128678795014953 1\,\epsilon^2 \\
&+ 15769.292489303665876826105281314356985844225946001 5\,\epsilon^3 \\
&+ 44094.429440956547374792202272314705574620829711083 5\,\epsilon^4 \\
&+ 262762.37829944337102984295461485332597004132760919 4\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{51}
$$



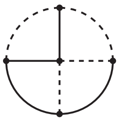
$T_{7,16}$

$$
\begin{aligned}
&= \frac{2}{\epsilon^2}\,\zeta_3 - \frac{1}{\epsilon}\left(\frac{\pi^4}{24} - 8\,\zeta_3 + 6\,s_2^2\right) \\
&+ 30.0908437\underline{8}2152469478220657021372064582648022901646 6 \\
&- 36.823069190453525435554049326216284608015940168948 6\,\epsilon \\
&+ 260.50782746146939082756161009600334601439242109067 8\,\epsilon^2 \\
&- 562.80964682660196589691793356296636488024402235792 1\,\epsilon^3 \\
&+ 2222.6758856584030945669034220933355009355096917922 7\,\epsilon^4 \\
&- 6072.2056453182248961854410227879720041202677189076 7\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{52}
$$



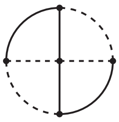
$T_{8,2}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 \\
&- 20.858597\underline{6}739540823192137169923590025170233079442547 \\
&+ 148.81432728918479088096075228386306672351460590192 2\,\epsilon \\
&- 657.41892071216819652652876811700126605403034966152 5\,\epsilon^2 \\
&+ 3182.4722178244908662027423159557983054619864181001 3\,\epsilon^3 \\
&- 13795.821136376369794182560837716030147461461656188 0\,\epsilon^4 + \mathcal{O}(\epsilon^5)\,,
\end{aligned} \tag{53}
$$



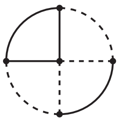
$T_{8,3}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 \\
&\quad - 19.111865\underline{7}0594|90732311103013898 \\
&\quad + 141.144175505|09485801299887229 8\,\epsilon \\
&\quad - 605.43119057|5217055408942299967\,\epsilon^2 \\
&\quad | + 2945.5804129314914976272043702 2\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,,
\end{aligned} \tag{54}
$$



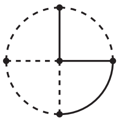
$T_{8,4}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 \\
&- 19.079375\underline{0}92707996031425740587562394152578910419784 3 \\
&+ 141.25224818610774716409263279748985033707084397584 1\,\epsilon \\
&- 605.0296212018702589046626714552065506456710350324 0\,\epsilon^2 \\
&+ 2946.4874043511733940960656451002747558364665277 2\,\epsilon^3 \\
&- 12664.231982725689641172668688917148051149093738\,\epsilon^4 \\
&+ 54825.719313643610729563127424198851586342860 91\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{55}
$$



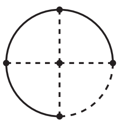
$T_{8,5}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 - 18.7699726\underline{9}|85410818615637417243 \\
&\quad | + 142.191941064595949485071666672\,\epsilon \\
&\quad | - 600.255511127362803430335175869\,\epsilon^2 \\
&\quad | + 2957.185588135948680830606207 64\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,,
\end{aligned} \tag{56}
$$



$T_{8,6}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 \\
&- 7.8587\underline{6}279518922242360822990783004542937882100674500 \\
&+ 128.531386676431303444223252316571121713818949997681\,\epsilon \\
&- 349.879176215675436895102723350659482216370181950816\,\epsilon^2 \\
&+ 2332.83399433790756728143973285449385093610438970117\,\epsilon^3 \\
&- 8346.39781820188664641354322284607837497614943630437\,\epsilon^4 \\
&+ 40158.5604096624536390379365615442757755373577157863\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{57}
$$



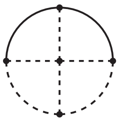
$T_{8,7}$

$$
\begin{aligned}
&= \frac{5}{\epsilon}\,\zeta_5 \\
&- 13.72063\underline{0}262592016207048734265855738878247496776770 6 \\
&+ 135.928906108952872038638377087143640185611677958787\,\epsilon \\
&- 497.764644552772376477970102393026516204883127610554\,\epsilon^2
\end{aligned}
$$

$$+\,2695.351253951946106329097294851399564343540331781\,37\,\epsilon^3$$

$$-\,11004.947885048168133876514716901992347129498548517\,0\,\epsilon^4$$

$$+\,49153.949047606384209078832848686274393477351977603\,1\,\epsilon^5+\mathcal{O}(\epsilon^6)\,, \tag{58}$$



$T_{8,8}$

$$= \frac{5}{\epsilon}\,\zeta_5$$

$$+\,0.3748\underline{4}41919268512749108051658710975669979422713212\,80$$

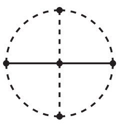
$$+\,141.683133328263640451220557622143025514756763189200\,\epsilon$$

$$-\,146.684014785607112711629455842851698248300684175120\,\epsilon^2$$

$$+\,2448.778678724447872602242562549580001626248566741\,33\,\epsilon^3$$

$$-\,5421.705877931039816193835609606287183328781550201\,20\,\epsilon^4$$

$$+\,38306.610649411986605898099039553274146620601442752\,2\,\epsilon^5+\mathcal{O}(\epsilon^6)\,, \tag{59}$$



$T_{8,9}$

$$= \frac{5}{\epsilon}\,\zeta_5$$

$$+\,7.304459\underline{6}850865993470513029498232853650567313505750\,4$$

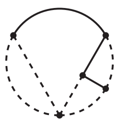
$$+\,141.443417856878944115040221346329269896865316839719\,\epsilon$$

$$+\,40.4773793831579631350040651321790566085488080065966\,\epsilon^2$$

$$+\,2244.390627166621545265263606316282455379487122033\,04\,\epsilon^3$$

$$-\,2188.789995259847936806807100774250469126689993503\,76\,\epsilon^4$$

$$+\,31134.483806707361785786161517384587214204944333793\,3\,\epsilon^5+\mathcal{O}(\epsilon^6)\,, \tag{60}$$



$T_{8,10}$

$$= \frac{3}{2\,\epsilon^2}\,\zeta_3-\frac{1}{\epsilon}\,\left(\frac{\pi^4}{20}-\frac{15}{2}\,\zeta_3+6\,s_2^2\right)$$

$$+\,40.477\underline{3}90115082924780660914736191421037448530788460\,8$$

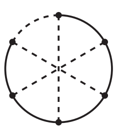
$$-\,100.628977840966481693832556085825783093037821912010\,\epsilon$$

$$+\,679.056339067891403282049708893797069643756722410073\,\epsilon^2$$

$$-\,2387.604520827586356320618397101522335835037315922\,10\,\epsilon^3$$

$$+\,11259.259506858455686211634245395867714093329427260\,1\,\epsilon^4$$

$$-\,44413.954963639907061815335771527293113528482741363\,8\,\epsilon^5+\mathcal{O}(\epsilon^6)\,, \tag{61}$$



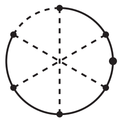
$T_{9,3}$

$$= -\,2.4269563\underline{3}|9537700735$$

$$|+\,4.01554669961524192\,\epsilon$$

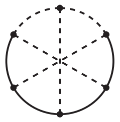
$$|-\,43.6962533603324647\,\epsilon^2$$

$$| + 128.936157875347890\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \tag{62}$$



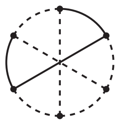
$T_{9,4}$

$$\begin{aligned} = &+ 0.4736|11472272364450 \\ &| + 1.09585342206826990\,\epsilon \\ &| + 5.37764333252884269\,\epsilon^2 \\ &| + 8.82896457590640998\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{63}$$



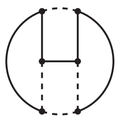
$T_{9,5}$

$$\begin{aligned} = &- 3.7114026\underline{4}|536682392682628965373 \\ &| - 2.115205995458772647561352618 04\,\epsilon \\ &| - 71.989945138982949183067462955 0\,\epsilon^2 \\ &| + 41.188129417430924441786441946 8\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{64}$$



$T_{9,6}$

$$\begin{aligned} = &- 6.7284\underline{7}05600856|8105547188977521 \\ &- 26.08764659996|66155389659770717\,\epsilon \\ &- 214.6477179124|11362028052727052\,\epsilon^2 \\ &- 613.715203096|626075654874908838\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{65}$$



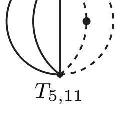
$T_{9,7}$

$$\begin{aligned} = &- 3.449751\underline{1}|317390349922288 \\ &| + 6.3127694885459812824115\,\epsilon \\ &| - 63.668771344187502234181\,\epsilon^2 \\ &| + 196.344026126273593229 23\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,, \end{aligned} \tag{66}$$

with the symbol s_2 defined by:

$$s_2 = \mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \mathrm{Im}\left[\mathrm{Li}_2(e^{i\,\frac{\pi}{3}})\right] = 1.01494160640965362 50\ldots\,. \tag{67}$$

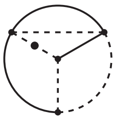
In ref. [4] the set of master integrals has been chosen slightly different. Instead of the topologies $T_{5,9}$ and $T_{7,11}$ the topologies $T_{5,11}$ and $T_{7,17}$ have been chosen as master integrals. These different master integrals can be related to each other by the IBP-relations. For these integrals we also give the analytical results, which we obtained for the leading ϵ-orders:



$T_{5,11}$

$$\begin{aligned} = &-\frac{1}{4\,\epsilon^4} - \frac{9}{8\,\epsilon^3} - \frac{1}{2\,\epsilon^2}\left(5 + \frac{\pi^2}{6}\right) + \frac{1}{\epsilon}\left(\frac{15}{8} - \frac{3}{8}\pi^2 - 3\sqrt{3}\,s_2 + \frac{\zeta_3}{3}\right) \\ &+ 30.025\underline{3}835218317623192659472127405301377841888425075 \\ &+ 254.1110319647040935393758262065251340722560040571 09\,\epsilon \\ &+ 1805.3659336642529001240935993234616478719646513929 9\,\epsilon^2 \end{aligned}$$

$$
\begin{aligned}
&+ 8898.2467236228295903253851867289518740270118769956\,8\,\epsilon^3 \\
&+ 43751.255181800144471762511535023621445607185642568\,4\,\epsilon^4 \\
&+ 193367.01231793643565369136417546665762856476049082\,4\,\epsilon^5 + \mathcal{O}(\epsilon^6)\,,
\end{aligned} \tag{68}
$$



$T_{7,17}$

$$
\begin{aligned}
&= -\frac{1}{8\,\epsilon^4} - \frac{3}{4\,\epsilon^3} - \frac{1}{8\,\epsilon^2}\left(25 - \frac{\pi^2}{3}\right) - \frac{1}{\epsilon}\left(\frac{45}{4} - \frac{7}{12}\,\pi^2 + \frac{4}{3}\,\zeta_3\right) \\
&- \frac{301}{8} + \frac{27}{8}\,\pi^2 + \frac{29}{720}\,\pi^4 - \frac{55}{3}\,\zeta_3 - 4\,\sqrt{3}\,s_2 + 7\,s_2^2 \\
&- 17.7874\underline{4}259404|76975226672333232\,\epsilon \\
&- 175.317651447|398940503032885592\,\epsilon^2 \\
&+ 313.98310535|3518264983361554717\,\epsilon^3 \\
&|- 2142.79466723532874375769322841\,\epsilon^4 + \mathcal{O}(\epsilon^5)\,.
\end{aligned} \tag{69}
$$

All results are in full agreement with those published in [4].
An overview over the different members of the two sets of master integrals discussed in section 2 and 3 is given in table 1.

Finally, we want to make a few comments about the results for the master integrals in the standard basis as collected in eqs. (27)-(69).

- For the calculation of the ρ-parameter the coefficients of the master integrals shown in eqs. (27)-(69) suffer from not more than a $1/\epsilon^2$-pole. On general grounds it is clear that in any foreseeable *four*-loop calculation one would always need the analytical terms, displayed in eqs. (27)-(69) of the ϵ-expansion of a master integral and additionally up to at most the order ϵ^2 for the ρ-parameter. Thus all other higher order ϵ-terms, shown in eqs. (27)-(69), will become physically relevant only for five and more loops.

- The use of the ϵ-finite basis allows for an easy and automatically *analytical* determination of a “trivial” part (that is eventually expressible through the three-loop tadpoles) of the ϵ-expansion of a four-loop tadpole master integral[14]. This is just enough to control the cancellation of the UV poles during the renormalization procedure in completely algebraic way. In order to have the final physical result completely analytically one needs to know at most one more term in the ϵ-expansion of the members of the ϵ-finite basis in an analytical form. In relatively simple cases this could be achieved with analytical summation of auxiliary series constructed with the help of either the differential relations [29–32] or the difference ones [1, 2, 11–13]. For recent progress in this direction see e.g. [18, 19, 33, 34] and references therein.

Set of master integrals, which are equal in both basis	$M_{eql} = \{T_{6,11},\ T_{7,5},\ T_{7,7},\ T_{7,8},\ T_{7,9},\ T_{7,10},\ T_{7,12},\ T_{7,13},\ T_{7,16},\ T_{8,1},\ T_{8,2},\ T_{8,3},\ T_{8,4},\ T_{8,5},\ T_{8,6},\ T_{8,7},\ T_{8,8},\ T_{8,9},\ T_{8,10},\ T_{9,1},\ T_{9,3},\ T_{9,5},\ T_{9,6},\ T_{9,7}\}$
Set of additional members of the standard basis	$M_{std} = \{T_{5,2},\ T_{5,4},\ T_{5,8},\ T_{5,9},\ T_{5,10},\ T_{6,1},\ T_{6,2},\ T_{6,4},\ T_{6,9},\ T_{6,10},\ T_{6,12},\ T_{6,13},\ T_{6,14},\ T_{6,15},\ T_{6,16},\ T_{6,17},\ T_{6,18},\ T_{6,19},\ T_{7,1},\ T_{7,2},\ T_{7,6},\ T_{7,11},\ T_{7,14},\ T_{7,15},\ T_{9,4}\}$
Set of additional members of the ϵ-finite basis	$M_{epf} = \{T^f_{5,2},\ T^f_{5,4},\ T^f_{5,8},\ T^f_{5,9},\ T^f_{5,10},\ T^f_{6,1},\ T^f_{6,2},\ T^f_{6,4},\ T^f_{6,9},\ T^f_{6,10},\ T^f_{6,12},\ T^f_{6,13},\ T^f_{6,14},\ T^f_{6,15},\ T^f_{6,16},\ T^f_{6,17},\ T^f_{6,18},\ T^f_{6,19},\ T^f_{7,1},\ T^f_{7,2},\ T^f_{7,6},\ T^f_{7,11},\ T^f_{7,14},\ T^f_{7,15},\ T^f_{9,4}\}$
Set of master integrals, known analytically to sufficient orders in ϵ	$M_{ana} = \{T_{4,1},\ T_{5,1},\ T_{5,3},\ T_{5,5},\ T_{5,6},\ T_{5,7},\ T_{6,3},\ T_{6,5},\ T_{6,6},\ T_{6,7},\ T_{6,8},\ T_{7,3},\ T_{7,4},\ T_{9,2}\}$

Table 1: The standard basis is given by $M_{eql} \cup M_{std} \cup M_{ana}$. The ϵ-finite basis is given by $M_{eql} \cup M_{epf} \cup M_{ana}$. Note that the simple master integrals in the set M_{ana} have not been replaced in the case of the existence of a spurious pole in the coefficient function, since these master integrals are known analytically to high orders in ϵ. The master integrals of the table, which have not been given in this work can be found in ref. [10, 14].

- One should, however, keep in mind that the numerical accuracy of the Laporta method of computing master integrals is huge (30-50 digits!). This means that the availability of a completely analytical result is absolutely irrelevant for phenomenology.

4 Conclusions

We have constructed an ϵ-finite basis for problems related to the ρ-parameter. The calculation of the elements of this basis via the Padé method allows to obtain a lot of exact analytical relations among various terms of the Taylor expansion in ϵ of the master integrals. In particular we found analytically the pole parts of the master integrals in both bases, the standard and the ϵ-finite one. For some members of the standard basis analytical results are also obtained for orders beyond the pole part. With the help of the ϵ-finite basis the cancellation of all divergences during the renormalization procedure can be done analytically. These relations are indispensable for any attempts to provide, say, the four-loop QCD contributions to the ρ parameter in a completely analytical form. They also serve as a powerful check of available numerical results. We find full agreement with the results of the standard basis also calculated in ref. [4]. The conversion between the two descriptions as well as all the results presented in this work will be made available in computer readable form under the URL `http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp06/ttp06-30`.

Acknowledgments

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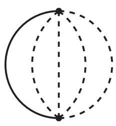
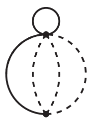
A Master integrals excluded in the construction of the ϵ-finite basis

In this appendix we give for completeness the result of the integrals, which have been excluded in the construction of an ϵ-finite basis, since they are known analytically to higher powers in the ϵ-expansion. The results read:



$T_{5,5}$

$$= 3\, e^{4\,\epsilon\,\gamma_E}\, \Gamma(5-2\,d)\, \Gamma\!\left(3-\tfrac{3\,d}{2}\right)\, \Gamma^3\!\left(\tfrac{d}{2}-1\right)\,, \tag{70}$$

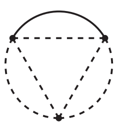


$T_{5,6}$

$$= -3\, e^{4\,\epsilon\,\gamma_E}\, (d-4)\, \Gamma(2-d)\, \Gamma\!\left(3-\tfrac{3}{2}\,d\right)\, \Gamma\!\left(3-\tfrac{d}{2}\right)\, \Gamma^2\!\left(\tfrac{d}{2}-2\right)\,, \tag{71}$$

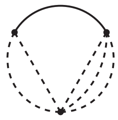
$T_{5,7}$

$$= -\frac{32\, e^{4\,\epsilon\,\gamma_E}\, \Gamma(2-d)\, \Gamma^3\!\left(3-\tfrac{d}{2}\right)\, \Gamma\!\left(\tfrac{d}{2}-2\right)}{(d-4)^2\, (d-2)^2}\,, \tag{72}$$



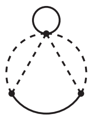
$T_{6,5}$

$$= -\frac{8\, e^{4\,\epsilon\,\gamma_E}\, \Gamma(6-2\,d)\, \Gamma\!\left(6-\tfrac{3\,d}{2}\right)\, \Gamma^2\!\left(3-\tfrac{d}{2}\right)\, \Gamma^4\!\left(\tfrac{d}{2}-1\right)\, \Gamma\!\left(\tfrac{3\,d}{2}-5\right)}{(d-4)^2\, (d-2)\, \Gamma(4-d)\, \Gamma^2(d-2)}\,, \tag{73}$$



$T_{6,6}$

$$= \frac{12\, e^{4\,\epsilon\,\gamma_E}\, \Gamma(8-2\,d)\; \Gamma(4-d)\; \Gamma\big(3-\tfrac{d}{2}\big)\; \Gamma(2\,d-8)\; \Gamma^4\big(\tfrac{d}{2}-1\big)}{(d-3)\,\Gamma(d-2)\; \Gamma\big(\tfrac{3\,d}{2}-2\big)}\,, \quad (74)$$

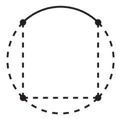


$T_{6,7}$

$$= -\frac{256\, e^{4\,\epsilon\,\gamma_E}\, \Gamma\big(6-\tfrac{3}{2}\,d\big)\; \Gamma^3\big(\tfrac{d}{2}\big)\; \Gamma^3\big(3-\tfrac{d}{2}\big)\; \Gamma\big(\tfrac{3}{2}\,d-5\big)}{(d-4)^3\,(d-2)^3\,\Gamma^2(d-1)}\,, \quad (75)$$

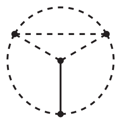
$T_{6,8}$

$$= 4\, e^{4\,\epsilon\,\gamma_E}\, \Gamma^2(2-d)\; \Gamma^2\big(3-\tfrac{d}{2}\big)\; \Gamma^2\big(\tfrac{d}{2}-2\big)\;, \quad (76)$$



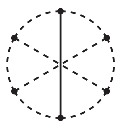
$T_{7,3}$

$$= -\frac{32\, e^{4\,\epsilon\,\gamma_E}\, \Gamma(8-2\,d)\; \Gamma^3\big(3-\tfrac{d}{2}\big)\; \Gamma(2\,d-8)\; \Gamma^5\big(\tfrac{d}{2}-1\big)}{(d-4)^2\,(d-2)\,\Gamma^3(d-2)}\,, \quad (77)$$



$T_{7,4}$

$$\begin{aligned} &= \frac{1}{12\,\epsilon^4} + \frac{2}{3\,\epsilon^3} + \frac{1}{36\,\epsilon^2}\,\big(117+7\,\pi^2\big) + \frac{1}{18\,\epsilon}\,\big(213+28\,\pi^2+10\,\zeta_3\big) \\ &+ \frac{1}{180}\,\big(5655+1365\,\pi^2+64\,\pi^4+800\,\zeta_3\big) \\ &+ \frac{\epsilon}{270}\,\big(8910+7455\,\pi^2+768\,\pi^4+7470\,\zeta_3+350\,\pi^2\,\zeta_3+8082\,\zeta_5\big) \\ &+ \frac{\epsilon^2}{1620}\Big(-480195+118755\,\pi^2+22626\,\pi^4+1093\,\pi^6+273600\,\zeta_3 \\ &\qquad +16800\,\pi^2\,\zeta_3-40740\,\zeta_3^2+387936\,\zeta_5\Big) + \mathcal{O}(\epsilon^3)\,, \end{aligned} \quad (78)$$



$T_{9,2}$

$$\begin{aligned} &= \frac{5\,\zeta_5}{\epsilon} + \frac{1}{378}\,\big(5\,\pi^6+6426\,\zeta_3^2+5670\,\zeta_5\big) + \frac{\epsilon}{630}\Big(25\,\pi^6+357\,\pi^4\,\zeta_3 \\ &+ 32130\,\zeta_3^2+22050\,\zeta_5+7350\,\pi^2\,\zeta_5+70875\,\zeta_7\Big) + \mathcal{O}(\epsilon^2)\,. \end{aligned} \quad (79)$$

The integrals (70)-(77) are easily found via a repeated application of well-known analytical formulas for a one-loop massive tadpole and a one-loop massless propagator. A less simple diagram (78) can be extracted from [35–37] while the most complicated non-planar one (79) from [38, 39].

B Relations among particular orders of different master integrals of the standard basis

In this appendix we present the relations among particular orders of different master integrals as discussed in section 3:

$$T^{(0)}_{6,13} - T^{(0)}_{6,18} = -\frac{11561}{128} - \frac{1685\,\pi^2}{288} - \frac{\pi^4}{4} + \frac{9\,\sqrt{3}\,s_2}{2} + \frac{166\,\zeta_3}{9}, \tag{80}$$

$$T^{(0)}_{7,10} - T^{(0)}_{7,11} = \frac{567}{4} + \frac{179\,\pi^2}{36} + \frac{187\,\pi^4}{360} + 4\,\sqrt{3}\,s_2 + 2\,s_2^2 - \frac{388\,\zeta_3}{9}, \tag{81}$$

$$2\,T^{(0)}_{6,13} - 3\,T^{(0)}_{6,14} = \frac{89}{4} + \frac{31\,\pi^2}{36} + \frac{13\,\pi^4}{60} + 12\,\sqrt{3}\,s_2 + 6\,s_2^2 - \frac{131\,\zeta_3}{18}, \tag{82}$$

$$2\,T^{(0)}_{5,10} - 3\,T^{(0)}_{6,10} = \frac{3651}{16} + 5\,\pi^2 + \frac{11\,\pi^4}{40} - 9\,\sqrt{3}\,s_2 + 12\,s_2^2 - \frac{379\,\zeta_3}{6}, \tag{83}$$

$$2\,T^{(0)}_{5,10} - T^{(0)}_{6,9} = \frac{1731}{16} + \frac{13\,\pi^2}{9} + \frac{11\,\pi^4}{30} - 19\,\sqrt{3}\,s_2 + 16\,s_2^2 - \frac{211\,\zeta_3}{6}, \tag{84}$$

$$\begin{aligned} &6\,T^{(1)}_{5,9} + T^{(0)}_{5,10} + T^{(0)}_{6,11} = \\ &\quad \frac{2097}{64} + \frac{241\,\pi^2}{48} - \frac{35\,\pi^4}{72} + 20\,\sqrt{3}\,s_2 - 20\,s_2^2 - \frac{\zeta_3}{9}, \end{aligned} \tag{85}$$

$$\begin{aligned} &6\,T^{(1)}_{5,9} - T^{(0)}_{5,10} + 3\,T^{(0)}_{6,12} = \\ &\quad -\frac{5435}{64} - \frac{247\,\pi^2}{48} - \frac{\pi^4}{2} + 9\,\sqrt{3}\,s_2 - 24\,s_2^2 + \frac{43\,\zeta_3}{6}, \end{aligned} \tag{86}$$

$$\begin{aligned} &4\,T^{(0)}_{6,13} - 2\,T^{(1)}_{6,13} + 3\,T^{(1)}_{6,14} - T^{(0)}_{8,10} = \\ &\quad -\frac{323}{4} + \frac{71\,\pi^2}{12} - \frac{19\,\pi^4}{36} + \frac{95\,\zeta_3}{9} + \frac{86\,\pi^2\,\zeta_3}{27} + \frac{328\,\zeta_5}{5}, \end{aligned} \tag{87}$$

$$\begin{aligned} &T^{(0)}_{5,10} + T^{(0)}_{6,17} + T^{(0)}_{7,5} = \\ &\quad \frac{50447}{192} + \frac{2671\,\pi^2}{144} + \frac{71\,\pi^4}{180} - 9\,\sqrt{3}\,s_2 - \frac{45\,\zeta_3}{2} + \frac{\pi^2\,\zeta_3}{9} - \frac{437\,\zeta_5}{9}, \end{aligned} \tag{88}$$

$$\begin{aligned} &5\,T^{(1)}_{5,8} - 21\,T^{(1)}_{5,9} - 2\,T^{(1)}_{7,14} = \\ &\quad -\frac{2716223}{2592} - \frac{3533\,\pi^2}{216} - \frac{293\,\pi^4}{180} - \frac{467\,s_2}{\sqrt{3}} \\ &\quad -\frac{4\,s_2^2}{3} + \frac{2585\,\zeta_3}{9} + \frac{46\,\pi^2\,\zeta_3}{27} + \frac{263\,\zeta_5}{5}, \end{aligned} \tag{89}$$

$$
\begin{aligned}
&T^{(1)}_{5,8} - 9\,T^{(1)}_{5,9} - 2\,T^{(1)}_{7,10} + 2\,T^{(1)}_{7,11} = \\
&\qquad -\frac{3471265}{2592} - \frac{4543\,\pi^2}{216} - \frac{229\,\pi^4}{45} - \frac{217\,s_2}{\sqrt{3}} \\
&\qquad +\frac{40\,s_2^2}{3} + \frac{893\,\zeta_3}{3} - \frac{28\,\pi^2\,\zeta_3}{27} + \frac{2133\,\zeta_5}{5}, \qquad (90)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(1)}_{5,8} - 15\,T^{(1)}_{5,9} - 2\,T^{(1)}_{6,16} = \\
&\qquad \frac{8415131}{10368} - \frac{39595\,\pi^2}{864} + \frac{1039\,\pi^4}{180} - \frac{349\,s_2}{\sqrt{3}} + \frac{298\,s_2^2}{3} + \frac{64\,\pi^2\,\log^2(2)}{3} \\
&\qquad -\frac{64\,\log^4(2)}{3} - 512\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{11135\,\zeta_3}{18} + \frac{10\,\pi^2\,\zeta_3}{3} + \frac{51\,\zeta_5}{5}, \qquad (91)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(1)}_{5,8} + 15\,T^{(1)}_{5,9} + 2\,T^{(0)}_{5,10} + 4\,T^{(1)}_{5,10} + 2\,T^{(1)}_{6,11} - 6\,T^{(1)}_{6,12} + 2\,T^{(0)}_{7,16} = \\
&\qquad \frac{957005}{2592} + \frac{11027\,\pi^2}{216} - \frac{23\,\pi^4}{30} + \frac{53\,s_2}{\sqrt{3}} - \frac{140\,s_2^2}{3} \\
&\qquad -\frac{238\,\zeta_3}{9} - \frac{101\,\pi^2\,\zeta_3}{27} + \frac{974\,\zeta_5}{15}, \qquad (92)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(1)}_{5,8} + 15\,T^{(1)}_{5,9} - 2\,T^{(0)}_{5,10} + 4\,T^{(1)}_{5,10} + 2\,T^{(1)}_{6,11} - 6\,T^{(1)}_{6,12} + 12\,T^{(0)}_{6,13} - 4\,T^{(1)}_{6,13} \\
&\qquad +6\,T^{(1)}_{6,14} + 6\,T^{(0)}_{7,8} = \frac{2290913}{2592} + \frac{14459\,\pi^2}{216} - \frac{79\,\pi^4}{45} + \frac{179\,s_2}{\sqrt{3}} \\
&\qquad -\frac{212\,s_2^2}{3} + \frac{100\,\zeta_3}{9} + \frac{53\,\pi^2\,\zeta_3}{27} + \frac{634\,\zeta_5}{5}, \qquad (93)
\end{aligned}
$$

$$
\begin{aligned}
&4\,T^{(1)}_{5,8} - 12\,T^{(1)}_{5,9} - 5\,T^{(0)}_{5,10} - 2\,T^{(1)}_{5,10} - T^{(1)}_{6,11} + 3\,T^{(1)}_{6,12} + 12\,T^{(0)}_{6,13} \\
&\qquad -4\,T^{(1)}_{6,13} + 6\,T^{(1)}_{6,14} - 3\,T^{(0)}_{7,9} = -\frac{964973}{1296} - \frac{6613\,\pi^2}{216} - \frac{43\,\pi^4}{72} \\
&\qquad -\frac{184\,s_2}{\sqrt{3}} + \frac{52\,s_2^2}{3} + \frac{1618\,\zeta_3}{9} + \frac{29\,\pi^2\,\zeta_3}{27} + 16\,\zeta_5, \qquad (94)
\end{aligned}
$$

$$
\begin{aligned}
&T^{(1)}_{5,8} + 15\,T^{(1)}_{5,9} + T^{(0)}_{5,10} + 4\,T^{(1)}_{5,10} + 2\,T^{(1)}_{6,11} - 6\,T^{(1)}_{6,12} + 6\,T^{(0)}_{6,13} - 4\,T^{(1)}_{6,13} \\
&\qquad +6\,T^{(1)}_{6,14} + 3\,T^{(0)}_{6,17} - 3\,T^{(0)}_{7,7} = \frac{1284721}{5184} + \frac{29521\,\pi^2}{432} - \frac{313\,\pi^4}{180} \\
&\qquad +\frac{8\,s_2}{\sqrt{3}} - \frac{176\,s_2^2}{3} - \frac{2551\,\zeta_3}{18} + \frac{35\,\pi^2\,\zeta_3}{27} + \frac{5372\,\zeta_5}{15}, \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
&7\,T^{(1)}_{5,8} - 31\,T^{(1)}_{5,9} - 3\,T^{(0)}_{5,10} - 2\,T^{(1)}_{5,10} - T^{(1)}_{6,11} + 3\,T^{(1)}_{6,12} + 20\,T^{(0)}_{6,13} - 8\,T^{(1)}_{6,13} \\
&\quad + 12\,T^{(1)}_{6,14} + 2\,T^{(0)}_{9,3} + 6\,T^{(0)}_{9,4} = -\frac{590641}{2592} + \frac{2627\,\pi^2}{216} + \frac{707\,\pi^4}{360} \\
&\quad - \frac{508\,s_2}{\sqrt{3}} + \frac{376\,s_2^2}{3} + \frac{53\,\pi^4\,\log(2)}{15} + \frac{4\,\pi^2\,\log^2(2)}{3} - \frac{4\,\log^4(2)}{3} \\
&\quad + \frac{16\,\pi^2\,\log^3(2)}{3} - \frac{16\,\log^5(2)}{5} - 32\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) \\
&\quad + 384\,\mathrm{Li}_5\left(\tfrac{1}{2}\right) + \frac{1817\,\zeta_3}{18} + \frac{47\,\pi^2\,\zeta_3}{18} - \frac{2127\,\zeta_5}{5}. \qquad (96)
\end{aligned}
$$

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