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Decoupling of heavy quarks in HQET

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Abstract: Decoupling of c-quark loops in b-quark HQET is considered. The decoupling coefficients for the HQET heavy-quark field and the heavy–light quark current are calculated with three-loop accuracy. The last result can be used to improve the accuracy of extracting f_B from HQET lattice simulations (without c-quark loops). The decoupling coefficient for the flavour-nonsinglet QCD current with n antisymmetrized γ-matrices is also obtained at three loops; the result for the tensor current ($n = 2$) is new.

Keywords: QCD, NLO Computations, Heavy Quark Physics.

Contents

1. Introduction

Let us consider QCD with a heavy flavour, say c. It is well-known that processes with light quarks and gluons having characteristic momenta $\ll m_c$ can be described by an effective low-energy theory — QCD without c-quarks. The renormalized light-quark and gluon fields $q_i(\mu)$, $A(\mu)$ in the full theory are related to the corresponding fields in the effective theory, up to corrections suppressed by powers of $1/m_c$, by the decoupling relations

$$q_i(\mu) = \zeta_q^{1/2}(\mu) q_i'(\mu)\,, \qquad A(\mu) = \zeta_A^{1/2}(\mu) A'(\mu)\,, \tag{1.1}$$

where all quantities in the low-energy theory are denoted by primes (we use the $\overline{\text{MS}}$ renormalization scheme throughout this paper). Similarly, the coupling constant, the gauge-fixing parameter and the light-quark masses in the two theories are related by

$$\alpha_s(\mu) = \zeta_\alpha(\mu)\alpha_s'(\mu)\,, \qquad a(\mu) = \zeta_A(\mu) a'(\mu)\,, \qquad m_i(\mu) = \zeta_m(\mu) m_i'(\mu)\,. \tag{1.2}$$

The QCD decoupling coefficients are known at three [1] and even four loops [2, 3]. Various operators of full QCD, e.g., light–light quark currents, can be expressed via operators of the low-energy effective theory, similarly to (1.1).

Now let us consider the b-quark HQET. If the characteristic residual momentum of b, as well as characteristic momenta of light quarks and gluons, are $\ll m_c$, then all processes can be described by the low-energy effective theory — HQET without c-quarks. The decoupling relations for the light fields (1.1) and the parameters of the Lagrangian (1.2) are exactly the same as in QCD. The static b-quark field $\tilde{Q}$ in the "full" theory (HQET with c-quarks) and in the effective theory (HQET without c-quarks) are related by

$$\tilde{Q}(\mu) = \tilde{\zeta}_Q^{1/2}(\mu) \tilde{Q}'(\mu)\,. \tag{1.3}$$

Various operators of the "full" HQET can be expressed via operators of the low-energy HQET. For example, for the heavy–light quark currents $\tilde{\jmath} = \bar{q}\Gamma\tilde{Q}$ we have

$$\tilde{\jmath}(\mu) = \tilde{\zeta}_j(\mu) \tilde{\jmath}'(\mu)\,, \tag{1.4}$$

up to corrections suppressed by powers of $1/m_c$. The HQET-specific decoupling coefficients $\tilde{\zeta}_Q$, $\tilde{\zeta}_j$ have been calculated in [4] with two-loop accuracy[1]. Decoupling in HQET is also discussed in [7], Sects. 4.7 and 5.5.

In the present paper, we shall calculate $\tilde{\zeta}_Q$ and $\tilde{\zeta}_j$ up to three loops. To do so, we need to calculate on-shell HQET propagator diagrams containing a massive quark loop. Reduction of scalar Feynman integrals of this type to master integrals is considered in Sect. 2; the master integrals are calculated in Sect. 3. The HQET decoupling coefficients are obtained in Sects. 4 and 5. In Appensix A we derive the three-loop decoupling coefficient for the QCD flavour-nonsinglet quark current with n antisymmetrized γ-matrices, as a generic formula containing n; the result for the tensor current ($n = 2$) is new.

[1]One more HQET decoupling coefficient, that for the b-quark chromomagnetic interaction, has been found in [5], based on the calculations of [6], also at two loops.

2. Reduction of Feynman integrals

2.1 General remarks

In this subsection, we discuss some general properties and simple particular cases of on-shell HQET integrals with a massive quark loop.

Let us consider the integrals (Fig. 1)

$$F(a_1,a_2) = \int \frac{f(k^2)\, d^d k}{E_1^{a_1} E_2^{a_2}}\,,$$
$$E_1 = -2k\cdot v - i0\,, \qquad E_2 = -k^2 - i0 \quad (2.1)$$

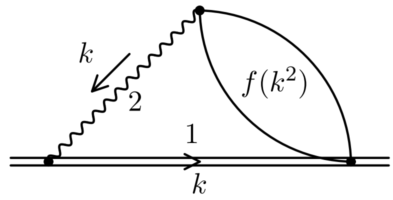


Figure 1: Diagram with a single HQET line

with some $f(k^2)$. Following the strategy of integration by parts [8] and integrating the identity

$$\frac{\partial}{\partial k}\cdot\left(k - 2\frac{E_2}{E_1}v\right)\frac{f(k^2)}{E_1^{a_1}E_2^{a_2}} = \left[d - a_1 - 2 - 4(a_1+1)\frac{E_2}{E_1^2}\right]\frac{f(k^2)}{E_1^{a_1}E_2^{a_2}}\,, \tag{2.2}$$

we obtain the recurrence relation

$$(d - a_1 - 2)F(a_1,a_2) = 4(a_1+1)\mathbf{1}^{++}\mathbf{2}^{-}F(a_1,a_2)\,. \tag{2.3}$$

Its solution is

$$F(a_1,a_2) = \begin{cases} (-4)^{-a_1/2}\dfrac{\Gamma\left(\frac{d}{2}\right)}{\Gamma(\frac{d-a_1}{2})}\dfrac{\Gamma\left(\frac{1-a_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}F\Big(0, a_2 + \dfrac{a_1}{2}\Big)\,, & \text{even } a_1,\\ 2^{1-a_1}\dfrac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{a_1+1}{2}\right)\Gamma(\frac{d-a_1}{2})}F\Big(1, a_2 + \dfrac{a_1-1}{2}\Big)\,, & \text{odd } a_1 > 0,\\ 0, & \text{odd } a_1 < 0, \end{cases} \tag{2.4}$$

The result for even $a_1 \le 0$ can easily be derived by averaging over v directions in the d-dimensional Euclidean space after the Wick rotation; it was supposed in [9] that the same formula can be used for even $a_1 > 0$, and this method was used, e.g., in [6, 4], but no proof existed. Now we see that this averaging trick follows from (2.3). For odd $a_1 < 0$, averaging over v directions gives 0, but this result does not extend to odd $a_1 > 0$.

Now we shall discuss the two-loop integrals (Fig. 2)

$$F(a_1,a_2,a_3,a_4,a_5) = \frac{1}{(i\pi^{d/2})^2}\int \frac{d^d k_1\, d^d k_2}{E_1^{a_1}E_2^{a_2}E_3^{a_3}E_4^{a_4}E_5^{a_5}} \tag{2.5}$$

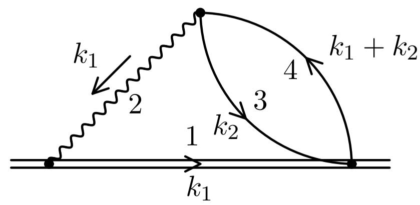


Figure 2: Two-loop diagram

with four denominators

$$E_1 = -2k_1\cdot v - i0\,, \qquad E_2 = -k_1^2 - i0\,,$$
$$E_3 = 1 - k_2^2 - i0\,,$$
$$E_4 = 1 - (k_1+k_2)^2 - i0 \tag{2.6}$$

(we have put the quark mass $m = 1$; the power of m
can easily be restored by dimensionality), and one numerator

$$E_5 = (2k_2 + k_1) \cdot v \tag{2.7}$$

(a_5 is always ≤ 0). Integrals with $a_3 \leq 0$ or $a_4 \leq 0$ vanish. We have

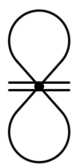
$$F(a_1, a_2, a_3, a_4, a_5) = (-1)^{a_5} F(a_1, a_2, a_4, a_3, a_5)\,. \tag{2.8}$$

These integrals can be subdivided into two disjoint subsets, with even and odd $a_1 + a_5$. We shall call them "apparently even" and "apparently odd" (they would be even and odd with respect to $v \to -v$ if there were no $-i0$ in E_1). Apparently odd integrals with $a_1 \leq 0$ vanish, because we can omit $-i0$ in the numerator. These two classes are not mixed by any recurrence relations, therefore, we have two disjoint problems.

These integrals can be calculated as follows. First, we get rid of the numerator $E_5^{-a_5}$. We have a contraction of $-a_5$ vectors v with the one-loop tensor integral which depends on k_1. Decomposing this integral into all possible tensor structures, we obtain an expression containing only E_1 (no E_5). Using α-parametrization, we obtain

$$\begin{aligned}&F(a_1, a_2, a_3, a_4, 0) = \\ &\frac{\Gamma\left(\frac{a_1}{2}\right)\Gamma\left(\frac{a_1-d}{2} + a_2 + a_3\right)\Gamma\left(\frac{a_1-d}{2} + a_2 + a_4\right)\Gamma\left(\frac{a_1}{2} + a_2 + a_3 + a_4 - d\right)\Gamma\left(\frac{d-a_1}{2} - a_2\right)}{2\Gamma(a_1)\Gamma(a_3)\Gamma(a_4)\Gamma(a_1 + 2a_2 + a_3 + a_4 - d)\Gamma\left(\frac{d-a_1}{2}\right)}\,.\end{aligned} \tag{2.9}$$

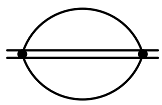
In full accordance with (2.4), integrals $F(a_1, a_2, a_3, a_4, 0)$ with even a_1 reduce to $F(0, a_2 + a_1/2, a_3, a_4, 0)$ (this is a well-known two-loop vacuum integral [10]); those with odd $a_1 > 0$ reduce to $F(1, a_2 + (a_1 - 1)/2, a_3, a_4, 0)$; and those with odd $a_1 < 0$ vanish. This is a strong check. All apparently even integrals are proportional to the single master integral

$I_0^2 =$  , (2.10)

where

$$I_0 = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{1 - k^2 - i0} = \Gamma(1 - d/2) \tag{2.11}$$

is the one-loop vacuum integral. All apparently odd integrals are proportional to the single master integral

$J_0 =$  $= 2^{2d-9}\pi^2 \dfrac{\Gamma(5 - 2d)}{\Gamma(2 - d/2)}\,.$ (2.12)

In order to perform our decoupling calculation, we need two classes of three-loop integrals. They will be reduced to master integrals using integration by parts [8]. A similar problem (off-shell HQET integrals without massive quarks) has been solved earlier [11].

However, constructing reduction algorithms by hand is tedious and time-consuming. Several algorithmic methods to solve reduction problems for Feynman integrals have been recently suggested; they are discussed, e.g., in [12]. Some approaches are based on the use of Gröbner bases [13, 14, 15]. This is the last algorithm in this list [14, 15] that we have used in the present calculations.

2.2 Integrals of class 1

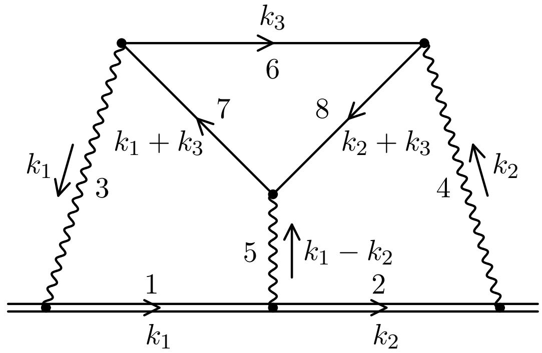


Figure 3: Diagram of class 1

These integrals (Fig. 3),

$$F_1(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = \frac{1}{(i\pi^{d/2})^3} \int \frac{d^d k_1\, d^d k_2\, d^d k_3}{E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_5^{a_5} E_6^{a_6} E_7^{a_7} E_8^{a_8} E_9^{a_9}}, \tag{2.13}$$

have 8 denominators,

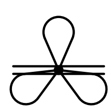
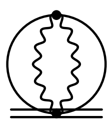
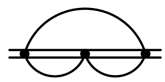
$$\begin{aligned} &E_1 = -2k_1 \cdot v\,, \qquad E_2 = -2k_2 \cdot v\,, \\ &E_3 = -k_1^2\,, \qquad E_4 = -k_2^2\,, \qquad E_5 = -(k_1 - k_2)^2\,, \\ &E_6 = 1 - k_3^2\,, \qquad E_7 = 1 - (k_1 + k_3)^2\,, \qquad E_8 = 1 - (k_2 + k_3)^2 \end{aligned} \tag{2.14}$$

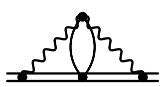
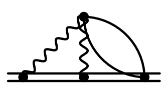
(here and in what follows, $-i0$ is implied in all denominators), and one numerator,

$$E_9 = 2k_3 \cdot v \tag{2.15}$$

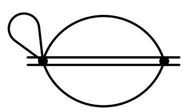
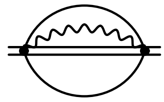
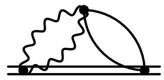
(a_9 is always ≤ 0). This integral is symmetric with respect to $(1 \leftrightarrow 2, 3 \leftrightarrow 4, 7 \leftrightarrow 8)$. It vanishes if indices of the following subsets of lines are non-positive: $\{5, 7\}$, $\{5, 8\}$, $\{6, 7\}$, $\{6, 8\}$, $\{7, 8\}$, $\{3, 4, 6\}$.

We applied the method of [14, 15] to construct an algorithm for reducing integrals of this class to master integrals. Recurrence relations do not mix apparently even integrals (with even $a_1 + a_2 + a_9$) with apparently odd ones. All apparently even integrals reduce to linear combinations of the following master integrals:

$I_0^3 =$  , $I_1 =$  , $I_2 =$  ,

$I_3 =$  , $I_4 =$  . (2.16)

All apparently odd integrals reduce to the following master ones:

$I_0 J_0 =$  , $\quad J_1 =$  , $\quad J_2 =$  . (2.17)

2.3 Integrals of class 2

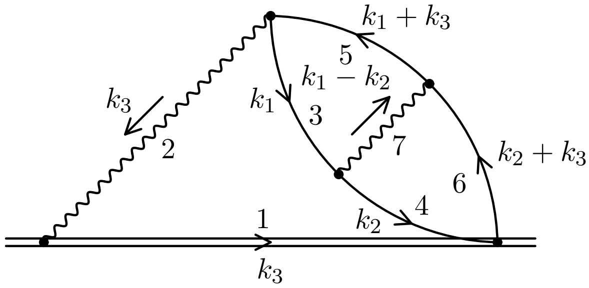


Figure 4: Diagram of class 2

These integrals (Fig. 4),

$$F_2(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = \frac{1}{(i\pi^{d/2})^3} \int \frac{d^d k_1\, d^d k_2\, d^d k_3}{E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_5^{a_5} E_6^{a_6} E_7^{a_7} E_8^{a_8} E_9^{a_9}}\,, \tag{2.18}$$

have 7 denominators,

$$\begin{gathered} E_1 = -2k_3 \cdot v\,, \qquad E_2 = -k_3^2\,, \qquad E_3 = 1 - k_1^2\,, \qquad E_4 = 1 - k_2^2\,, \\ E_5 = 1 - (k_1 + k_3)^2\,, \qquad E_6 = 1 - (k_2 + k_3)^2\,, \qquad E_7 = -(k_1 - k_2)^2\,, \end{gathered} \tag{2.19}$$

and two numerators,

$$E_8 = (2k_1 + k_3) \cdot v\,, \qquad E_9 = (2k_2 + k_3) \cdot v \tag{2.20}$$

(a_8 and a_9 are always ≤ 0). This integral is symmetric with respect to $(3 \leftrightarrow 4, 5 \leftrightarrow 6, 8 \leftrightarrow 9)$, and

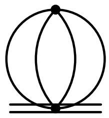
$$F_2(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = (-1)^{a_8 + a_9} F_2(a_1, a_2, a_5, a_6, a_3, a_4, a_7, a_8, a_9)\,. \tag{2.21}$$

It vanishes if indices of the following subsets of lines are non-positive: $\{3, 5\}$, $\{4, 6\}$, $\{3, 4\}$, $\{5, 6\}$.

We have used the method of [14, 15] to construct an algorithm reducing these integrals to master ones. For apparently even integrals (with even $a_1 + a_8 + a_9$) we also used a method similar to subsection 2.1. Namely, we have a contraction of $-a_8 - a_9$ vectors v with a two-loop tensor integral which depends on k_3. Decomposing this integral into all possible tensor structures, we get rid of the numerator $E_8^{-a_8} E_9^{-a_9}$. Then we use the averaging formula (2.4), and the problem reduces to three-loop massive vacuum integrals

B_M [16]. A REDUCE package RECURSOR [16] reduces them to two master integrals, I_0^3 (see (2.16)) and

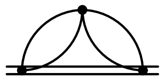
$$I_5 = \quad . \tag{2.22}$$



Agreement of results produced by these two ways serves as a strong check.

Apparently odd integrals of this class are expressed via $I_0 J_0$, J_1 (see (2.17)), and

$$J_3 = \quad . \tag{2.23}$$



3. Master integrals

3.1 Simple master integrals

Simplest master integrals can exactly be expressed via Γ functions:

$$\begin{aligned}
I_1 &= -\frac{\Gamma(d/2-1)\Gamma(1-d/2)\Gamma^2(3-d)\Gamma(4-3d/2)}{\Gamma(6-2d)}\,,\\
J_1 &= 2^{8d-17}\pi^2\frac{\Gamma(d/2-1)\Gamma(5-2d)\Gamma(7-3d)}{\Gamma(2-d/2)\Gamma(3-d)\Gamma(4-3d/2)}\,,\\
J_2 &= \frac{\pi^{1/2}\Gamma^2(7/2-d)\Gamma(2-d/2)\Gamma^2(d/2-1)\Gamma((3(3-d))/2)\Gamma(d-5/2)}{2\Gamma(7-2d)\Gamma(d-2)\Gamma((d-1)/2)}\,.
\end{aligned} \tag{3.1}$$

The vacuum integral I_5 (2.22) has been investigated in [16]. It can be written as

$$\begin{aligned}
I_5 &= -4\frac{(d-2)^2(3d-11)}{(d-3)(3d-8)(3d-10)}I_0^3 - 7\frac{d-4}{2d-7}I_1\\
&\quad + 64\frac{(d-4)^2}{(d-2)(d-3)(3d-8)(3d-10)}\Gamma^3(1+\varepsilon)B_4(\varepsilon)\,,
\end{aligned} \tag{3.2}$$

where $B_4(\varepsilon)$ can be expressed via ${}_3F_2$ of unit argument, and

$$\begin{aligned}
B_4(\varepsilon) &= B_4 + \mathcal{O}(\varepsilon)\,,\\
B_4 &= 16\,\mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{2}{3}\log^2 2\left(\log^2 2 - \pi^2\right) - \frac{13}{180}\pi^4\,;
\end{aligned} \tag{3.3}$$

its expansion up to the ε^3 term was presented in [17].

3.2 Integral I_2

We evaluated all the complicated master integrals (i.e. which are not expressed in terms of gamma functions for general d) by the method of MB representation which is based on the following formula

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\Gamma(\lambda+z)\Gamma(-z)\frac{Y^z}{X^{\lambda+z}}\,. \tag{3.4}$$

Here the contour of integration is chosen in the standard way: the poles with a $\Gamma(\ldots+z)$ dependence (let us call them *left* poles, for brevity) are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence (*right* poles) are to the right of it. This formula is used to replace a sum of terms raised to some power by their products raised to some powers at the cost of having an extra integration.

Two systematic strategies for evaluating MB integrals in a Laurent expansion in ε were suggested in [18, 19]. A description of this method is presented in [20] and Chap. 4 of [12].

The simplest variant of using (3.4) is to write down a massive propagator in terms of massless ones [21]:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z) \; . \tag{3.5}$$

To have more checks, it is very useful to derive such MB representations for general indices. To evaluate I_2 we derived a MB representation for $F_1(a_1,a_2,0,0,0,a_6,a_7,a_8,0)$. We applied (3.5) to the last two factors in the integrand (with a_7 and a_8) and evaluated the two resulting integrals over k_1 and k_2 and the resulting integral over k_3 by well-known one loop integration formulae to obtain

$$\begin{aligned}
&F_1(a_1,a_2,0,0,0,a_6,a_7,a_8,0) = \frac{1}{4\sqrt{\pi}\prod\Gamma(a_i)} \\
&\times\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \mathrm{d}z_1\mathrm{d}z_2\, \Gamma(a_{12}/2+a_{678}+3\varepsilon-6+z_1+z_2) \\
&\times\frac{\Gamma(a_1/2+a_7+\varepsilon-2+z_1)\Gamma(a_2/2+a_8+\varepsilon-2+z_2)}{\Gamma((a_1+a_2-7)/2+a_7+a_8+2\varepsilon+z_1+z_2)} \\
&\times\Gamma((a_1-3)/2+a_7+\varepsilon+z_1)\Gamma((a_2-3)/2+a_8+\varepsilon+z_2) \\
&\times\Gamma(2-a_7-\varepsilon-z_1)\Gamma(2-a_8-\varepsilon-z_2)\Gamma(-z_1)\Gamma(-z_2) \; ,
\end{aligned} \tag{3.6}$$

where $a_{12}=a_1+a_2$ etc.

It was more convenient to obtain a Laurent expansion in ε $F_1(1,1,0,0,0,1,1,1,0)$ from a Laurent expansion of the integral $F_1(2,2,0,0,0,1,2,2,0)$ and its reduction, in particular, to I_2. For this auxiliary integral, we used (3.6). The resulting twofold MB integral can be expanded immediately in ε because there is no gluing of poles of different nature when $\varepsilon\to 0$. Them one can close the integration contours over z_1 and z_2 to the right and obtain a double series. Its summation gives the following result:

$$F_1(2,2,0,0,0,1,2,2,0) = \Gamma^3(1+\varepsilon)\left(\frac{\pi^2}{18}-\frac{1}{3}-\frac{\pi^2}{9}\varepsilon+\mathcal{O}(\varepsilon^2)\right) \tag{3.7}$$

which leads to

$$I_2 = -\Gamma^3(1+\varepsilon)\,\frac{\pi^2}{6}\left(\frac{1}{\varepsilon}+\frac{5}{2}+\mathcal{O}(\varepsilon)\right) \; . \tag{3.8}$$

Integrals $F_1(a_1,a_2,0,0,0,a_6,a_7,a_8,0)$ can be calculated also in the coordinate space. After continuation to Euclidean time, we have

$$F_1(a_1,a_2,0,0,0,a_6,a_7,a_8,0) = \frac{2^{3d/2-a_1-a_2-a_6-a_7-a_8+3}}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_6)\Gamma(a_7)\Gamma(a_8)}$$

$$
\times \int_0^\infty dt_1 \int_0^\infty dt_2 t_1^{a_1+a_7-d/2-1} t_2^{a_2+a_8-d/2-1} (t_1+t_2)^{a_6-d/2}
K_{d/2-a_7}(t_1) K_{d/2-a_8}(t_2) K_{d/2-a_6}(t_1+t_2)\,. \tag{3.9}
$$

If we expand the Bessel functions $K_{d/2-a_7}(t_1)$ and $K_{d/2-a_8}(t_2)$ in t_1 and t_2, then the integrals can be calculated, and we obtain a double series. However, this series is not very convenient for expansion in ε, because even for convergent integrals separate terms contain $1/\varepsilon^2$. We have verified that the $\mathcal{O}(1)$ term in (3.7) is reproduced by numerical integration in (3.9) with 12 digits accuracy, thus providing a good check.

3.3 Integral I_3

To evaluate I_3 we derived a general MB representation for $F_1(a_1, a_2, a_3, a_4, 0, 0, a_7, a_8, 0)$, where $I_3 = F_1(1, 1, 1, 1, 0, 0, 1, 1, 0)$. To do this, we used an alpha representation and then applied (3.4) twice in an appropriate way. We obtained

$$
F_1(a_1, a_2, a_3, a_4, 0, 0, a_7, a_8, 0) = \frac{\Gamma(a_{12}/2 + a_{347} + 2\varepsilon - 4)\Gamma(a_{12}/2 + a_{348} + 2\varepsilon - 4)}{4\sqrt{\pi}\Gamma((4 - a_{12} - 2\varepsilon)/2)\Gamma(a_{12} + 2a_{3478} + 4\varepsilon - 8)}
\times \frac{\Gamma(a_{12}/2 + a_{3478} + 3\varepsilon - 6)}{\prod \Gamma(a_i)} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \mathrm{d}z_1 \mathrm{d}z_2 \, \frac{\Gamma(a_1/2 + z_2)\Gamma(1/2 + a_1/2 + z_2)}{\Gamma(1/2 - z_1)}
\times \Gamma(a_{12}/2 + z_1 + z_2)\Gamma(a_{34} + \varepsilon - 2 - z_1)\Gamma(2 - a_{12}/2 - a_4 - \varepsilon - z_2)
\Gamma(-a_1/2 - z_1 - z_2)\Gamma(1/2 - a_1/2 - z_1 - z_2)\Gamma(2 - a_3 - \varepsilon + z_1 + z_2)\Gamma(-z_2)\,. \tag{3.10}
$$

As in the previous case, we preferred to calculate, instead of I_3, another integral of this family, $F_1(2, 2, 1, 1, 0, 0, 1, 1, 0)$ and then obtain I_3 using our reduction procedure. We have, after some changes of variables,

$$
F_1(2, 2, 1, 1, 0, 0, 1, 1, 0) = \frac{\Gamma(3\varepsilon)\Gamma(1 + 2\varepsilon)^2}{4\sqrt{\pi}\Gamma(-\varepsilon)\Gamma(2 + 4\varepsilon)}
\times \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \mathrm{d}z_1 \mathrm{d}z_2 \, \frac{\Gamma(3/2 + z_1)\Gamma(1 + z_2)\Gamma(3/2 + z_2)\Gamma(1 + z_1)}{\Gamma(5/2 + z_1 + z_2)}
\times \Gamma(2 + \varepsilon + z_1 + z_2)\Gamma(-1 - \varepsilon - z_1)\Gamma(-z_1)\Gamma(-1 - \varepsilon - z_2)\Gamma(-z_2)\,. \tag{3.11}
$$

After the replacement

$$
\Gamma(2 + \varepsilon + z_1 + z_2) \to (1 + \varepsilon + z_1 + z_2)\Gamma(1 + \varepsilon + z_1 + z_2)
$$

the integral can be decomposed into two integrals where the integration can be performed using the first Barnes lemma. Resulting onefold integrals can be also evaluated with the help of the first and the second Barnes lemmas so that we obtain a result in terms of gamma functions for general d:

$$
I_3 = \frac{\Gamma(1/2 - \varepsilon)\Gamma(-\varepsilon)\Gamma(2\varepsilon)^2\Gamma(1 + \varepsilon)\Gamma(3\varepsilon - 1)}{4\Gamma(3/2 - \varepsilon)\Gamma(4\varepsilon)}
\times [\psi(1/2 - \varepsilon) + \psi(1 - \varepsilon) - 2\log 2 + 2\gamma_{\mathrm{E}}]\,. \tag{3.12}
$$

3.4 Integral I_4

To obtain a MB representation for the integrals $F_1(a_1, a_2, 0, a_4, a_5, a_6, a_7, 0, 0)$, with $I_4 = F_1(1, 1, 0, 1, 1, 1, 1, 0, 0)$, we replace the internal integral over k_2 by a onefold MB integral, using Feynman parameters straightforwardly. After this the resulting integral over k_1 and k_3 can be evaluated in terms of gamma functions and we obtain a general onefold MB representation which gives

$$I_4 = \frac{\Gamma(2\varepsilon)^2\Gamma(3\varepsilon-1)}{4\Gamma(4\varepsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \frac{\Gamma(1+z)\Gamma(1/2+\varepsilon+z)\Gamma(1+\varepsilon+z)}{\Gamma(3/2+\varepsilon+z)\Gamma(1-2\varepsilon-z)} \times\Gamma(-2\varepsilon-z)\Gamma(-\varepsilon-z)\Gamma(-z) \; . \tag{3.13}$$

This integral can straightforwardly be evaluated by expanding the integrand in ε, closing the integration contour to the right and summing up resulting series. We obtain

$$\begin{aligned} I_4 \, = \, -\Gamma^3(1+\varepsilon)\Bigg[& \frac{\pi^2}{9\varepsilon^2} - \frac{6\zeta_3 - 5\pi^2}{9\varepsilon} + \frac{11}{270}\pi^4 - \frac{10}{3}\zeta_3 + \frac{19}{9}\pi^2 \\ & + \left(-\frac{8}{3}\zeta_5 + \frac{8}{9}\pi^2\zeta_3 + \frac{11}{54}\pi^4 - \frac{38}{3}\zeta_3 + \frac{65}{9}\pi^2\right)\varepsilon + \mathcal{O}(\varepsilon^2)\Bigg] \; . \end{aligned} \tag{3.14}$$

3.5 Integral J_3

To obtain a MB representation for the integrals $F_2(a_1, a_2, a_3, a_4, a_5, a_6, 0, 0, 0)$, with $J_3 = F_2(1, 0, 1, 1, 1, 1, 0, 0, 0)$, we use the following onefold MB representation for a self-energy one-loop integral with two equal masses which can straightforwardly be derived using Feynman parameters:

$$\begin{aligned} & \int \frac{\mathrm{d}^d k}{(-k^2+m^2)^{a_1}[-(q-k)^2+m^2]^{a_2}} = \frac{1}{\Gamma(a_1)\Gamma(a_2)(m^2)^{a_1+a_2+\varepsilon-2}} \\ & \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \left(\frac{-q^2}{m^2}\right)^z \Gamma(-z) \frac{\Gamma(a_1+z)\Gamma(a_2+z)\Gamma(a_1+a_2+\varepsilon-2+z)}{\Gamma(a_1+a_2+2z)} \; . \end{aligned} \tag{3.15}$$

Writing down the subintegral over k_2 using (3.15) we obtain an integral which can explicitly be evaluated in terms of gamma functions for general ε so that we obtain only onefold MB representation for the integral under consideration:

$$\begin{aligned} F_2(a_1, a_2, a_3, a_4, a_5, a_6, 0, 0, 0) \; = \; & \frac{\Gamma(a_1/2)}{2\prod_{l\neq 2}\Gamma(a_l)\Gamma(2-a_1/2-\varepsilon)(m^2)^{a_1/2+a_{23456}-6+3\varepsilon}} \\ & \times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \, \frac{\Gamma(a_4+z)\Gamma(a_6+z)\Gamma(-z)\Gamma(a_1/2+a_{235}+2\varepsilon-4-z)}{\Gamma(a_{46}+2z)\Gamma(a_{12235}+2\varepsilon-4-2z)} \\ & \times\Gamma(2-a_1/2-a_2-\varepsilon+z)\Gamma(a_{46}+\varepsilon-2+z) \\ & \times\Gamma(a_1/2+a_{23}+\varepsilon-2-z)\Gamma(a_1/2+a_{25}+\varepsilon-2-z) \; . \end{aligned} \tag{3.16}$$

This gives

$$J_3 = \frac{\pi^{3/2}}{4^\varepsilon\Gamma(3/2-\varepsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z \frac{\Gamma(1+z)\Gamma(3/2-\varepsilon+z)\Gamma(\varepsilon+z)}{\Gamma(3/2+z)\Gamma(\varepsilon-z)} \times\Gamma(-1/2+\varepsilon-z)\Gamma(-3/2+2\varepsilon-z)\Gamma(-z) \; . \tag{3.17}$$

For our purposes, it was sufficient to evaluate only the leading term of this integral (using the strategy of [18]) in expansion in ε:

$$J_3 = -\frac{32\pi^2}{3} + \mathcal{O}(\varepsilon) \,. \tag{3.18}$$

4. Decoupling for the heavy-quark field

4.1 General formulae

It is convenient to find the relation between the bare fields in the "full " and the low-energy HQET,

$$\tilde{Q}_0 = \left(\tilde{\zeta}_Q^0\right)^{1/2} \tilde{Q}'_0 \,, \tag{4.1}$$

first; then

$$\tilde{\zeta}_Q(\mu) = \frac{\tilde{Z}'_Q(\alpha'_s(\mu), a'(\mu))}{\tilde{Z}_Q(\alpha_s(\mu), a(\mu))} \tilde{\zeta}_Q^0 \,. \tag{4.2}$$

The renormalization constant $\tilde{Z}_Q$ can be reconstructed from the three-loop anomalous dimension of the HQET heavy-quark field $\tilde{Q}$ which has been found in [22] by an on-shell massive QCD calculation and confirmed by an independent HQET calculation in [23].

The bare heavy-quark propagators in the two theories near the mass shell ($\omega \to 0$) behave as

$$\tilde{S}(\omega) = \frac{\tilde{Z}_Q^{\mathrm{os}}}{\omega} + \cdots \,, \qquad \tilde{S}'(\omega) = \frac{\tilde{Z}'^{\mathrm{os}}_Q}{\omega} + \cdots \,, \tag{4.3}$$

where $\tilde{Z}_Q^{\mathrm{os}}$ and $\tilde{Z}'^{\mathrm{os}}_Q$ are the on-shell renormalization constants of the heavy-quark field in these theories. Therefore,

$$\tilde{\zeta}_Q^0 = \frac{\tilde{Z}_Q^{\mathrm{os}}}{\tilde{Z}'^{\mathrm{os}}_Q} \,. \tag{4.4}$$

The bare heavy-quark propagator can be written as

$$\tilde{S}(\omega) = \frac{1}{\omega - \tilde{\Sigma}(\omega)} \,, \tag{4.5}$$

where $-i\tilde{\Sigma}(\omega)$ is the sum of all one-particle-irreducible self-energy diagrams. Therefore,

$$\tilde{Z}_Q^{\mathrm{os}} = \frac{1}{1 - \left(d\tilde{\Sigma}(\omega)/d\omega\right)_{\omega=0}} \,, \tag{4.6}$$

and similarly

$$\tilde{Z}'^{\mathrm{os}}_Q = \frac{1}{1 - \left(d\tilde{\Sigma}'(\omega)/d\omega\right)_{\omega=0}} \,. \tag{4.7}$$

All diagrams for $(d\tilde{\Sigma}'(\omega)/d\omega)_{\omega=0}$ contain no scale and hence vanish, and

$$\tilde{Z}'^{\mathrm{os}}_Q = 1 \,. \tag{4.8}$$

4.2 Bare calculation in full HQET

Only diagrams with (at least one) c-quark loop contribute to $(d\tilde{\Sigma}(\omega)/d\omega)_{\omega=0}$. Such diagrams first appear at two loops (Fig. 5a). This diagram has been calculated in [9]:

$$\left(\frac{d\tilde{\Sigma}(\omega)}{d\omega}\right)^{(2)}_{\omega=0} = -C_F T_F \frac{g_0^4 m_{c0}^{-4\varepsilon}}{(4\pi)^d} I_0^2 \frac{(d-1)(d-2)(d-6)}{2(d-5)(d-7)} \tag{4.9}$$

(of course, g_0 and the bare c-quark mass m_{c0} are those of the full theory).

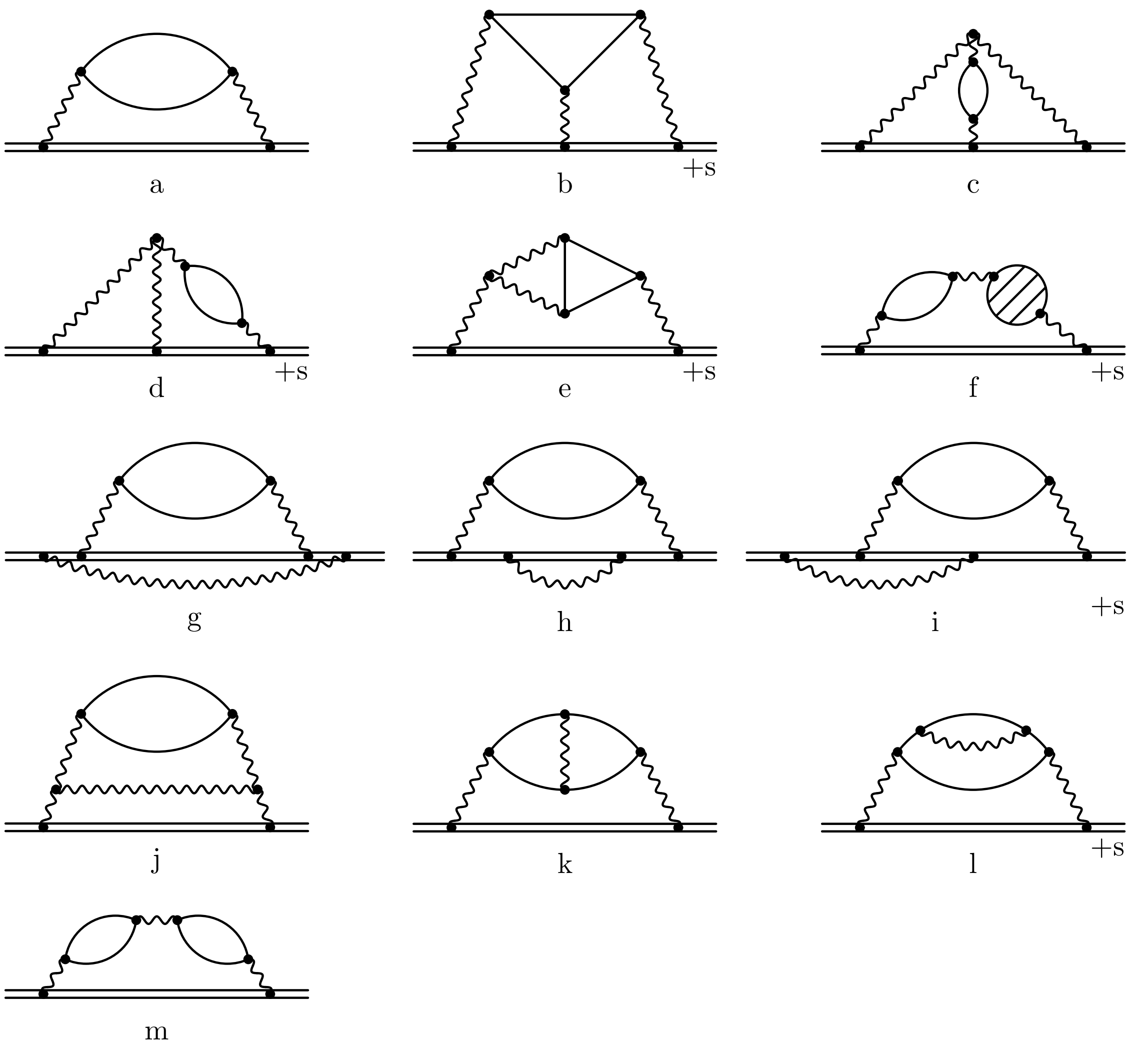


Figure 5: Diagrams for $(d\tilde{\Sigma}(\omega)/d\omega)_{\omega=0}$

Three-loop diagrams contributing to $(d\tilde{\Sigma}(\omega)/d\omega)_{\omega=0}$ are shown in Fig. 5b–m, where the shaded blob means the sum of the massless one-loop insertions (light quark, gluon, ghost), and "+s" means that there is also a mirror-symmetric diagram. The diagram Fig. 5i has three HQET denominators depending on only two loop momenta; these denominators are linearly dependent, and we can eliminate one HQET line by taking partial fractions. After that, all diagrams of Fig. 5b–j can be expressed via scalar integrals of class 1 (Sect. 2.2); those of Fig. 5k–m can be expressed via scalar integrals of class 2 (Sect. 2.3).

The three-loop result can be written in the form

$$\left(\frac{d\tilde{\Sigma}(\omega)}{d\omega}\right)^{(3)}_{\omega=0} = C_F T_F \frac{g_0^6 m_{c0}^{-6\varepsilon}}{(4\pi)^{3d/2}} \sum_{i=1}^{4}\sum_{j=1}^{5} C_i \bar{I}_j s_{ij}\,, \tag{4.10}$$

with the colour structures

$$C_1 = C_F\,, \qquad C_2 = C_A\,, \qquad C_3 = T_F n_l\,, \qquad C_4 = T_F\,, \tag{4.11}$$

and the integral structures

$$\begin{aligned}
\bar{I}_1 &= \frac{(d-2)^2}{24(d-3)(d-4)^2(d-5)^3(d-6)(d-7)(d-8)(d-10)} I_0^3\,,\\
\bar{I}_2 &= \frac{(3d-8)(3d-10)}{16(d-3)(d-4)(d-5)^2(2d-7)(2d-9)(2d-11)} I_1\,,\\
\bar{I}_3 &= \frac{3d-10}{(d-5)(2d-9)(2d-11)} I_3\,, \qquad \bar{I}_4 = \frac{3d-10}{(d-5)(2d-9)(2d-11)} I_4\,,\\
\bar{I}_5 &= \frac{(3d-8)(3d-10)}{16(d-4)^2(d-6)(d-8)(d-10)} I_5\,.
\end{aligned} \tag{4.12}$$

All the non-zero coefficients are:

$$\begin{aligned}
s_{11} &= 6(d-5)^2(d-10)(2d^8 - 75d^7 + 1212d^6 - 11042d^5 + 62070d^4 - 220131d^3\\
&\qquad + 478504d^2 - 576108d + 287408)\,,\\
s_{12} &= 2(d-2)(d-3)(d-5)^2(2d-11)(2d^4 - 29d^3 + 157d^2 - 366d + 288)\,,\\
s_{15} &= -2s_{25} = -2(d-10)(d^4 - 20d^3 + 141d^2 - 398d + 328)\,,\\
s_{21} &= (d-7)(d-10)(15d^8 - 555d^7 + 8858d^6 - 79542d^5 + 438503d^4 - 1514103d^3\\
&\qquad + 3179372d^2 - 3678156d + 1767408)\,,\\
s_{22} &= -(d-3)(2d^8 - 63d^7 + 864d^6 - 6725d^5 + 32377d^4 - 98156d^3 + 181403d^2\\
&\qquad - 184490d + 77536) + 2(d-1)(d-3)^2(d-4)(d-5)^3(1-a_0)\,,\\
s_{23} &= -s_{24} = 2(2d^2 - 19d + 47)\,, \qquad s_{32} = 16(d-2)(d-3)(d-4)(d-5)^3\,,\\
s_{41} &= 12(d-4)(d-5)^2(9d^6 - 277d^5 + 3387d^4 - 20943d^3 + 68428d^2 - 110236d + 66352)\,,\\
s_{45} &= 4(d-4)(d^3 - 15d^2 + 54d - 8)\,.
\end{aligned} \tag{4.13}$$

The coefficient of $C_A I_1$ is not gauge invariant. Therefore, the on-shell renormalization constant of the HQET heavy-quark field (4.6) is not gauge-invariant, starting from three loops. The same phenomenon has been observed in QCD [22]. In the abelian case, the on-shell renormalization constant is gauge invariant to all orders of perturbation theory; the gauge dependence only appears in the non-abelian colour structure C_A.

4.3 Renormalized decoupling coefficient

Now we calculate the renormalized decoupling coefficient (4.2). To this end, we re-express $\tilde{Z}_Q$ via g_0^2, a_0; $\tilde{Z}'_Q$ via $g_0'^2$, a'_0, which are re-expressed via g_0^2, a_0 using the bare decoupling relations; combine all factors in (4.2); and, finally, re-express the result via $\alpha_s(\mu)$, $a(\mu)$ and expand the coefficients in ε. We arrive at

$$\begin{aligned}
\tilde{\zeta}_Q(\mu) = {}& 1 + C_F T_F \left(2L^2 - \frac{16}{3}L + \frac{52}{9}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \\
&+ C_F T_F \left\{ \left[\frac{1}{3}\left(a(\mu) + \frac{44}{3}\right) C_A - \frac{16}{9} T_F (n_l + 2)\right] L^3 \right. \\
&\quad + \left[-18 C_F - \frac{1}{3}\left(\frac{13}{2}a(\mu) - 31\right) C_A + \frac{64}{9} T_F\right] L^2 \\
&\quad + \left[6(8\zeta_3 - 1) C_F + \left(\frac{121}{18}a(\mu) - 48\zeta_3 + \frac{155}{9}\right) C_A - \frac{16}{27} T_F (36 n_l + 31)\right] L \\
&\quad + \left(16 B_4 - \frac{4}{5}\pi^4 + 106\zeta_3 - \frac{475}{6}\right) C_F \\
&\quad + \left[\frac{1}{3}\left(8\zeta_3 - \frac{2387}{72}\right) a(\mu) - 8 B_4 + \frac{52}{45}\pi^4 - \frac{413}{9}\zeta_3 + \frac{262}{243}\right] C_A \\
&\quad \left. - \frac{32}{9}\left(4\zeta_3 - \frac{179}{27}\right) T_F n_l + \frac{8}{9}\left(28\zeta_3 - \frac{395}{27}\right) T_F \right\} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 + \cdots
\end{aligned} \tag{4.14}$$

where

$$L = 2 \log \frac{\mu}{m_c(\mu)} \tag{4.15}$$

($m_c(\mu)$ is the $\overline{\text{MS}}$ renormalized mass), and B_4 is given by (3.3).

5. Decoupling for the heavy–light current

5.1 General formulae

The bare heavy–light current $\tilde{\jmath}_0 = \bar{q}_0 \Gamma \tilde{Q}_0$ in the "full" HQET is related to the corresponding operator $\tilde{\jmath}'_0$ in the low-energy theory by

$$\tilde{\jmath}_0 = \tilde{\zeta}_j^0 \tilde{\jmath}'_0 \,, \tag{5.1}$$

up to $1/m_c$ corrections. The renormalized operators are related by (1.4), where

$$\tilde{\zeta}_j(\mu) = \frac{\tilde{Z}'_j(\alpha'_s(\mu))}{\tilde{Z}_j(\alpha_s(\mu))} \tilde{\zeta}_j^0 \,. \tag{5.2}$$

The renormalization constant $\tilde{Z}_j$ can be reconstructed from the three-loop anomalous dimension of the HQET current $\tilde{\jmath}$ which has been calculated in [23].

The Green function of $\tilde{\jmath}_0$, $\bar{\tilde{Q}}_0$, and q_0 can be written as $S(p)\tilde{\Gamma}(p,\omega)\Gamma\tilde{S}(\omega)$ (Fig. 6), where $\tilde{\Gamma}(p,\omega)\Gamma$ is the proper vertex — the sum of all one-particle-irreducible diagrams (here Γ is the Dirac matrix in the current; due to the HQET Feynman rules, no γ-matrices can appear to the right from Γ). The Green functions in the two theories are related by

$$S(p)\tilde{\Gamma}(p,\omega)\Gamma\tilde{S}(\omega) = \left(\tilde{\zeta}_Q^0 \zeta_q^0\right)^{1/2} \tilde{\zeta}_j^0 S'(p)\tilde{\Gamma}'(p,\omega)\Gamma\tilde{S}'(\omega) + \mathcal{O}\left(\frac{p,\omega}{m_c}\right) . \tag{5.3}$$

Recalling the relations

$$S(p) = \zeta_q^0 S'(p) , \qquad \tilde{S}(\omega) = \tilde{\zeta}_Q^0 \tilde{S}'(\omega) \tag{5.4}$$

(up to $1/m_c$ corrections), we obtain

$$\tilde{\Gamma}(p,\omega) = \left(\tilde{\zeta}_Q^0 \zeta_q^0\right)^{-1/2} \tilde{\zeta}_j^0 \tilde{\Gamma}'(p,\omega) + \mathcal{O}\left(\frac{p,\omega}{m_c}\right) . \tag{5.5}$$

It is most convenient to use the point $p = 0$, $\omega = 0$, then power corrections may be omitted. The vertices $\tilde{\Gamma}(0,0)$, $\tilde{\Gamma}'(0,0)$ can have Dirac structures 1 and $\not{p}$; if the light quark q is massless, the number of γ-matrices in them is always even, and the second structure does not appear. So, these vertices are scalar, and

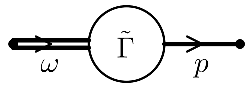


Figure 6: Green function

$$\tilde{\zeta}_j^0 = \left(\tilde{\zeta}_Q^0 \zeta_q^0\right)^{1/2} \frac{\tilde{\Gamma}(0,0)}{\tilde{\Gamma}'(0,0)} . \tag{5.6}$$

All loop corrections to $\tilde{\Gamma}'(0,0)$ contain no scale and hence vanish:

$$\tilde{\Gamma}'(0,0) = 1 . \tag{5.7}$$

Therefore,

$$\tilde{\zeta}_j^0 = \left(\tilde{Z}_Q^{\mathrm{os}} Z_q^{\mathrm{os}}\right)^{1/2} \tilde{\Gamma}(0,0) . \tag{5.8}$$

The right-hand side is an on-shell matrix element of the gauge-invariant operator $\tilde{\jmath}_0$, and hence is gauge-invariant; the quantities $\tilde{\Gamma}(0,0)$, $\tilde{Z}_Q^{\mathrm{os}}$, Z_q^{os} taken separately are not gauge-invariant starting from three loops.

5.2 Bare calculation in full HQET

Only diagrams with (at least one) c-quark loop contribute to $\tilde{\Lambda}(0,0) = \tilde{\Gamma}(0,0) - 1$. Such diagrams first appear at two loops (Fig. 7). As we know, the result is scalar; so, we can take $\frac{1}{4}$ Tr of the γ-matrices associated with the light-quark line. In this diagram, there is only one light-quark propagator $S(k)$ and the vertex, and taking the trace gives k^μ. The c-quark loop inserted into the gluon propagator is transverse, and the diagram vanishes [9]:

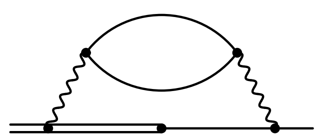


Figure 7: Diagram for $\tilde{\Lambda}(0,0)$

$$\tilde{\Lambda}^{(2)}(0,0) = 0 . \tag{5.9}$$

Three-loop diagrams for $\tilde{\Lambda}(0,0)$ can be obtained from those in Fig. 5b–m by inserting the heavy–to–light vertex into all possible places along the heavy-quark line; of course, mirror-symmetric diagrams denoted by "+s" in Fig. 5 should be considered separately. A number of diagrams with a single light-quark propagator vanish for the same reason as the two-loop diagram (Fig. 7). For some diagrams with three HQET lines, we use partial fractioning. After that, all diagrams produced from those of Fig. 5b–j are expressed via scalar integrals of class 1 (Sect. 2.2); diagrams produced from Fig. 5k–m are expressed via scalar integrals of class 2 (Sect. 2.3).

The three-loop result can be written in the form

$$\tilde{\Lambda}^{(3)}(0,0) = C_F T_F \frac{g_0^6 m_{c0}^{-6\varepsilon}}{(4\pi)^{3d/2}} \sum_{i=1}^{2} \sum_{j=2}^{4} C_i \bar{I}_j \lambda_{ij} \tag{5.10}$$

(see (4.11) and (4.12)). All the non-zero coefficients are:

$$\begin{aligned}
\lambda_{12} &= -2(d-2)^2(d-4)^2(d-5)^3\,, \qquad \lambda_{13} = \lambda_{14} = 2(d-2)(d-5)^2\,,\\
\lambda_{22} &= (d-3)^2(d-4)(d-5)^3(d+2) - 2(d-1)(d-3)^2(d-4)(d-5)^3(1-a_0)\,,\\
\lambda_{23} &= -(d-5)^2\,, \qquad \lambda_{24} = -(d-3)(d-5)^2\,.
\end{aligned} \tag{5.11}$$

Here again the coefficient of $C_A I_1$ is not gauge invariant. The on-shell matrix element (5.8) is gauge-invariant; this is a strong check of our calculation.

5.3 Renormalized decoupling coefficient

Now we calculate the renormalized decoupling coefficient (5.2). To this end, we re-express $\alpha_s(\mu)$ in $\tilde{Z}_Q$ via g_0^2; $\alpha_s'(\mu)$ in $\tilde{Z}_Q'$ via $g_0'2$, which is, in its turn, re-expressed via g_0^2 using the bare decoupling coefficient ζ_α^0; combine all factors in (5.2); and, finally, re-express the result via $\alpha_s(\mu)$ and expand the coefficients in ε. We arrive at

$$\begin{aligned}
\tilde{\zeta}_j(\mu) = 1 &+ C_F T_F \left(L^2 - \frac{5}{3}L + \frac{89}{36}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\\
&+ C_F T_F \Biggl\{ \frac{2}{9}\left[11C_A - 4T_F(n_l+2)\right]L^3 + \frac{1}{9}\left[16(\pi^2-6)C_F - (4\pi^2-39)C_A + 20T_F\right]L^2\\
&\quad + \frac{1}{3}\Biggl[\left(88\zeta_3 - \frac{112}{9}\pi^2 - \frac{173}{6}\right)C_F - \left(76\zeta_3 - \frac{28}{9}\pi^2 - \frac{401}{6}\right)C_A\\
&\qquad - \frac{2}{3}T_F\left(53n_l + \frac{124}{3}\right)\Biggr]L\\
&\quad + \left(8B_4 - \frac{86}{405}\pi^4 + \frac{1427}{27}\zeta_3 + \frac{1600}{243}\pi^2 - \frac{7219}{162}\right)C_F\\
&\quad - \left(4B_4 - \frac{43}{81}\pi^4 + \frac{1471}{54}\zeta_3 + \frac{400}{243}\pi^2 + \frac{3845}{486}\right)C_A\\
&\quad - \frac{2}{9}\left(32\zeta_3 - \frac{1327}{27}\right)T_F n_l + \frac{1}{9}\left(112\zeta_3 - \frac{1685}{27}\right)T_F\Biggr\}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 + \cdots
\end{aligned} \tag{5.12}$$

Here L is given by (4.15) and B_4 by (3.3). This is our main result. The two-loop part agrees with [4].

At the normalization scale μ_c defined as the root of the equation

$$m_c(\mu_c) = \mu_c \tag{5.13}$$

we have, for the physical $SU(3)$ colour group,

$$\begin{aligned}
\tilde{\zeta}_j(\mu_c) &= 1 + \frac{89}{864}\left(\frac{\alpha_s(\mu_c)}{\pi}\right)^2 \\
&\quad + \frac{1}{9}\biggl[-2\operatorname{Li}_2\left(\frac{1}{2}\right) - \frac{1}{12}\log^2 2\left(\log^2 2 - \pi^2\right) + \frac{427}{3240}\pi^4 - \frac{815}{1728}\zeta_3 + \frac{175}{486}\pi^2 - \frac{877}{108} \\
&\qquad - \frac{1}{3}\left(\zeta_3 - \frac{1327}{864}\right) n_l\biggr]\left(\frac{\alpha_s(\mu_c)}{\pi}\right)^3 \\
&\approx 1 + 0.1030\left(\frac{\alpha_s(\mu_c)}{\pi}\right)^2 + (0.7828 + 0.0124 n_l)\left(\frac{\alpha_s(\mu_c)}{\pi}\right)^3,
\end{aligned} \tag{5.14}$$

where the number of light flavours $n_l = 3$ includes neither b nor c.

It is easy to re-write (5.12) via the on-shell mass m_c^{os} instead of the $\overline{\mathrm{MS}}$ mass $m_c(\mu)$. Substituting the well-known one-loop relation between them, we obtain

$$\begin{aligned}
\tilde{\zeta}_j(\mu) &= 1 + C_F T_F\left(L_{\mathrm{os}}^2 - \frac{5}{3}L_{\mathrm{os}} + \frac{89}{36}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \\
&+ C_F T_F\biggl\{\frac{2}{9}\left[11C_A - 4T_F(n_l+2)\right]L_{\mathrm{os}}^3 + \frac{1}{9}\left[4(4\pi^2+3)C_F - (4\pi^2-39)C_A + 20T_F\right]L_{\mathrm{os}}^2 \\
&\quad + \frac{1}{3}\biggl[\left(88\zeta_3 - \frac{112}{9}\pi^2 - \frac{65}{6}\right)C_F - \left(76\zeta_3 - \frac{28}{9}\pi^2 - \frac{401}{6}\right)C_A \\
&\qquad - \frac{2}{3}T_F\left(53n_l + \frac{124}{3}\right)\biggr]L_{\mathrm{os}} \\
&\quad + \left(8B_4 - \frac{86}{405}\pi^4 + \frac{1427}{27}\zeta_3 + \frac{1600}{243}\pi^2 - \frac{9379}{162}\right)C_F \\
&\quad - \left(4B_4 - \frac{43}{81}\pi^4 + \frac{1471}{54}\zeta_3 + \frac{400}{243}\pi^2 + \frac{3845}{486}\right)C_A \\
&\quad - \frac{2}{9}\left(32\zeta_3 - \frac{1327}{27}\right)T_F n_l + \frac{1}{9}\left(112\zeta_3 - \frac{1685}{27}\right)T_F\biggr\}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 + \cdots
\end{aligned} \tag{5.15}$$

where

$$L_{\mathrm{os}} = 2\log\frac{\mu}{m_c^{\mathrm{os}}}\,. \tag{5.16}$$

At $\mu = m_c^{\mathrm{os}}$ we have, for the physical $SU(3)$ colour group,

$$\begin{aligned}
\tilde{\zeta}_j(m_c^{\mathrm{os}}) &= 1 + \frac{89}{864}\left(\frac{\alpha_s(m_c^{\mathrm{os}})}{\pi}\right)^2 \\
&\quad + \frac{1}{9}\biggl[-2\operatorname{Li}_2\left(\frac{1}{2}\right) - \frac{1}{12}\log^2 2\left(\log^2 2 - \pi^2\right) + \frac{427}{3240}\pi^4 - \frac{815}{1728}\zeta_3 + \frac{175}{486}\pi^2 - \frac{1057}{108}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{3}\left(\zeta_3 - \frac{1327}{864}\right) n_l \Bigg] \left(\frac{\alpha_s(m_c^{\text{os}})}{\pi}\right)^3 \\
&\approx 1 + 0.1030 \left(\frac{\alpha_s(m_c^{\text{os}})}{\pi}\right)^2 + (0.5976 + 0.0124 n_l) \left(\frac{\alpha_s(m_c^{\text{os}})}{\pi}\right)^3 . \qquad (5.17)
\end{aligned}
$$

6. Conclusion

The results on decoupling c-quark loops in the b-quark HQET obtained in the present paper can be used to improve the accuracy of extracting f_B from lattice HQET simulations. It requires the following steps (at least conceptually; in practice, some of them can be grouped together):

- Matching the lattice HQET to the continuum HQET in the $\overline{\text{MS}}$ scheme at a low scale $\mu \sim 1/a$, where a is the lattice spacing. It can be done using the lattice perturbation theory, or non-perturbatively.
- Running up to $\mu \sim m_c$ in the low-energy HQET with 3 active flavours.
- Matching the low-energy HQET to HQET with 4 active flavours. This is the subject of the present paper.
- Running up to $\mu \sim m_b$ in HQET with 4 active flavours.
- Matching to QCD with 5 active flavours, including b.

Currently, all steps (except the first one, which is lattice-specific) can be done at the next-to-next-to-leading order: the QCD/HQET matching is known at two loops [9, 4], and the HQET heavy–light current anomalous dimension (which determines running in both HQETs) — at three loops [23]. The present paper is the first step towards the N^{3}L order calculation: the step 3 can now be done with the three-loop accuracy. To complete this program, we need the QCD/HQET matching coefficients at three loops and the anomalous dimension at four loops. The first task does not seem impossible: the QCD/HQET matching at three loops can be calculated using the methods of [22]. Prospects of obtaining the four-loop anomalous dimension are doubtful. However, the intervals of μ (from $1/a$ to m_c and from m_c to m_b) are not really wide. Though formally the N^{3}L order calculation requires four-loop running, its effect is likely to be small, and three-loop running should be sufficient.

We have also obtained the generic formula (A.15) for the three-loop decoupling coefficient for the flavour-nonsinglet QCD current with n antisymmetrized γ-matrices. The expressions for $n = 0$, 1, 3, 4 can be obtained from published results; the formula for $n = 2$ is new.

The method of calculation of three-loop on-shell HQET diagrams with massive-quark loops can be applied to other physical problems. We hope to consider them in a future publication.

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A. Decoupling in QCD

A.1 Decoupling for the light-quark field

In order to find the decoupling coefficient for the heavy–light current, we need the decoupling coefficients for both the heavy-quark field (Sect. 4) and the light-quark one. The last coefficient has been calculated in [1]; however, only the result for $N_c = 3$ has been presented in the paper. Here we re-calculate this quantity. At $N_c = 3$, our result coincides with [1][2].

Similarly to Sect. 4, the bare decoupling coefficient for the light-quark field is given by

$$\zeta_q^0 = \frac{Z_q^{\rm os}}{Z_q^{\prime\rm os}}\,, \tag{A.1}$$

where

$$Z_q^{\rm os} = \frac{1}{1-\Sigma_V(0)}\,, \qquad Z_q^{\prime\rm os} = \frac{1}{1-\Sigma'_V(0)} = 1\,, \tag{A.2}$$

and the massless-quark self-energy is $\Sigma(p) = \not{p}\Sigma_V(p^2)$. The renormalized decoupling coefficient is

$$\zeta_q(\mu) = \frac{Z'_q(\alpha'_s(\mu), a'(\mu))}{Z_q(\alpha_s(\mu), a(\mu))}\zeta_q^0\,. \tag{A.3}$$

Only diagrams with a massive quark loop contribute to $\Sigma_V(0)$; they can be obtained from Fig. 5 by replacing the HQET line by a massless quark line. We made minimal replacements in the code calculating the HQET self-energy, so that the correctness of the light-quark result provides an additional check of the HQET calculation. The result is

$$\begin{aligned}\Sigma_V(0) = {}& C_F T_F \frac{g_0^4 m_{c0}^{-4\varepsilon}}{(4\pi)^d} I_0^2 \frac{(d-1)(d-2)(d-4)(d-6)}{2d(d-5)(d-7)} \\ & - C_F T_F \frac{g_0^6 m_{c0}^{-6\varepsilon}}{(4\pi)^{3d/2}} \frac{d-4}{d} \sum_{i=1}^{4} \sum_{j=1,2,5} C_i \bar{I}_j \sigma_{ij} + \cdots\,,\end{aligned} \tag{A.4}$$

where all non-zero coefficients are

$$\begin{aligned}&\sigma_{11} = s_{11}\,, \qquad \sigma_{15} = -2\sigma_{25} = s_{15}\,, \qquad \sigma_{32} = s_{32}\,, \qquad \sigma_{41} = s_{41}\,, \qquad \sigma_{45} = s_{45}\,,\\
&\sigma_{12} = 2(d-2)(d-3)(d-5)^2(4d^5 - 83d^4 + 673d^3 - 2646d^2 + 4952d - 3368)\,,\\
&\sigma_{21} = 3(d-5)(d-7)(d-10)(5d^7 - 156d^6 + 2040d^5 - 14470d^4 + 59897d^3 - 143640d^2\end{aligned}$$

[2] K.G. Chetyrkin has kindly provided an unpublished three-loop formula for $\zeta_q(\mu)$ for the $SU(N_c)$ colour group; our result coincides with it.

$$
\begin{aligned}
&\quad + 182340d - 92176)\,,\\
\sigma_{22} &= -(d-3)(d-5)(2d^7 - 53d^6 + 588d^5 - 3557d^4 + 12727d^3 - 26983d^2 + 31224d\\
&\quad - 14932) - 2d(d-1)(d-3)^2(d-5)^3(1-a_0)
\end{aligned}
\tag{A.5}
$$

(see (4.13)). The on-shell renormalization constant of the light-quark field (A.2) is not gauge-invariant, starting from three loops; the a_0-dependent term in it is the same as in the on-shell renormalization constant of the HQET field (4.6), see (4.10) and (4.13).

Now we calculate the renormalized decoupling coefficient (A.3):

$$
\begin{aligned}
&\zeta_q(\mu) = 1 + C_F T_F \left(2L - \frac{5}{6}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\\
&+ C_F T_F \biggl\{ \frac{1}{3} C_A a(\mu) L^3 + \left[2C_F - \left(\frac{13}{6}a(\mu) + 1\right) C_A - \frac{8}{3} T_F\right] L^2\\
&\quad + \left[-15 C_F + \frac{1}{9}\left(\frac{121}{2}a(\mu) + 232\right) C_A - \frac{20}{9} T_F n_l\right] L\\
&\quad + \left(8\zeta_3 + \frac{155}{18}\right) C_F + \left[\frac{1}{3}\left(8\zeta_3 - \frac{2387}{72}\right) a(\mu) - 8\zeta_3 - \frac{1187}{81}\right] C_A\\
&\quad - \frac{70}{81} T_F (2n_l + 1) \biggr\} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 + \cdots
\end{aligned}
\tag{A.6}
$$

A.2 Decoupling for light–light currents

Here we shall consider decoupling for the flavour non-singlet QCD currents

$$
j_0 = \bar{q}_0 \Gamma \tau q_0\,, \qquad \Gamma = \gamma^{[\alpha_1} \cdots \gamma^{\alpha_n]}\,,
\tag{A.7}
$$

where τ is a flavour matrix with $\operatorname{Tr}\tau = 0$, for an arbitrary n at three loops (the two-loop result has been obtained in [4]).

The proper vertex $\Gamma(p,p') = \Gamma + \Lambda(p,p')$ at $p = p' = 0$ has the structure $\Gamma(0,0) = \Gamma_n \Gamma$, where Γ is the Dirac matrix in (A.7), and $\Gamma_n = 1 + \Lambda_n$ is scalar. Similarly to Sect. 5, the bare decoupling coefficient is

$$
\zeta_n^0 = Z_q^{\mathrm{os}} \Gamma_n\,,
\tag{A.8}
$$

because $Z_q'^{\mathrm{os}} = 1$, $\Gamma_n' = 1$. This is nothing but the on-shell matrix element of the current in the full theory, and it must be gauge invariant. The renormalized decoupling coefficient is

$$
\zeta_n(\mu) = \frac{Z_n'(\alpha_s'(\mu))}{Z_n(\alpha_s(\mu))} \zeta_n^0\,,
\tag{A.9}
$$

where $j_0 = Z_n(\alpha_s(\mu)) j(\mu)$, and Z_n can be reconstructed from the anomalous dimension γ_n, which has been calculated at three loops for a generic n in [24].

The diagrams for $\Lambda^{(3)}(0,0)$ can be obtained from the HQET vertex diagrams (Sect. 5.2) by replacing the HQET line by a massless quark line. To find the result for an arbitrary n, we follow the method used in [9]. If we make no assumptions about the properties of the matrix Γ, then the three-loop vertex has the structure

$$
\Lambda^{(3)}(0,0) = x_1 \Gamma + x_2 \gamma^{\mu_1}\gamma^{\mu_2}\Gamma\gamma_{\mu_2}\gamma_{\mu_1} + x_3 \gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\Gamma\gamma_{\mu_4}\gamma_{\mu_3}\gamma_{\mu_2}\gamma_{\mu_1}\,,
\tag{A.10}
$$

because the number of γ-matrices to the left of Γ and to the right of it has to be even (for the massless quark). Taking separately traces of γ-matrices to the left of Γ with L_i, and of those to the right with R_i, where $L_i \times R_i = 1 \times 1$, $\gamma^{\nu_1}\gamma^{\nu_2} \times \gamma_{\nu_2}\gamma_{\nu_1}$, $\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4} \times \gamma_{\nu_4}\gamma_{\nu_3}\gamma_{\nu_2}\gamma_{\nu_1}$, and solving the linear system, we can find x_i via these double traces. We may apply this procedure to the integrands of the diagrams, thus expressing x_i via scalar Feynman integrals. Now we specialize to Γ being the antisymmetrized product of n γ-matrices (A.7):

$$\gamma^{\mu_1}\gamma^{\mu_2}\Gamma\gamma_{\mu_2}\gamma_{\mu_1} = h\Gamma\,, \qquad h = (d-2n)^2\,, \tag{A.11}$$

and obtain

$$\Lambda_n^{(3)} = x_1 + x_2 h + x_3 h^2\,. \tag{A.12}$$

We made minimal replacements in the code calculating the HQET vertex. For the vector current ($n = 1$), the Ward identity $\Lambda_1 = -\Sigma_V(0)$ gives $\zeta_1^0 = 1$, to all orders of perturbation theory. Therefore, corrections in ζ_n^0 (A.8) are proportional to $h - (d-2)^2$. We obtain

$$\begin{aligned}\zeta_n^0 = 1 + C_F T_F \frac{g_0^4 m_{c0}^{-4\varepsilon}}{(4\pi)^d}\left[h - (d-2)^2\right]\Bigg[&-I_0^2\frac{(d-2)(d-6)}{2d(d-5)(d-7)}\\ &+ \frac{g_0^2 m_{c0}^{-2\varepsilon}}{(4\pi)^{d/2}}\frac{1}{d(d-1)}\sum_{i=1}^{4}\sum_{j=1,2,5} C_i \bar{I}_j v_{ij} + \cdots\Bigg]\,,\end{aligned} \tag{A.13}$$

where all non-zero coefficients are

$$\begin{aligned} v_{11} &= \sigma_{11}\,, \qquad v_{15} = \sigma_{15}\,, \qquad v_{21} = \sigma_{21}\,, \qquad v_{25} = \sigma_{25}\,,\\ v_{32} &= \sigma_{32}\,, \qquad v_{41} = \sigma_{41}\,, \qquad v_{45} = \sigma_{45}\,,\\ v_{12} &= -\frac{1}{2}(d-3)(d-5)^2\Big[(d-1)(d-4)^2(d-5)(d-9)h\\ &\quad - 4\left(4d^6 - 94d^5 + 882d^4 - 4209d^3 + 10681d^2 - 13532d + 6736\right)\Big]\,,\\ v_{22} &= -(d-3)(d-5)\Big[(d-1)(d-4)^2(d-5)^2 h\\ &\quad + 2d^7 - 55d^6 + 629d^5 - 3878d^4 + 13904d^3 - 28928d^2 + 32274d - 14932\Big] \end{aligned} \tag{A.14}$$

The result is gauge invariant, as expected.

Now we calculate the renormalized decoupling coefficient (A.9):

$$\begin{aligned}\zeta_n(\mu) = 1 &+ \frac{1}{3}C_F T_F(n-1)\left[2(n-3)L^2 - \frac{2}{3}(n-15)L + \frac{1}{18}(85n-267)\right]\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\\ &+ C_F T_F(n-1)\Bigg\{\frac{4}{27}(n-3)\left[11C_A - 4T_F(n_l+2)\right]L^3\\ &\quad + \frac{2}{3}\left[(n-3)(5n^2-20n-8)C_F - \frac{1}{3}(n-3)(6n^2-24n-29)C_A + \frac{4}{9}(n-15)T_F\right]L^2\\ &\quad + \left[\left(16(n-3)\zeta_3 + \frac{1}{9}(17n^3+n^2-275n+117)\right)C_F\right.\end{aligned}$$

$$
\begin{aligned}
&\quad - \left(16(n-3)\zeta_3 + \frac{1}{81}(18n^3+306n^2-2003n+405)\right) C_A \\
&\quad - \frac{4}{81}(163n-477)T_F n_l - \frac{496}{81}(n-3)T_F\Bigg] L \\
&+ \frac{1}{3}\Bigg[16(n-3)B_4 - \frac{4}{5}(n-3)\pi^4 + \frac{2}{3}(187n-513)\zeta_3 \\
&\quad + \frac{1}{54}(1727n^3-11681n^2+15946n+12294)\Bigg] C_F \\
&- \frac{1}{3}\Bigg[8(n-3)B_4 - \frac{4}{5}(n-3)\pi^4 + \frac{1}{9}(725n-1887)\zeta_3 \\
&\quad + \frac{1}{243}(3087n^3-21393n^2+22565n+49881)\Bigg] C_A \\
&- \frac{4}{27}\left[32(n-3)\zeta_3 - \frac{1}{27}(1171n-3981)\right] T_F n_l \\
&+ \frac{2}{27}\left[112(n-3)\zeta_3 - \frac{1}{27}(1841n-5055)\right] T_F \Bigg\} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3 + \cdots
\end{aligned} \tag{A.15}
$$

The vector current does not renormalize ($Z_1 = 1$, $Z'_1 = 1$), and we have $\zeta_1(\mu) = 1$ to all orders. This is to be expected; for example, for a diagonal flavour matrix τ the integral of the vector current is a combination of the differences of the full numbers of quarks and antiquarks of several flavours, and these differences are some integers which are the same in both theories. Decoupling of the scalar current is related to that of the mass:

$$
\zeta_0(\mu) = \zeta_m^{-1}(\mu)\,, \tag{A.16}
$$

to all orders. Our result (A.15) at $n = 0$ reproduces the three-loop mass decoupling [1].

The currents $j(\mu)$ with $n = 4$, 3 differ from those with $n = 0$, 1 by insertion of the 't Hooft–Veltman γ_5^{HV}. They differ from the corresponding currents with the anticommuting γ_5^{AC} by finite renormalizations:

$$
\begin{aligned}
\left(\bar{q}\gamma_5^{\mathrm{AC}}\tau q\right)_\mu &= Z_P(\alpha_s(\mu))\left(\bar{q}\gamma_5^{\mathrm{HV}}\tau q\right)_\mu\,, \\
\left(\bar{q}\gamma_5^{\mathrm{AC}}\gamma^\alpha\tau q\right)_\mu &= Z_A(\alpha_s(\mu))\left(\bar{q}\gamma_5^{\mathrm{HV}}\gamma^\alpha\tau q\right)_\mu\,.
\end{aligned} \tag{A.17}
$$

The finite renormalization constants Z_P, Z_A are known with three-loop accuracy [25, 26]. Inserting γ_5^{AC} does not change the decoupling coefficient. Therefore,

$$
\frac{\zeta_4(\mu)}{\zeta_0(\mu)} = \frac{Z'_P(\alpha'_s(\mu))}{Z_P(\alpha_s(\mu))}\,, \qquad \frac{\zeta_3(\mu)}{\zeta_1(\mu)} = \frac{Z'_A(\alpha'_s(\mu))}{Z_A(\alpha_s(\mu))}\,. \tag{A.18}
$$

We have checked that our result (A.15) satisfies these strong checks.

The result for the tensor current is new:

$$
\begin{aligned}
\zeta_2(\mu) &= 1 + \frac{1}{3}C_F T_F\left(-2L^2 + \frac{26}{3}L - \frac{97}{18}\right)\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \\
&+ C_F T_F\Bigg\{-\frac{4}{27}\left[11C_A - 4T_F(n_l+2)\right]L^3 + \frac{2}{3}\left(28C_F - \frac{53}{3}C_A - \frac{52}{9}T_F\right)L^2
\end{aligned}
$$

$$
+\frac{1}{9}\left[-(144\zeta_3+293)C_F+\left(144\zeta_3+\frac{2233}{9}\right)C_A+\frac{604}{9}T_F n_l+\frac{496}{9}T_F\right]L
$$
$$
-\frac{1}{3}\left(16B_4-\frac{4}{5}\pi^4+\frac{278}{3}\zeta_3-\frac{5639}{27}\right)C_F+\frac{1}{3}\left(8B_4-\frac{4}{5}\pi^4+\frac{437}{9}\zeta_3-\frac{34135}{243}\right)C_A
$$
$$
+\frac{4}{27}\left(32\zeta_3-\frac{1639}{27}\right)T_F n_l-\frac{2}{27}\left(112\zeta_3-\frac{1373}{27}\right)T_F\Bigg\}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3+\cdots \qquad \text{(A.19)}
$$

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