SFB/CPP-06-21
TTP06-15
hep-ph/0604234

Four-loop moments of the heavy quark vacuum polarization function in perturbative QCD

K. G. Chetyrkin [a, 1], J. H. Kühn [a] and C. Sturm [b]

[a] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany*

[b] *Dipartimento di Fisica Teorica, Università di Torino, I-10125 Torino, Italy* & *INFN, Sezione di Torino, Italy*

Abstract

New results at four-loop order in perturbative QCD for the first two Taylor coefficients of the heavy quark vacuum polarization function are presented. They can be used to perform a precise determination of the charm- and bottom-quark mass. Implications for the value of the quark masses are briefly discussed.

[1] On leave from Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, 117312, Russia.

1 Introduction

Two-point correlators are of central importance for many theoretical and phenomenological investigations in Quantum Field Theory. As a consequence they are studied in great detail in the framework of perturbative calculations. Due to simple kinematics (only one external momentum) even multi-loop calculations can be performed. The results for all physically interesting diagonal and non-diagonal correlators including *full* quark mass dependence are available up to $\mathcal{O}(\alpha_s^2)$ [1–3].

At four-loop order the two-point correlators can be considered in two limits: In the high energy limit massless propagators need to be calculated, in the low energy limit vacuum integrals ("tadpole diagrams") arise. The evaluation of the latter in three-loop approximation has been pioneered in ref. [4] and automated in ref. [5].

Recently first results for physical quantities, which are related to four-loop tadpole diagrams, have been obtained. The four-loop matching condition for the strong coupling constant α_s at a heavy quark threshold has been calculated in ref. [6, 7]. The four-loop QCD contribution to the electroweak ρ-parameter induced by the singlet diagrams of the Z-boson self-energy has been computed in ref. [8].

The detailed knowledge of the heavy quark correlator is important for the precise determination of heavy quark masses with the help of QCD sum rules. As known from ref. [9, 10], the determination of the charm- and bottom-quark mass is further improved, if the four-loop corrections, hence $\mathcal{O}(\alpha_s^3)$, for the lowest Taylor coefficients of the vacuum polarization function are available. The subset of four-loop contributions to the lowest two moments, which involve two internal loops from massive and massless fermions coupled to gluons, hence of $\mathcal{O}(\alpha_s^3\, n_f^2)$, has already been calculated in ref. [11]. The symbol n_f denotes the number of active quark-flavors, contributing through fermion-loops inserted into gluon lines. The terms being proportional to $\alpha_s^j\, n_l^{j-1}$ are even known to all orders j in perturbative QCD [12]. The symbol n_l denotes the number of light quarks, considered as massless. In this paper the complete four-loop contributions originating from non-singlet diagrams for the first two Taylor coefficients are presented. Singlet contributions have been studied in ref. [13–15].

In general, the tadpole diagrams encountered in these calculations contain both massive and massless lines. As is well-known, the computation of the four-loop β-function can be reduced to the evaluation of four-loop tadpoles composed of completely massive propagators only. Calculations for this case have been performed in [16–18].

The outline of this paper is as follows. In section 2 we briefly introduce the notation and discuss generalities. In section 3 we discuss the reduction to master integrals, describe the solution of the linear system of equations, give the result for the lowest two moments and discuss the impact on the quark mass determination. Our conclusions and a brief summary are given in section 4.

2 Notation and Generalities

The correlator $\Pi^{\mu\nu}(q)$ of two currents is defined as

$$\Pi^{\mu\nu}(q,j) = i \int dx\, e^{iqx} \langle 0|T j^\mu(x) j^\nu(0)|0\rangle \,, \tag{1}$$

with the current $j^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)$ being composed out of the heavy quark fields $\Psi(x)$. The function $\Pi^{\mu\nu}(q)$ is conveniently written in the form:

$$\Pi^{\mu\nu}(q) = \left(-q^2\, g^{\mu\nu} + q^\mu\, q^\nu\right)\, \Pi(q^2)\,. \tag{2}$$

The vanishing of the longitudinal contribution, as well as the confirmation of ${\Pi^\mu}_\mu(q^2=0)=0$, has been used as a check of the calculation. The function $\Pi(q^2)$ is of phenomenological interest, because it can be related to the ratio $R(s) = \sigma(e^+\,e^- \to \text{hadrons})/\sigma(e^+\,e^- \to \mu^+\mu^-)$ with the help of dispersion relations:

$$\Pi(q^2) = \frac{1}{12\,\pi^2}\int_0^\infty ds\,\frac{R(s)}{(s-q^2)} \quad \text{mod subtr.} \tag{3}$$

Performing the derivative of eq. (3) with respect to q^2 one obtains on the one hand "experimental" moments

$$\mathcal{M}_n^{\text{exp}} = \int ds\,\frac{R(s)}{s^{n+1}}\,, \tag{4}$$

which can be evaluated from the R-ratio. On the other hand one can define "theoretical" moments

$$\mathcal{M}_n^{\text{th}} = Q_q^2\,\frac{9}{4}\left(\frac{1}{4\,\overline{m}_q^2}\right)^n \overline{C}_n\,, \tag{5}$$

which are related to the Taylor coefficients $\overline{C}_n$ of the vacuum polarization function:

$$\overline{\Pi}(q^2) = \frac{3\,Q_q^2}{16\,\pi^2}\sum_{n\geq 0}\overline{C}_n\,\overline{z}^n\,, \tag{6}$$

with $\overline{z} = q^2/(4\,\overline{m}^2)$. Symbols carrying a bar denote that renormalization has been performed in $\overline{\text{MS}}$-scheme. The Taylor expansion in q^2 around $q^2=0$ leads to massive tadpole integrals. The first and higher derivatives can be used for a precise determination of the charm- and bottom-quark mass. However, also

the lowest expansion coefficient $\overline{C}_0$ has an interesting physical meaning, since it relates the coupling of the electromagnetic interaction in different renormalization schemes.
Sum rules as tool for the determination of the charm- and bottom-quark mass have been suggested since long in ref. [19]. This method has then been applied later also to the determination of the bottom-quark mass [20]. One of the most precise determinations of the charm- and bottom-quark mass being based on sum rules in connection with the calculation of three-loop moments in perturbative QCD has been performed in ref. [9], with the values of $\overline{m}_c(\overline{m}_c) = 1.304(27)$ GeV and $\overline{m}_b(\overline{m}_b) = 4.191(51)$ GeV as results for the charm- and the bottom-quark mass.

It is convenient to define the expansion of the Taylor coefficients $\overline{C}_n$ of the vacuum polarization function in the strong coupling constant α_s as

$$\overline{C}_n = \overline{C}_n^{(0)} + \left(\frac{\alpha_s}{\pi}\right)^1 \overline{C}_n^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \overline{C}_n^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3 \overline{C}_n^{(3)} + \ldots . \tag{7}$$

Due to the distinct mass hierarchy of the quarks in the Standard-Model, one can consider for the above moments one species of quarks as massive ($n_h = 1$) and all lighter ones as massless. The number of active quarks n_f is then decomposed according to $n_f = n_l + n_h$. For convenience the symbol n_h is kept explicitly in the following.

3 Calculations and Results

The Feynman-diagrams have been generated with the help of the program `QGRAF` [21]. After performing the expansion in the external momentum q all integrals can be expressed in terms of 55 independent vacuum topologies with additional increased powers of propagators and additional irreducible scalar products.
In order to reduce this host of integrals to a small set of master integrals the traditional Integration-by-parts(IBP) method has been used in combination with Laporta's algorithm [22, 23]. The resulting system of linear equations has been solved with a `FORM3` [24–26] based program, in which partially also ideas described in ref. [23, 27, 28] have been implemented. The rational functions in the space-time dimension d, which arise in this procedure, have been simplified with the program `FERMAT` [29]. Masking of large integral coefficients has been implemented, a strategy also adopted in the program `AIR` [30]. In order to achieve the reduction to 13 master integrals more than 31 million IBP-equations have been generated and solved. This leads to integral-tables with solutions for around five million integrals. Furthermore all symmetries of the 55 independent topologies have been taken into account in an automated way, by reshuffling the powers of

the propagators of a given topology in a unique way. Taking into account symmetries is important in order to keep the size of the integral-tables under control. The first four master integrals shown in fig. 1 can be calculated completely analytically in terms of Γ-functions. The fifth integral (T_{52}) can be obtained from results of ref. [4, 31].

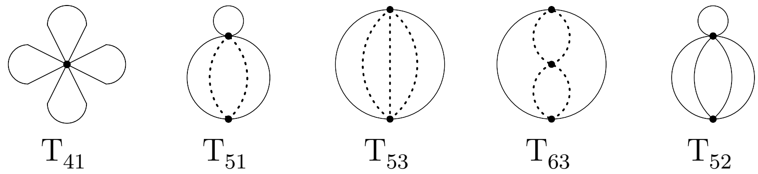

Figure 1: Factorized or analytically known master integrals. The solid (dashed) lines denote massive (massless) propagators.

The remaining eight master integrals (fig. 2) have been determined with high precision numerics with the difference equation method [23] in ref. [32]. Independently they have been determined in ref. [33] by constructing an ε-finite basis. Some of these master integrals have also been calculated in ref. [8, 11, 34, 35].

Inserting the master integrals and performing renormalization of the strong

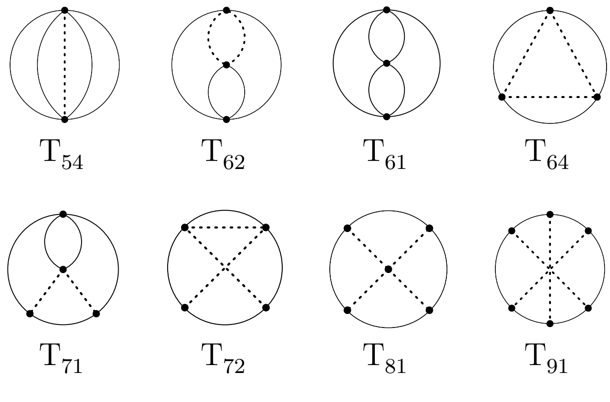

Figure 2: Master integrals where only a few terms of their ε-expansion are known analytically. The solid (dashed) lines denote massive (massless) propagators.

coupling constant α_s, the external current and the mass $m = \overline{m}(\mu)$ leads to the following result ($\mu = \overline{m}$):

$$\begin{aligned}
\overline{C}_0^{(3)} &= n_l\, n_h \Bigg(-\frac{2}{9}\, a_4 + \frac{7043}{34992} - \frac{1}{108}\, \log^4(2) + \frac{\pi^2}{108}\, \log^2(2) + \frac{49}{12960}\, \pi^4 \\
&\qquad -\frac{127}{324}\, \zeta_3 \Bigg) + n_h^2 \left(\frac{610843}{2449440} - \frac{661}{2835}\, \zeta_3 \right) + n_l^2 \left(\frac{17897}{69984} - \frac{31}{162}\, \zeta_3 \right) \\
&+ n_l \Bigg(-\frac{50}{81}\, a_4 - \frac{71629}{46656} - \frac{25}{972}\, \log^4(2) + \frac{25}{972}\, \log^2(2)\, \pi^2 + \frac{8533}{116640}\, \pi^4
\end{aligned}$$

$$
\begin{aligned}
& -\frac{21343}{3888}\,\zeta_3\Big) \\
& + \; n_h\left(-\frac{28364}{405}\,a_4-\frac{83433703}{8164800}-\frac{7091}{2430}\,\log^4(2)+\frac{7091}{2430}\,\log^2(2)\,\pi^2\right. \\
& \qquad \left.+\frac{14873}{18225}\,\pi^4-\frac{14509529}{340200}\,\zeta_3+\frac{5}{3}\,\zeta_5\right) \\
& + \; \frac{64}{3}\,a_5+\frac{12007}{243}\,a_4-\frac{8572423579}{604661760}+\frac{12007}{5832}\,\log^4(2)-\frac{8}{45}\,\log^5(2) \\
& - \; \frac{12007}{5832}\,\log^2(2)\,\pi^2+\frac{8}{27}\,\log^3(2)\,\pi^2-\frac{1074967}{699840}\,\pi^4+\frac{34}{135}\,\log 2\,\pi^4+\frac{\pi^6}{486} \\
& + \; \frac{53452189}{349920}\,\zeta_3-\frac{28}{243}\,\zeta_3^2-\frac{37651}{648}\,\zeta_5-\frac{1}{432}\,T_{54,3}+\frac{7}{1296}\,T_{62,2}\,, \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
\overline{C}_1^{(3)} = \; & n_l\,n_h\left(-\frac{116}{243}\,a_4+\frac{262877}{787320}-\frac{29}{1458}\,\log^4(2)+\frac{29}{1458}\,\log^2(2)\,\pi^2+\frac{1421}{174960}\,\pi^4\right. \\
& \left.-\frac{38909}{58320}\,\zeta_3\right)+n_h^2\left(\frac{163868}{295245}-\frac{3287}{7290}\,\zeta_3\right)+n_l^2\left(\frac{42173}{98415}-\frac{112}{405}\,\zeta_3\right) \\
& + \; n_h\left(-\frac{1394804}{8505}\,a_4-\frac{27670774337}{1414551600}-\frac{348701}{51030}\,\log^4(2)\right. \\
& \qquad \left.+\frac{348701}{51030}\,\log^2(2)\,\pi^2+\frac{1447057}{765450}\,\pi^4-\frac{95617883401}{943034400}\,\zeta_3+\frac{128}{27}\,\zeta_5\right) \\
& + \; n_l\left(-\frac{4793}{7290}\,a_4-\frac{9338899}{2099520}-\frac{4793}{174960}\,\log^4(2)+\frac{4793}{174960}\,\log^2(2)\,\pi^2\right. \\
& \qquad \left.+\frac{372689}{839808}\,\pi^4-\frac{48350497}{1399680}\,\zeta_3\right) \\
& - \; \frac{127168}{1215}\,a_5-\frac{22152385}{61236}\,a_4+\frac{237787820456749}{380936908800}-\frac{22152385}{1469664}\,\log^4(2) \\
& + \; \frac{15896}{18225}\,\log^5(2)+\frac{22152385}{1469664}\,\log^2(2)\,\pi^2-\frac{15896}{10935}\,\log^3(2)\,\pi^2 \\
& - \; \frac{29962031}{176359680}\,\pi^4-\frac{67558}{54675}\,\log 2\,\pi^4+\frac{60701}{1071630}\,\pi^6+\frac{282830677079}{881798400}\,\zeta_3 \\
& - \; \frac{242804}{76545}\,\zeta_3^2-\frac{653339}{2430}\,\zeta_5-\frac{5849}{272160}\,T_{54,3}+\frac{60701}{408240}\,T_{62,2}\,, \qquad (9)
\end{aligned}
$$

where Riemann's zeta-function ζ_n and the polylogarithm-function $\mathrm{Li}_n(1/2)$ are defined by

$$
\zeta_n=\sum_{k=1}^{\infty}\frac{1}{k^n} \quad \text{and} \quad a_n=\mathrm{Li}_n(1/2)=\sum_{k=1}^{\infty}\frac{1}{2^k k^n}\,. \qquad (10)
$$

The two numerical constants that appear in eq. (8) and eq. (9) were obtained in eq. (19) and eq. (20) of ref. [33]:

$$
T_{54,3}=-8445.8046390310298\ldots \quad \text{and} \quad T_{62,2}=-4553.4004372195263\ldots\,. \qquad (11)
$$

Numerically the coefficients $\overline{C}_0$ and $\overline{C}_1$ are given by:

$$\begin{aligned}\overline{C}_0 &= \left(\frac{\alpha_s}{\pi}\right) 1.4444 + \left(\frac{\alpha_s}{\pi}\right)^2 (1.5863 + 0.1387\, n_h + 0.3714\, n_l) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 (0.0252\, n_h\, n_l + 0.0257\, n_l^2 - 0.0309\, n_h^2 - 3.3426\, n_h - 1.2112\, n_l \\ &\qquad +1.4186) + \ldots \end{aligned} \tag{12}$$

and

$$\begin{aligned}\overline{C}_1 &= 1.0667 + \left(\frac{\alpha_s}{\pi}\right) 2.5547 + \left(\frac{\alpha_s}{\pi}\right)^2 (0.2461 + 0.2637\, n_h + 0.6623\, n_l) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 (0.1658\, n_h\, n_l + 0.0961\, n_l^2 + 0.0130\, n_h^2 - 6.4188\, n_h - 2.9605\, n_l \\ &\qquad +8.2846) + \ldots . \end{aligned} \tag{13}$$

The coefficient $\overline{C}_1$ can be used to evaluate the influence of the four-loop correction on the value of the charm- and bottom-quark mass given in section 2. The $\overline{\text{MS}}$-charm-quark mass is reduced by about 2 MeV, for bottom-quarks we find a reduction by about 4 MeV.

4 Summary and Conclusion

The calculation of Taylor expansion coefficients of the vacuum polarization function is important for a precise determination of the charm- and bottom-quark mass. Within this work we have presented a new result at four-loop order in perturbative QCD for the first two coefficients of the Taylor expansion. For the computation the traditional IBP-method in combination with Laporta's algorithm has been used in order to reduce all appearing integrals on a small set of master integrals. The new four-loop contributions decrease the value of the $\overline{\text{MS}}$-charm- (bottom-)quark mass by around 2 MeV (4 MeV). With the knowledge of the four-loop contributions the theoretical uncertainty is well under control in the view of the current and foreseeable precision of experimental data.

Acknowledgments:
The authors would like to thank M. Faisst for numerous discussions and for providing numerical cross checks of partial results. The work was supported by the Deutsche Forschungsgemeinschaft through the SFB/TR-9 "Computational Particle Physics". The work of C.S. was also partially supported by MIUR under contract 2001023713_006.

References

[1] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, Nucl. Phys. **B482** (1996) 213.

[2] K. G. Chetyrkin, R. Harlander and M. Steinhauser, Phys. Rev. **D58** (1998) 014012.

[3] K. G. Chetyrkin, J. H. Kühn and M. Steinhauser, Nucl. Phys. **B505** (1997) 40.

[4] D. J. Broadhurst, Z. Phys. **C54** (1992) 599.

[5] M. Steinhauser, Comput. Phys. Commun. **134** (2001) 335.

[6] K. G. Chetyrkin, J. H. Kühn and C. Sturm, hep-ph/0512060.

[7] Y. Schröder and M. Steinhauser, JHEP **01** (2006) 051.

[8] Y. Schröder and M. Steinhauser, Phys. Lett. **B622** (2005) 124.

[9] J. H. Kühn and M. Steinhauser, Nucl. Phys. **B619** (2001) 588.

[10] J. H. Kühn and M. Steinhauser, JHEP **10** (2002) 018.

[11] K. G. Chetyrkin, J. H. Kühn, P. Mastrolia and C. Sturm, Eur. Phys. J. **C40** (2005) 361.

[12] A. G. Grozin and C. Sturm, Eur. Phys. J. **C40** (2005) 157.

[13] S. Groote and A. A. Pivovarov, Eur. Phys. J. **C21** (2001) 133.

[14] J. Portoles and P. D. Ruiz-Femenia, J. Phys. **G29** (2003) 349.

[15] J. Portoles and P. D. Ruiz-Femenia, Eur. Phys. J. **C24** (2002) 439.

[16] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. **B400** (1997) 379.

[17] K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, JHEP **04** (2003) 036.

[18] M. Czakon, Nucl. Phys. **B710** (2005) 485.

[19] V. A. Novikov, B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Phys. Rept. **41** (1978) 1.

[20] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. **127** (1985) 1.

[21] P. Nogueira, J. Comput. Phys. **105** (1993) 279.

[22] S. Laporta and E. Remiddi, Phys. Lett. **B379** (1996) 283.

[23] S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087.

[24] J. A. M. Vermaseren, New features of FORM, math-ph/0010025 (2000).

[25] J. A. M. Vermaseren, Nucl. Phys. Proc. Suppl. **116** (2003) 343.

[26] M. Tentyukov and J. A. M. Vermaseren, cs.sc/0604052.

[27] P. Mastrolia and E. Remiddi, Nucl. Phys. Proc. Suppl. **89** (2000) 76.

[28] Y. Schröder, Nucl. Phys. Proc. Suppl. **116** (2003) 402.

[29] R. H. Lewis, Fermat's User Guide, http://www.bway.net/~lewis/ .

[30] C. Anastasiou and A. Lazopoulos, JHEP **07** (2004) 046.

[31] D. J. Broadhurst, hep-th/9604128.

[32] Y. Schröder and A. Vuorinen, JHEP **06** (2005) 051.

[33] K. G. Chetyrkin, M. Faisst, C. Sturm and M. Tentyukov, hep-ph/0601165.

[34] S. Laporta, Phys. Lett. **B549** (2002) 115.

[35] B. A. Kniehl and A. V. Kotikov, hep-ph/0508238.