SFB/CPP-06-48
TTP/06-27
WUB/06-05
LTH723
NSF-KITP-06-86

# Four-loop $\beta$ function and mass anomalous dimension in Dimensional Reduction

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**Abstract**

Within the framework of QCD we compute renormalization constants for the strong coupling and the quark masses to four-loop order. We apply the $\overline{\rm DR}$ scheme and put special emphasis on the additional couplings which have to be taken into account. This concerns the $\varepsilon$-scalar–quark Yukawa coupling as well as the vertex containing four $\varepsilon$-scalars. For a supersymmetric Yang Mills theory, we find, in contrast to a previous claim, that the evanescent Yukawa coupling equals the strong coupling constant through three loops as required by supersymmetry.

PACS numbers: 11.25.Db 11.30.Pb 12.38.Bx

# 1 Introduction

Dimensional regularisation [1, 2], (DREG) is a remarkably elegant procedure which completely dominates the radiative corrections industry associated with the standard model. Advocates of alternative regularisation methods rarely proceed beyond one loop (or exceptionally two). The fundamental reason for the DREG hegemony is that (with little increase in calculational difficulty) it preserves gauge invariance; that is to say, when the effective action is separated into a finite part (which is retained) and an "infinite" part (or more precisely, a part which tends to infinity in the limit that $D = 4 - 2\epsilon \to 4$) then the finite effective action satisfies the Ward identities of the gauge symmetry, without the necessity of introducing additional finite local counter-terms.

DREG is, however, less well-suited for supersymmetric theories because invariance of a given action with respect to supersymmetric transformations only holds in general for specific values of the space-time dimension $D$. An elegant attempt to modify DREG so as to render it compatible with supersymmetry (SUSY) was made by Siegel [3]. The essential difference between Siegel's method (DRED[1]) and DREG is that the continuation from 4 to $D$ dimensions is made by *compactification*, or *dimensional reduction*. Thus while the momentum (or space-time) integrals are $D$-dimensional in the usual way, the number of field components remains unchanged and consequently SUSY is undisturbed. (A pedagogical introduction to DRED was given by Capper et al [4].)

As pointed out by Siegel himself, [5] there remain potential problems with DRED. One manifestation of this was demonstrated in Ref. [6], where it was shown that the variation $\delta S$ of the action of a pure (no chiral matter) supersymmetric gauge theory is nonzero even with DRED. If $\delta S$ gives a nonzero result when inserted in a Greens function this creates an apparent violation of supersymmetric Ward identities. With DREG this happens at one loop, but with DRED all explicit calculations to date have found zero for such insertions. For discussion see Refs. [7, 8].

We turn now to the application of DRED to non-supersymmetric theories. That DRED is a viable alternative to DREG in the non-supersymmetric case was claimed early on [4]. Subsequently it was adopted occasionally, motivated, for example, by the fact that Dirac matrix algebra is easier in four dimensions and in particular by the desire to use Fierz identities. One must, however, be very careful in applying DRED to non-supersymmetric theories because of the existence of *evanescent couplings*. These were first described [9] in 1979, and later independently by van Damme and 't Hooft [10]. After dimensional reduction the four-dimensional vector field can be decomposed into a $D$-dimensional vector field and a $2\epsilon$-dimensional which transforms under gauge transformations as a scalar and is hence known as an $\varepsilon$-scalar. Couplings involving the $\varepsilon$-scalar are called evanescent couplings; in a non-supersymmetric theory they renormalise in a manner different from the "real" couplings with which we may be tempted to associate them. It has been conclusively demonstrated [11, 12] that there exists a set of transformations whereby the $\beta$–functions of a particular theory (calculated using DRED) may be related to the $\beta$–functions of the same theory (calculated using DREG) by means of coupling constant reparametrisation. The evanescent couplings play a crucial role in this analysis, but in the literature on non-supersymmetric DRED their significance is often ignored, and there have been few calculations which explicitly take them into account. In a recent paper [13], four of us described in detail the evanescent coupling structure of QCD and calculated the gauge $\beta$-function, $\beta$, and the mass anomalous dimension $\gamma_m$ through three loops. We found that at three loops $\beta$ depends on the $\varepsilon$-scalar Yukawa coupling $g_e$, while $\gamma_m$

[1]Dimensional reduction in combination with modified minimal subtraction is often known as $\overline{\mathrm{DR}}$; we will use the terms DRED and $\overline{\mathrm{DR}}$ interchangeably.

depends on both $g_e$ and the $\varepsilon$-scalar quartic self couplings, $\lambda_r$. In this paper we extend these calculations to the four-loop level, when $\beta$ also depends on $\lambda_r$. These results bring the precision of our knowledge of these quantities in DRED up to that in DREG [14–17].

## 2 Evanescent couplings

The technical framework of our calculation is described in detail in Ref. [13]. Let us at this point emphasise once more the role of the evanescent couplings and in particular elaborate on the $\varepsilon\varepsilon\varepsilon\varepsilon$ vertex. The part of the Lagrange density describing the latter is given by

$$\mathcal{L} \;=\; \ldots - \frac{1}{4}\sum_{r=1}^{R} \lambda_r H_r^{abcd} \varepsilon_\sigma^a \varepsilon_{\sigma'}^c \varepsilon_\sigma^b \varepsilon_{\sigma'}^d + \ldots\,, \tag{1}$$

where the $\varepsilon_\sigma^a$ are the $\varepsilon$-scalar fields, and $\sigma$, $\sigma'$ are $2\varepsilon$-dimensional indices. For the gauge group $SU(N)$ the dimensionality $R$ of the basis for rank-four tensors is given by $R = 3$ for $SU(2)$, $R = 8$ for $SU(3)$ and $R = 9$ for $SU(N)$, $N \geq 4$; for tensors $H_r^{abcd}$ symmetric with respect to $(a, b)$ and $(c, d)$ exchange these numbers become $R = 2$, $R = 3$ and $R = 4$ respectively [11]. We will restrict ourselves to $SU(3)$; our basis choice reads

$$\begin{aligned}
H_1^{abcd} &= \frac{1}{2}\left(f^{ace}f^{bde} + f^{ade}f^{bce}\right)\,,\\
H_2^{abcd} &= \delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\,,\\
H_3^{abcd} &= \frac{1}{2}\left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) - \delta^{ab}\delta^{cd}\,.
\end{aligned} \tag{2}$$

Note that dimensional reduction of the original action yields $\lambda_1 = g^2$, $\lambda_2 = \lambda_3 = 0$, but, as we have already emphasised, this situation is not preserved by renormalisation.

Once the tensors $H_r^{abcd}$ are chosen, the Feynman rules are fixed in a unique way. It is straightforward to relate the result obtained from a different choice of the $H_r^{abcd}$ to each other. For example, for

$$\begin{aligned}
\tilde{H}_1^{abcd} &= \frac{1}{2}\delta^{ab}\delta^{cd}\,,\\
\tilde{H}_2^{abcd} &= \frac{1}{2}\left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right)\,,\\
\tilde{H}_3^{abcd} &= \frac{1}{2}\left(f^{ace}f^{bde} + f^{ade}f^{bce}\right)\,,
\end{aligned} \tag{3}$$

one obtains

$$\begin{aligned}
\lambda_1 &= \tilde{\lambda_3}\,,\\
\lambda_2 &= \frac{1}{6}\left(\tilde{\lambda_1} + 2\tilde{\lambda_2}\right)\,,\\
\lambda_3 &= \frac{1}{3}\left(-\tilde{\lambda_1} + \tilde{\lambda_2}\right)\,.
\end{aligned} \tag{4}$$

The renormalization constants for the evanescent couplings are defined through

$$
\begin{aligned}
g_e^0 &= \mu^\epsilon Z_e g_e\,, & \sqrt{\lambda_r^0} &= \mu^\epsilon Z_{\lambda_r}\sqrt{\lambda_r}\,, & \varepsilon_\sigma^{0,a} &= \sqrt{Z_3^\varepsilon}\varepsilon_\sigma^a\,,\\
\Gamma^0_{q\bar{q}\varepsilon} &= Z_1^\varepsilon \Gamma_{q\bar{q}\varepsilon}\,, & \Gamma^{r,0}_{\varepsilon\varepsilon\varepsilon\varepsilon} &= Z_1^r\Gamma^r_{\varepsilon\varepsilon\varepsilon\varepsilon}\,, & & \qquad (5)
\end{aligned}
$$

where $\Gamma_{q\bar{q}\varepsilon}$ and $\Gamma_{\varepsilon\varepsilon\varepsilon\varepsilon}$ are the one-particle irreducible $\varepsilon$-scalar–quark and four-$\varepsilon$-scalar Green functions, respectively, the superscript "0" denotes bare quantities, and $\mu$ is the renormalization scale. The charge renormalization constants are obtained from the following relations

$$
Z_e = \frac{Z_1^\varepsilon}{Z_2\sqrt{Z_3^\varepsilon}}\,, \qquad Z_{\lambda_r} = \frac{\sqrt{Z_1^r}}{Z_3^\varepsilon}\,, \tag{6}
$$

with $Z_2$ being the wave function renormalization constant of the quark fields.

Let us introduce the couplings

$$
\alpha_s = \frac{g_s^2}{4\pi}\,, \quad \alpha_e = \frac{g_e^2}{4\pi} \quad \text{and} \quad \eta_r = \frac{\lambda_r}{4\pi}\,, \tag{7}
$$

and define the corresponding $\beta$ functions in the $\overline{\text{DR}}$ scheme:

$$
\begin{aligned}
\beta_s^{\overline{\text{DR}}}(\alpha_s^{\overline{\text{DR}}},\alpha_e,\{\eta_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\\
&= -\left[\epsilon\frac{\alpha_s^{\overline{\text{DR}}}}{\pi} + 2\frac{\alpha_s^{\overline{\text{DR}}}}{Z_s^{\overline{\text{DR}}}}\left(\frac{\partial Z_s^{\overline{\text{DR}}}}{\partial\alpha_e}\beta_e + \sum_r\frac{\partial Z_s^{\overline{\text{DR}}}}{\partial\eta_r}\beta_{\eta_r}\right)\right]\left(1+2\frac{\alpha_s^{\overline{\text{DR}}}}{Z_s^{\overline{\text{DR}}}}\frac{\partial Z_s^{\overline{\text{DR}}}}{\partial\alpha_s^{\overline{\text{DR}}}}\right)^{-1}\\
&= -\epsilon\frac{\alpha_s^{\overline{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m}\beta^{\overline{\text{DR}}}_{ijklm}\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{\eta_1}{\pi}\right)^k\left(\frac{\eta_2}{\pi}\right)^l\left(\frac{\eta_3}{\pi}\right)^m\,, \qquad (8)\\
\beta_e(\alpha_s^{\overline{\text{DR}}},\alpha_e,\{\eta_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha_e}{\pi}\\
&= -\left[\epsilon\frac{\alpha_e}{\pi} + 2\frac{\alpha_e}{Z_e}\left(\frac{\partial Z_e}{\partial\alpha_s^{\overline{\text{DR}}}}\beta_s^{\overline{\text{DR}}} + \sum_r\frac{\partial Z_e}{\partial\eta_r}\beta_{\eta_r}\right)\right]\left(1+2\frac{\alpha_e}{Z_e}\frac{\partial Z_e}{\partial\alpha_e}\right)^{-1}\\
&= -\epsilon\frac{\alpha_e}{\pi} - \sum_{i,j,k,l,m}\beta^{e}_{ijklm}\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{\eta_1}{\pi}\right)^k\left(\frac{\eta_2}{\pi}\right)^l\left(\frac{\eta_3}{\pi}\right)^m\,, \qquad (9)\\
\beta_{\eta_r}(\alpha_s^{\overline{\text{DR}}},\alpha_e,\{\eta_r\}) &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\eta_r}{\pi}\\
&= -\left[\epsilon\frac{\eta_r}{\pi} + 2\frac{\eta_r}{Z_{\lambda_r}}\left(\frac{\partial Z_{\lambda_r}}{\partial\alpha_s^{\overline{\text{DR}}}}\beta_s^{\overline{\text{DR}}} + \frac{\partial Z_{\lambda_r}}{\partial\alpha_e}\beta_e + \sum_{r'\neq r}\frac{\partial Z_{\lambda_r}}{\partial\eta_{r'}}\beta_{\eta_{r'}}\right)\right]\left(1+2\frac{\eta_r}{Z_{\lambda_r}}\frac{\partial Z_{\lambda_r}}{\partial\eta_r}\right)^{-1}\\
&= -\epsilon\frac{\eta_r}{\pi} - \sum_{i,j,k,l,m}\beta^{\eta_r}_{ijklm}\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{\eta_1}{\pi}\right)^k\left(\frac{\eta_2}{\pi}\right)^l\left(\frac{\eta_3}{\pi}\right)^m\,. \qquad (10)
\end{aligned}
$$

Here and in the following we do not explicitely display the dependence on the renormalization scale $\mu$, i.e., $\alpha_s \equiv \alpha_s(\mu)$ etc. Note that in the second line of Eq. (8), the $\mathcal{O}(\epsilon)$ terms of $\beta_e$ and $\beta_{\eta_r}$ contribute to the finite part of $\beta_s^{\overline{\rm DR}}$, and similarly for Eqs. (9) and (10). As we will see below, in order to compute the four-loop term of $\beta^{\overline{\rm DR}}$ one needs $\beta_e$ to two-loop and $\beta_{\eta_r}$ $(r = 1, 2, 3)$ to one-loop order.

In analogy to the $\beta$ functions we introduce the quark mass anomalous dimension which is defined through

$$\begin{aligned}
\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}}, \alpha_e, \{\eta_r\}) &= \frac{\mu^2}{m^{\overline{\rm DR}}} \frac{\mathrm{d}}{\mathrm{d}\mu^2} m^{\overline{\rm DR}} \\
&= -\pi\beta_s^{\overline{\rm DR}} \frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm DR}}} - \pi\beta_e \frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial \alpha_e} - \pi \sum_r \beta_{\eta_r} \frac{\partial \ln Z_m^{\overline{\rm DR}}}{\partial \eta_r} \\
&= -\sum_{i,j,k,l,m} \gamma_{ijklm}^{\overline{\rm DR}} \left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i \left(\frac{\alpha_e}{\pi}\right)^j \left(\frac{\eta_1}{\pi}\right)^k \left(\frac{\eta_2}{\pi}\right)^l \left(\frac{\eta_3}{\pi}\right)^m . \qquad (11)
\end{aligned}$$

From this equation one can see that for the four-loop term of $\gamma_m^{\overline{\rm DR}}$, the beta functions $\beta_e$ and $\beta_{\eta_r}$ are needed to three- and one-loop order, respectively, since the dependence of $Z_m^{\overline{\rm DR}}$ on $\alpha_e$ $(\eta_r)$ starts at one-loop (three-loop) order [13].

The one-loop terms for $\beta_{\eta_r}$ and the three-loop expression for $\beta_e$ can be computed using standard techniques (see e.g. Ref. [18]), leading to the following non-vanishing coefficients in Eqs. (9) and (10):

$$\begin{aligned}
\beta^e_{04000} &= -\frac{55}{432}-\frac{91}{48}\zeta_3-\left(\frac{725}{1152}-\frac{17}{96}\zeta_3\right)n_f+\frac{55}{768}n_f^2\,,\\
\beta^e_{13000} &= \frac{2423}{1728}+\frac{5}{36}\zeta_3-\left(\frac{313}{288}+\frac{5}{24}\zeta_3\right)n_f+\frac{9}{64}n_f^2\,,\\
\beta^e_{22000} &= \frac{189157}{13824}-\frac{11}{16}\zeta_3-\left(\frac{35543}{9216}-\frac{73}{32}\zeta_3\right)n_f+\frac{55}{768}n_f^2\,,\\
\beta^e_{31000} &= \frac{4589}{512}+\left(\frac{1157}{6912}-\frac{5}{3}\zeta_3\right)n_f-\frac{415}{5184}n_f^2\,,\\
\beta^e_{03100} &= -\frac{9}{64}+\frac{243}{128}n_f\,,\qquad \beta^e_{03010} = \frac{5}{8}-\frac{45}{64}n_f\,,\\
\beta^e_{03001} &= \frac{3}{32}-\frac{81}{64}n_f\,,\qquad \beta^e_{12100} = -\frac{219}{16}\,,\qquad \beta^e_{12010} = \frac{145}{48}\,,\\
\beta^e_{12001} &= \frac{73}{8}\,,\qquad \beta^e_{02200} = \frac{1413}{512}-\frac{729}{1024}n_f\,,\qquad \beta^e_{02020} = -\frac{115}{32}+\frac{135}{64}n_f\,,\\
\beta^e_{02002} &= -\frac{161}{256}-\frac{567}{512}n_f\,,\qquad \beta^e_{02110} = \frac{75}{8}\,,\qquad \beta^e_{02101} = -\frac{471}{128}+\frac{243}{256}n_f\,,\\
\beta^e_{02011} &= -\frac{85}{8}\,,\qquad \beta^e_{21100} = -\frac{1125}{1024}\,,\qquad \beta^e_{21010} = \frac{105}{128}\,,\qquad \beta^e_{21001} = \frac{615}{512}\,,\\
\beta^e_{11200} &= \frac{891}{128}\,,\qquad \beta^e_{11020} = -\frac{45}{4}\,,\qquad \beta^e_{11002} = \frac{693}{64}\,,\qquad \beta^e_{11101} = -\frac{297}{32}\,,\\
\beta^e_{01300} &= -\frac{1701}{1024}\,,\qquad \beta^e_{01003} = \frac{63}{128}\,,\qquad \beta^e_{01210} = -\frac{405}{128}\,,\qquad \beta^e_{01201} = \frac{1701}{512}\,,\\
\beta^e_{01120} &= \frac{135}{32}\,,\qquad \beta^e_{01021} = -\frac{315}{32}\,,\qquad \beta^e_{01102} = -\frac{81}{128}\,,\\
\beta^e_{01012} &= -\frac{315}{32}\,,\qquad \beta^e_{01111} = \frac{135}{16}\,,
\end{aligned}\tag{12}$$

$$\begin{aligned}
\beta^{\eta_1}_{20000} &= -\frac{3}{8}\,,\qquad \beta^{\eta_1}_{10100} = \frac{9}{2}\,,\qquad \beta^{\eta_1}_{02000} = \frac{1}{3}n_f\,,\qquad \beta^{\eta_1}_{01100} = -\frac{1}{2}n_f\,,\\
\beta^{\eta_1}_{00200} &= -\frac{11}{8}\,,\qquad \beta^{\eta_1}_{00110} = -2\,,\qquad \beta^{\eta_1}_{00101} = \frac{7}{2}\,,\\
\beta^{\eta_2}_{20000} &= -\frac{9}{16}\,,\qquad \beta^{\eta_2}_{10010} = \frac{9}{2}\,,\qquad \beta^{\eta_2}_{02000} = \frac{1}{24}n_f\,,\qquad \beta^{\eta_2}_{01010} = -\frac{1}{2}n_f\,,\\
\beta^{\eta_2}_{00200} &= \frac{3}{16}\,,\qquad \beta^{\eta_2}_{00110} = \frac{1}{2}\,,\qquad \beta^{\eta_2}_{00101} = -\frac{1}{2}\,,\\
\beta^{\eta_2}_{00020} &= -\frac{32}{3}\,,\qquad \beta^{\eta_2}_{00011} = -\frac{7}{6}\,,\qquad \beta^{\eta_2}_{00002} = \frac{7}{12}\,,\\
\beta^{\eta_3}_{10001} &= \frac{9}{2}\,,\qquad \beta^{\eta_3}_{01001} = -\frac{1}{2}n_f\,,\qquad \beta^{\eta_3}_{00110} = 2\,,\qquad \beta^{\eta_3}_{00101} = \frac{5}{2}\,,\\
\beta^{\eta_3}_{00020} &= \frac{10}{3}\,,\qquad \beta^{\eta_3}_{00011} = -\frac{20}{3}\,,\qquad \beta^{\eta_3}_{00002} = -\frac{7}{6}\,,
\end{aligned}\tag{13}$$

where $n_f$ is the number of active quark flavours and $\zeta_3 = \zeta(3) = 1.20206...$, where $\zeta$ is Riemann's zeta function. The one- and two-loop result of $\beta_e$ can be found in Ref. [13].

# 3 $\beta_e^{\rm SYM}$ and $\beta_s^{\rm SYM}$ to three loops

In order to check our technical setup, we calculated the $\beta$ functions of the strong and the evanescent coupling constant for a supersymmetric Yang Mills (SYM) theory.

Such a theory is obtained from massless QCD when replacing the quarks by the SUSY partner of the gluon, the so-called gluino. Technically, this amounts to changing the colour-matrix for the fermion-gluon coupling of QCD from the fundamental to the adjoint representation of the gauge group. In addition, closed fermion loops have to be multiplied by an extra factor 1/2 in order to take into account the Majorana character of the gluino.

SUSY requires that the gluino-gluon coupling $\alpha_s$ equals the gluino-$\varepsilon$-scalar coupling $\alpha_e$ to all orders of perturbation theory, and therefore $\beta_e^{\rm SYM} = \beta_s^{\rm SYM}$ for the respective $\beta$ functions. However, it was found in Ref. [19] that this equality is violated at the three-loop level. The result was interpreted [19,20] as a manifestation of explicit SUSY breaking by DRED, when employed in the component (as opposed to the superfield) formalism.

Using the approach described above, we re-calculated $\beta_s^{\rm SYM}$ and $\beta_e^{\rm SYM}$ through three-loop order. When we set $\alpha_e = \alpha_s$, the result for $\beta_s^{\rm SYM}$ agrees with Ref. [21] ($C_A$ is the Casimir of the adjoint representation of the gauge group):

$$\beta_s^{\rm SYM} = -\left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{3}{4}C_A + \frac{3}{8}C_A^2\frac{\alpha_s}{\pi} + \frac{21}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2\right] + \mathcal{O}(\alpha_s^5)\,. \tag{14}$$

However, in contrast to Ref. [19], we find that

$$\beta_e^{\rm SYM} = \beta_s^{\rm SYM} + \mathcal{O}(\alpha_s^5)\,. \tag{15}$$

We draw the following conclusions from this result: (i) The expression quoted in Ref. [19] for the three-loop expression of $\beta_e$ is incorrect; considering the fact that this calculation has been done almost 25 years ago, it is probably impossible to trace the origin of the difference to our result; (ii) in a supersymmetric Yang Mills theory, the evanescent coupling constant $\alpha_e$ renormalises in the same way as the strong coupling constant $\alpha_s$ up to three-loop level, as required by SUSY; (iii) the setup of our calculation has passed a strong consistency check.

# 4 $\beta^{\overline{\rm DR}}$ and $\gamma_m^{\overline{\rm DR}}$ to four loops

The direct way to compute the $\beta_s$ and $\gamma_m$ function is based on the evaluation of the corresponding renormalization constants. For such a calculation one can exploit that the divergent parts of a logarithmically divergent integral is independent of the masses and external momenta. Thus the latter can be chosen in a convenient way (provided no infrared divergences are introduced). Up to three loops this procedure is quite well established and automated programs exist to perform such calculations (see, e.g., Refs. [22,23]). Also at four-loop order this approach is feasible, however, technically quite challenging. Thus we decided to adopt the indirect method discussed in Refs. [13,24]. It is based on the following formulæ relating the quantities in DRED and DREG:

$$
\begin{aligned}
\beta_s^{\overline{\rm DR}} &= \beta_s^{\overline{\rm MS}} \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \beta_e \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \alpha_e} + \sum_r \beta_{\eta_r} \frac{\partial \alpha_s^{\overline{\rm DR}}}{\partial \eta_r}\,, \\
\gamma_m^{\overline{\rm DR}} &= \gamma_m^{\overline{\rm MS}} \frac{\partial \ln m^{\overline{\rm DR}}}{\partial \ln m^{\overline{\rm MS}}} + \frac{\pi \beta_s^{\overline{\rm MS}}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_s^{\overline{\rm MS}}} + \frac{\pi \beta_e}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \alpha_e} + \sum_r \frac{\pi \beta_{\eta_r}}{m^{\overline{\rm DR}}} \frac{\partial m^{\overline{\rm DR}}}{\partial \eta_r}\,. 
\end{aligned}
\tag{16}
$$

Let us in the following briefly discuss the order in perturbation theory up to which the individual building blocks are needed. Of course, the $\overline{\rm MS}$ quantities are needed to four-loop order; they can be found in Refs. [14–17]. The dependence of $\alpha_s^{\overline{\rm DR}}$ and $m^{\overline{\rm DR}}$ on $\alpha_e$ starts at two- and one-loop order [13], respectively. Thus, $\beta_e$ is needed up to the three-loop level (cf. Eq. (12)). On the other hand, both $\alpha_s^{\overline{\rm DR}}$ and $m^{\overline{\rm DR}}$ depend on $\eta_r$ starting from three loops and consequently only the one-loop term of $\beta_{\eta_r}$ enters in Eq. (16). It is given in Eq. (13).

Two further new ingredients are needed for the four-loop analysis, namely, the three-loop relations between $\alpha_s^{\overline{\rm DR}}$ and $\alpha_s^{\overline{\rm MS}}$ and between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$. The two-loop results have already been presented in Ref. [13]. Parametrising the three-loop terms by $\delta_\alpha^{(3)}$ and $\delta_m^{(3)}$, we have

$$
\begin{aligned}
\alpha_s^{\overline{\rm DR}} &= \alpha_s^{\overline{\rm MS}} \left[ 1 + \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{1}{4} + \left( \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \right)^2 \frac{11}{8} - \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{\alpha_e}{\pi} \frac{1}{12} n_f + \delta_\alpha^{(3)} + \ldots \right]\,, \\
m^{\overline{\rm DR}} &= m^{\overline{\rm MS}} \Bigg[ 1 - \frac{\alpha_e}{\pi} \frac{1}{3} + \left( \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \right)^2 \frac{11}{48} - \frac{\alpha_s^{\overline{\rm MS}}}{\pi} \frac{\alpha_e}{\pi} \frac{59}{72} \\
&\quad + \left( \frac{\alpha_e}{\pi} \right)^2 \left( \frac{1}{6} + \frac{1}{48} n_f \right) + \delta_m^{(3)} + \ldots \Bigg]\,,
\end{aligned}
\tag{17}
$$

where the dots denote higher orders in $\alpha_s^{\overline{\rm MS}}$, $\alpha_e$, and $\eta_r$. $\delta_\alpha^{(3)}$ and $\delta_m^{(3)}$ are obtained from the finite parts of three-loop diagrams (see Ref. [13] for details). We find

$$
\begin{aligned}
\delta_\alpha^{(3)} &= \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3 \left(\frac{3049}{384} - \frac{179}{864}n_f\right) \\
&\quad + \frac{\left(\alpha_s^{\overline{\rm MS}}\right)^2}{\pi^3} \left(-\eta_1 \frac{9}{256} + \eta_2 \frac{15}{32} + \eta_3 \frac{3}{128} - \alpha_e \frac{887}{1152} n_f\right) \\
&\quad + \frac{\alpha_s^{\overline{\rm MS}}}{\pi^3} \left[\eta_1^2 \frac{27}{256} - \eta_2^2 \frac{15}{16} - \eta_1\eta_3 \frac{9}{64} + \eta_3^2 \frac{21}{128} + \alpha_e^2 \left(\frac{43}{864} n_f + \frac{19}{1152} n_f^2\right)\right] , \\
\delta_m^{(3)} &= \left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^3 \left(\frac{2207}{864} + \frac{19}{648}n_f\right) - \frac{\left(\alpha_s^{\overline{\rm MS}}\right)^2 \alpha_e}{\pi^3} \left(\frac{62815}{20736} + \frac{253}{1728} n_f - \frac{25}{72} n_f^2\right) \\
&\quad + \frac{\alpha_s^{\overline{\rm MS}} \alpha_e^2}{\pi^3} \left[\frac{1973}{2592} - \frac{5}{36}\zeta_3 + \left(\frac{103}{1728} + \frac{5}{36}\zeta_3\right) n_f\right] \\
&\quad - \left(\frac{\alpha_e}{\pi}\right)^3 \left(\frac{7}{144} + \frac{5}{216}\zeta_3 + \frac{31}{576} n_f - \frac{5}{576} n_f^2\right) - \frac{\alpha_e^2 \eta_2}{\pi^3} \frac{5}{24} \\
&\quad - \frac{\alpha_e}{\pi^3} \left(\eta_1^2 \frac{9}{256} - \eta_2^2 \frac{15}{16} - \eta_1\eta_3 \frac{3}{64} + \eta_3^2 \frac{7}{128}\right) .
\end{aligned}
\tag{18}
$$

We performed the corresponding calculation for arbitrary gauge parameter and use the independence of the $\overline{\rm MS}$–$\overline{\rm DR}$ relation as a check of our result. Furthermore, let us stress that also the cancellation of the explicit $\ln \mu^2$ terms which occur at intermediate steps of the calculation is non-trivial.

Inserting Eq. (18) into Eq. (16) gives for the four-loop coefficients of the $\beta$ function

$$\begin{aligned}
\beta^{\overline{\rm DR}}_{50000} &= \frac{\beta_3}{256}+\frac{166861}{6144}-\frac{9109}{6912}n_f+\frac{457}{20736}n_f^2\,, \qquad \beta^{\overline{\rm DR}}_{41000} = -\frac{1667}{512}n_f+\frac{145}{2304}n_f^2\,,\\
\beta^{\overline{\rm DR}}_{32000} &= -\frac{409}{6912}n_f+\frac{1303}{4608}n_f^2\,, \qquad \beta^{\overline{\rm DR}}_{23000} = \frac{5}{1296}n_f-\frac{49}{3456}n_f^2-\frac{19}{2304}n_f^3\,,\\
\beta^{\overline{\rm DR}}_{40100} &= -\frac{171}{512}+\frac{3}{512}n_f\,, \qquad \beta^{\overline{\rm DR}}_{40010} = \frac{285}{64}-\frac{5}{64}n_f\,, \qquad \beta^{\overline{\rm DR}}_{40001} = \frac{57}{256}-\frac{1}{256}n_f\,,\\
\beta^{\overline{\rm DR}}_{31100} &= \frac{9}{512}n_f\,, \qquad \beta^{\overline{\rm DR}}_{31010} = -\frac{15}{64}n_f\,, \qquad \beta^{\overline{\rm DR}}_{31001} = -\frac{3}{256}n_f\,,\\
\beta^{\overline{\rm DR}}_{30200} &= \frac{2223}{2048}\,, \qquad \beta^{\overline{\rm DR}}_{30020} = -\frac{855}{64}\,, \qquad \beta^{\overline{\rm DR}}_{30002} = \frac{441}{256}\,, \qquad \beta^{\overline{\rm DR}}_{30110} = \frac{45}{128}\,,\\
\beta^{\overline{\rm DR}}_{30101} &= -\frac{801}{512}\,, \qquad \beta^{\overline{\rm DR}}_{30011} = -\frac{45}{64}\,, \qquad \beta^{\overline{\rm DR}}_{22100} = \frac{21}{128}n_f\,, \qquad \beta^{\overline{\rm DR}}_{22010} = -\frac{35}{192}n_f\,,\\
\beta^{\overline{\rm DR}}_{22001} &= -\frac{7}{64}n_f\,, \qquad \beta^{\overline{\rm DR}}_{21200} = -\frac{9}{64}n_f\,, \qquad \beta^{\overline{\rm DR}}_{21020} = \frac{5}{4}n_f\,, \qquad \beta^{\overline{\rm DR}}_{21002} = -\frac{7}{32}n_f\,,\\
\beta^{\overline{\rm DR}}_{21101} &= \frac{3}{16}n_f\,, \qquad \beta^{\overline{\rm DR}}_{20300} = -\frac{297}{1024}\,, \qquad \beta^{\overline{\rm DR}}_{20030} = 20\,, \qquad \beta^{\overline{\rm DR}}_{20003} = -\frac{49}{128}\,,\\
\beta^{\overline{\rm DR}}_{20210} &= -\frac{135}{128}\,, \qquad \beta^{\overline{\rm DR}}_{20201} = \frac{297}{512}\,, \qquad \beta^{\overline{\rm DR}}_{20120} = -\frac{45}{32}\,, \qquad \beta^{\overline{\rm DR}}_{20021} = \frac{105}{32}\,,\\
\beta^{\overline{\rm DR}}_{20102} &= \frac{63}{128}\,, \qquad \beta^{\overline{\rm DR}}_{20012} = -\frac{105}{32}\,, \qquad \beta^{\overline{\rm DR}}_{20111} = \frac{45}{16}\,,
\end{aligned} \tag{19}$$

where the four-loop $\overline{\rm MS}$ coefficient $\beta_3$ is given in Eq. (8) of Ref. [14].

Similarly we obtain for the four-loop coefficient of the mass anomalous dimension

$$
\begin{aligned}
\gamma^{\overline{\rm DR}}_{40000} &= \gamma_3 - \frac{18763}{2304} + \left(\frac{1}{6} + \frac{5}{8}\zeta_3\right) n_f + \frac{29}{5184} n_f^2\,,\\
\gamma^{\overline{\rm DR}}_{31000} &= -\frac{147659}{4608} + \frac{125}{48}\zeta_3 + \left(\frac{58253}{31104} + \frac{95}{216}\zeta_3\right) n_f + \frac{407}{7776} n_f^2\,,\\
\gamma^{\overline{\rm DR}}_{22000} &= -\frac{134147}{62208} - \frac{281}{432}\zeta_3 + \left(\frac{336497}{124416} + \frac{49}{432}\zeta_3\right) n_f - \left(\frac{181}{10368} + \frac{5}{216}\zeta_3\right) n_f^2\,,\\
\gamma^{\overline{\rm DR}}_{13000} &= -\frac{595}{7776} - \frac{25}{108}\zeta_3 - \left(\frac{1163}{10368} - \frac{5}{27}\zeta_3\right) n_f - \left(\frac{145}{3456} + \frac{5}{72}\zeta_3\right) n_f^2\,,\\
\gamma^{\overline{\rm DR}}_{04000} &= \frac{191}{2592} + \frac{67}{108}\zeta_3 + \left(\frac{301}{1728} - \frac{1}{24}\zeta_3\right) n_f + \frac{5}{384} n_f^2 - \frac{5}{768} n_f^3\,,\\
\gamma^{\overline{\rm DR}}_{30100} &= \frac{9}{256}\,, \quad \gamma^{\overline{\rm DR}}_{30010} = -\frac{15}{32}\,, \quad \gamma^{\overline{\rm DR}}_{30001} = -\frac{3}{128}\,, \quad \gamma^{\overline{\rm DR}}_{21100} = \frac{201}{512}\,,\\
\gamma^{\overline{\rm DR}}_{21010} &= -\frac{85}{64}\,, \quad \gamma^{\overline{\rm DR}}_{21001} = -\frac{107}{256}\,, \quad \gamma^{\overline{\rm DR}}_{20200} = -\frac{27}{256}\,, \quad \gamma^{\overline{\rm DR}}_{20020} = \frac{15}{16}\,,\\
\gamma^{\overline{\rm DR}}_{20002} &= -\frac{21}{128}\,, \quad \gamma^{\overline{\rm DR}}_{20101} = \frac{9}{64}\,, \quad \gamma^{\overline{\rm DR}}_{12100} = \frac{351}{64}\,, \quad \gamma^{\overline{\rm DR}}_{12010} = -\frac{365}{96}\,,\\
\gamma^{\overline{\rm DR}}_{12001} &= -\frac{117}{32}\,, \quad \gamma^{\overline{\rm DR}}_{11200} = -\frac{1563}{512}\,, \quad \gamma^{\overline{\rm DR}}_{11020} = \frac{1645}{96}\,, \quad \gamma^{\overline{\rm DR}}_{11002} = -\frac{3647}{768}\,,\\
\gamma^{\overline{\rm DR}}_{11101} &= \frac{521}{128}\,, \quad \gamma^{\overline{\rm DR}}_{03100} = -\frac{13}{64} - \frac{45}{64} n_f\,, \quad \gamma^{\overline{\rm DR}}_{03010} = \frac{55}{96} n_f\,,\\
\gamma^{\overline{\rm DR}}_{03001} &= \frac{13}{96} + \frac{15}{32} n_f\,, \quad \gamma^{\overline{\rm DR}}_{02200} = -\frac{223}{256} + \frac{153}{512} n_f\,, \quad \gamma^{\overline{\rm DR}}_{02020} = \frac{395}{144} - \frac{65}{32} n_f\,,\\
\gamma^{\overline{\rm DR}}_{02002} &= \frac{259}{1152} + \frac{119}{256} n_f\,, \quad \gamma^{\overline{\rm DR}}_{02110} = -\frac{155}{48}\,, \quad \gamma^{\overline{\rm DR}}_{02101} = \frac{233}{192} - \frac{51}{128} n_f\,,\\
\gamma^{\overline{\rm DR}}_{02011} &= \frac{545}{144}\,, \quad \gamma^{\overline{\rm DR}}_{01300} = \frac{333}{512}\,, \quad \gamma^{\overline{\rm DR}}_{01030} = -20\,, \quad \gamma^{\overline{\rm DR}}_{01003} = -\frac{7}{192}\,,\\
\gamma^{\overline{\rm DR}}_{01210} &= \frac{105}{64}\,, \quad \gamma^{\overline{\rm DR}}_{01201} = -\frac{333}{256}\,, \quad \gamma^{\overline{\rm DR}}_{01120} = -\frac{5}{16}\,, \quad \gamma^{\overline{\rm DR}}_{01021} = \frac{35}{48}\,,\\
\gamma^{\overline{\rm DR}}_{01102} &= \frac{3}{64}\,, \quad \gamma^{\overline{\rm DR}}_{01012} = \frac{245}{48}\,, \quad \gamma^{\overline{\rm DR}}_{01111} = -\frac{35}{8}\,,
\end{aligned}
\tag{20}
$$

where the four-loop $\overline{\rm MS}$ coefficient $\gamma_3$ can be found in Eq. (6) of Ref. [15].

# 5 The four-loop supersymmetric case

We saw in Section 3 that in the special case of SUSY, the relation $\alpha_s = \alpha_e$ is preserved by the $\beta$-functions through three loops. We now consider the supersymmetric case at the four loop level. The results in Section 4 were presented for the gauge group $SU(3)$; however, we have evaluated those parts that are not related to the evanescent couplings $\eta_2$ and $\eta_3$

also for a general Lie group $G$. It is well-known that a simple change of color factors,[2] in addition to the statistical factor 1/2 for closed fermion loops, translates these terms into a supersymmetric Yang Mills theory. In this way, we can compare our four-loop results to the gauge $\beta$-function $\beta_s^{\mathrm{SYM}}$ which was presented in 1998 [25]:

$$\beta_s^{\mathrm{SYM}} = -\left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{3}{4}C_A + \frac{3}{8}C_A^2\frac{\alpha_s}{\pi} + \frac{21}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 + \frac{51}{128}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}(\alpha_s^6)\,. \tag{21}$$

The method employed in Ref. [25] to obtain the four-loop result was very indirect, in particular relying on the NSVZ form [26, 27] of $\beta_s^{\mathrm{SYM}}$. It is therefore a remarkable check on our calculations, and indeed those of Ref. [25], that we obtain precise agreement with Eq. (21) when we adapt our calculation to the supersymmetric case, as described above. (As well as setting $\alpha_e = \alpha_s$ we must of course set $\eta_1 = \alpha_s, \eta_2 = \eta_3 = 0$).

Turning to the mass anomalous dimension we have a similar powerful check. In the supersymmetric case the fermion mass term breaks SUSY; but $\gamma_m$ (a.k.a. the gaugino $\beta$-function) is in fact given in terms of $\beta_s$ by the simple equation [28]:

$$\gamma_m^{\mathrm{SYM}} = \pi\alpha_s\frac{\mathrm{d}}{\mathrm{d}\alpha_s}\left[\frac{\beta_s^{\mathrm{SYM}}}{\alpha_s}\right]. \tag{22}$$

(This relationship[3] between $\gamma_m$ and $\beta$ holds in both DRED and NSVZ schemes.) Through four loops we have at once from Eq. (22) that

$$\gamma_m^{\mathrm{SYM}} = -\left(\frac{\alpha_s}{\pi}\right) \left[\frac{3}{4}C_A + \frac{3}{4}C_A^2\frac{\alpha_s}{\pi} + \frac{63}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 + \frac{51}{32}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}(\alpha_s^6)\,. \tag{23}$$

Quite remarkably, in the supersymmetric case we again find this agrees with our calculations. This is again striking confirmation of our methodology and of the exact result Eq. (22).

# 6 Conclusions

In this paper we have applied DRED to QCD, and calculated both the gauge $\beta$-function and the mass anomalous dimension to the four-loop level. These calculations required careful treatment of the evanescent Yukawa and quartic couplings of the $\varepsilon$-scalar. In the supersymmetric limit we explicitly verified that the $\beta$-function for the evanescent Yukawa coupling reproduces the gauge $\beta$-function through three loops.

[2]Note that this procedure differs from the one outlined in Section 3, where we modified the Feynman rules in color space and re-evaluated the color factor for each diagram. Here, we simply replace $C_F$ (fundamental Casimir) and $T$ (fundamental trace normalization) by $C_A$ (adjoint Casimir) in the final result.

[3]Note that our definition of $\gamma_m$ in Eq. (11) differs by a factor of two (and a factor of $M$) from the definition of $\beta_M$ in Ref. [28].

The popularity of the MSSM and the construction of the CERN Large Hardon Collider (LHC) has led to many increasingly precise calculations of sparticle production and decay processes, using DRED. The MSSM is a softly broken *supersymmetric* theory, so we might well expect its dimensionless coupling sector to renormalise like the underlying supersymmetric theory, without worrying about evanescent couplings; to test this (in the same manner as described above) will require a generalisation of our calculations to incorporate scalar fields. In the MSSM, however, there is in fact one evanescent quantity which must certainly be considered: the $\varepsilon$-scalar mass [29]. This exists also in QCD, but affects neither the gauge $\beta$-function nor the fermion mass anomalous dimension on simple dimensional grounds so we have not considered it here.

If, however, one wants to match MSSM calculations on to the Standard Model (or, for example, consider an intermediate energy theory produced by integrating out the squarks and sleptons [30]) then evidently the use of DRED inevitably means one must worry about evanescent couplings. Ref. [13] pointed out a couple of instances where naive treatment of the evanescent couplings has led to incorrect conclusions; we believe that careful treatment of the $\varepsilon$-scalars in higher order calculations as presented in Ref. [13] and here will prove invaluable in matching MSSM calculations to low energy physics.

### Acknowledgements

One of us (DRTJ) thanks Ian Jack for conversations, and both CERN and KITP (Santa Barbara) for hospitality. This work was supported by the DFG through SFB/TR 9 and the *Emmy Noether* program HA 2990/2-1, and by the National Science Foundation under Grant No. PHY99-07949.

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