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Dimensional Reduction applied to QCD at three loops

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Abstract

Dimensional Reduction is applied to QCD in order to compute various renormalization constants in the $\overline{\text{DR}}$ scheme at higher orders in perturbation theory. In particular, the β function and the anomalous dimension of the quark masses are derived to three-loop order. Special emphasis is put on the proper treatment of the so-called ε-scalars and the additional couplings which have to be considered.

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1 Introduction

The big success of the Standard Model of elementary particle physics is based to a large extent on precision calculations which sometimes reach three-, four- and even five-loop accuracy. Such calculations currently all rely on Dimensional Regularization (DREG) [1, 2] which is an elegant and powerful tool to parametrize the divergences occurring at intermediate steps of the calculations.

In DREG, the number of space-time dimensions is altered from four to $D = 4 - 2\epsilon$, which renders the loop integrations finite. It is clear, however, that if DREG is applied to a 4-dimensional supersymmetric theory, the number of bosonic and fermionic degrees of freedom in super-multiplets is no longer equal, such that supersymmetry (SUSY) is explicitly broken. In order to avoid this problem, Dimensional Reduction (DRED) has been suggested as an alternative regularization method [3]. Space-time is compactified to $D = 4-2\epsilon$ dimensions in DRED, such that the number of vector field components remains equal to four. Momentum integrations are D-dimensional, however, and divergences are parametrized in terms of $1/\epsilon$ poles, just like in DREG. Since it is assumed that $\epsilon > 0$, the four-dimensional vector fields can be decomposed in terms of D-dimensional ones plus so-called ε-scalars. The occurrence of these ε-scalars is therefore the only difference between DREG and DRED, so that all the calculational techniques developed for DREG are applicable also in DRED.

Nevertheless, it soon was realized that DRED suffers from mathematical inconsistencies in its original formulation [4]. Currently, it seems that they can only be avoided by interpreting the fields as living in an infinite dimensional space, which again leads to explicit SUSY breaking [5, 6]. A higher order calculation will therefore require similar SUSY restoring counter terms as they are needed in DREG in general. For some of the currently available two-loop results, however, it has been shown that these counter terms vanish [7].

Although DRED was originally constructed for applications in supersymmetric models, it has been shown that in certain cases it can be useful also in non-supersymmetric theories [8] like QCD. For example, since it is possible to turn QCD (with massless quarks) into a super-Yang-Mills theory by simply adjusting the colour factors, a calculation using DRED provides the possibility to use non-trivial Ward identities for a check of complicated calculations (see, e.g., Ref. [9]).

As mentioned before, DRED parametrizes ultra-violet divergences as poles in ϵ. One can therefore formulate a renormalization scheme analogous to the $\overline{\rm MS}$ scheme, usually called the $\overline{\rm DR}$ scheme. In this paper, we compute the beta function of the strong coupling and the anomalous dimension of the quark masses to three-loop accuracy within this scheme. An important issue turns out to be the renormalization of the $q\bar{q}\varepsilon$ vertex. It requires to introduce a new, so-called evanescent coupling constant α_e. A similar argument holds for the four-ε-scalar vertex, but at the order considered here, this vertex does not get renormalized. The proper treatment of α_e leads us to conclude that the three-loop result for the QCD β function available in the literature [10] is incorrect. The correct result is provided in Section 3.

Our calculation should also be seen as a preparation for higher order calculations in supersymmetric theories like the minimal supersymmetric extension of the Standard Model. A common basis for the SUSY parameters that enter such an enterprise was formulated in the so-called SPA project in Ref. [11]. However, as we will show, the identification $\alpha_e = \alpha_s^{\overline{\rm DR}}$ at every scale renders some of the formulas given in that paper inconsistent.

The outline of the paper is as follows. In Section 2 we provide the notation and set the general framework for the calculation. Subsequently, we describe in Sections 3 and 4 the calculation for the β and the γ_m function up to three loops. Section 5 contains the conclusions.

2 Framework

We consider QCD and apply Dimensional Reduction (DRED) as the regularization scheme. Thus, besides the usual QCD Feynman rules for quarks (q) and gluons (g), we have to consider additional vertices involving the so-called ε-scalars, namely $q\bar{q}\varepsilon, g\varepsilon\varepsilon, gg\varepsilon\varepsilon, \varepsilon\varepsilon\varepsilon\varepsilon$ (for the corresponding Lagrange density, see, e.g., Refs. [12,13]). In a non-supersymmetric theory, it is important to note that the $q\bar{q}\varepsilon$ and the $q\bar{q}g$ vertices renormalize differently. Therefore, one needs to distinguish the coupling constant g_e, multiplying the $q\bar{q}\varepsilon$ vertex, from the strong coupling g_s [8]. Also the $\varepsilon\varepsilon\varepsilon\varepsilon$ vertex renormalizes differently; in fact, in QCD one needs to allow for a more general colour structure of this vertex, leading to three additional coupling constants λ_r ($r = 1, 2, 3$). In order to fix the notation we display the relevant part of the Lagrange density [8]

$$\mathcal{L} \;=\; \ldots - \frac{1}{4}\sum_{r=1}^{3} \lambda_r H_r^{abcd} \varepsilon_\sigma^a \varepsilon_{\sigma'}^c \varepsilon_\sigma^b \varepsilon_{\sigma'}^d + \ldots\,, \tag{1}$$

where ε denotes the ε-scalar fields, and σ and σ' are 2ε-dimensional indices. For the $SU(3)$ gauge group, the H_r^{abcd} are three independent rank four tensors which are symmetric under the interchange of (ab) and (cd). Our choice

$$\begin{aligned} H_1^{abcd} &= \frac{1}{2}\left(f^{ace}f^{bde} + f^{ade}f^{bce}\right)\,, \\ H_2^{abcd} &= \delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\,, \\ H_3^{abcd} &= \frac{1}{2}\left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) - \delta^{ab}\delta^{cd}\,, \end{aligned} \tag{2}$$

fixes the Feynman rules in a unique way. Note that for $SU(N_c), N_c > 3$, there are four independent tensors H_r^{abcd}. At the order considered in this paper no renormalization constant for the $\varepsilon\varepsilon\varepsilon\varepsilon$ vertex has to be introduced. The vertices $g\varepsilon\varepsilon$ and $gg\varepsilon\varepsilon$, on the other hand, are renormalized according to g_s [8].

g_e and λ_r will be called "evanescent couplings" in what follows, and we define

$$\alpha_s = \frac{g_s^2}{4\pi}\,, \quad \alpha_e = \frac{g_e^2}{4\pi} \quad \text{and} \quad \eta_r = \frac{\lambda_r}{4\pi}\,. \tag{3}$$

The renormalization constants for the couplings g_s and g_e, the quark mass m, the QCD gauge parameter ξ, as well as for the fields and the vertices are introduced as

$$\begin{aligned} g_s^0 &= \mu^\epsilon Z_s g_s\,, & g_e^0 &= \mu^\epsilon Z_e g_e\,, & m^0 &= m Z_m\,, \\ 1-\xi^0 &= (1-\xi)\,Z_3\,, & q^0 &= \sqrt{Z_2}\,q\,, & G_\mu^{0,a} &= \sqrt{Z_3}\,G_\mu^a\,, \\ \varepsilon_\sigma^{0,a} &= \sqrt{Z_3^\varepsilon}\,\varepsilon_\sigma^a\,, & c^{0,a} &= \sqrt{\tilde{Z}_3}\,c^a\,, & \Gamma_{q\bar{q}G}^0 &= Z_1\Gamma_{q\bar{q}G}\,, \\ \Gamma_{q\bar{q}\varepsilon}^0 &= Z_1^\varepsilon\Gamma_{q\bar{q}\varepsilon}\,, & \Gamma_{ccG}^0 &= \tilde{Z}_1\Gamma_{ccG}\,, \end{aligned} \tag{4}$$

where μ is the renormalization scale, $D = 4 - 2\epsilon$ is the number of space-time dimensions, and the bare quantities are marked by the superscript "0". The quark, gluon, ε-scalar, and ghost fields are denoted by q, G_μ^a, ε^a and c^a, respectively, and Γ_{xyz} stands for the vertex functions involving the particles x, y and z (a is the colour index.). The gauge parameter ξ is defined through the gluon propagator,

$$D_g^{\mu\nu}(q) \;=\; -i\,\frac{g^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}}{q^2 + i\varepsilon}\,. \tag{5}$$

From the renormalization of the ghost-gluon or quark-gluon vertex one obtains the renormalization constant of the strong coupling

$$Z_s \;=\; \frac{\tilde{Z}_1}{\tilde{Z}_3\sqrt{Z_3}} \;=\; \frac{Z_1}{Z_2\sqrt{Z_3}}\,. \tag{6}$$

Similarly, the quark–ε-scalar vertex leads to the relation

$$Z_e \;=\; \frac{Z_1^\varepsilon}{Z_2\sqrt{Z_3^\varepsilon}}\,. \tag{7}$$

It is well-known that $Z_s \neq Z_e$ even at one-loop order. Furthermore, both Z_s and Z_e depend on g_s, g_e, and λ_r [8]; note, however, that Z_s depends on g_e and λ_r only starting from three- and four-loop order, respectively, while Z_e depends on g_e and λ_r already at one- and two-loop order, respectively.

Let us next introduce the β functions both for DREG and DRED. In DREG, of course, the ε-scalars are absent, and from the definition

$$\beta^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}) \;=\; \mu^2\frac{\rm d}{{\rm d}\mu^2}\frac{\alpha_s^{\overline{\rm MS}}}{\pi} \tag{8}$$

the usual relation between $\beta^{\overline{\rm MS}}$ and Z_s is obtained:

$$\beta^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}) \;=\; -\epsilon\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\left(1+2\alpha_s^{\overline{\rm MS}}\frac{\partial\ln Z_s^{\overline{\rm MS}}}{\partial\alpha_s^{\overline{\rm MS}}}\right)^{-1}\,, \tag{9}$$

where $Z_s^{\overline{\rm MS}}$ denotes Z_s evaluated in the $\overline{\rm MS}$ scheme, and $\alpha_s^{\overline{\rm MS}}$ is the usual definition of the strong coupling within this scheme [14]. $\beta^{\overline{\rm MS}}$ is known to four-loop order (see Refs. [15,16] and references therein). Due to the fact that we have five different couplings in DRED, the relations between Z_s and Z_e and the corresponding beta functions are slightly more involved. They are given by

$$\begin{aligned}
\beta_s^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) &= \mu^2\frac{\rm d}{{\rm d}\mu^2}\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\\
&= -\left(\epsilon\frac{\alpha_s^{\overline{\rm DR}}}{\pi}+2\frac{\alpha_s^{\overline{\rm DR}}}{Z_s^{\overline{\rm DR}}}\frac{\partial Z_s^{\overline{\rm DR}}}{\partial\alpha_e}\beta_e+2\frac{\alpha_s^{\overline{\rm DR}}}{Z_s^{\overline{\rm DR}}}\sum_r\frac{\partial Z_s^{\overline{\rm DR}}}{\partial\eta_r}\beta_{\eta_r}\right)\left(1+2\frac{\alpha_s^{\overline{\rm DR}}}{Z_s^{\overline{\rm DR}}}\frac{\partial Z_s^{\overline{\rm DR}}}{\partial\alpha_s^{\overline{\rm DR}}}\right)^{-1}\,,\\
\beta_e(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) &= \mu^2\frac{\rm d}{{\rm d}\mu^2}\frac{\alpha_e}{\pi}\\
&= -\left(\epsilon\frac{\alpha_e}{\pi}+2\frac{\alpha_e}{Z_e}\frac{\partial Z_e}{\partial\alpha_s^{\overline{\rm DR}}}\beta_s^{\overline{\rm DR}}+2\frac{\alpha_e}{Z_e}\sum_r\frac{\partial Z_e}{\partial\eta_r}\beta_{\eta_r}\right)\left(1+2\frac{\alpha_e}{Z_e}\frac{\partial Z_e}{\partial\alpha_e}\right)^{-1}\,,
\end{aligned} \tag{10}$$

where it is understood that the renormalization constants Z_e and $Z_s^{\overline{\rm DR}}$ in Eq. (10) are evaluated within DRED with (modified) minimal subtraction, and $\alpha_s^{\overline{\rm DR}}$ is the corresponding strong coupling constant in this scheme. As in the $\overline{\rm MS}$ scheme, the coefficients of the single poles fully determine the β functions. Let us remark that the terms proportional to β_{η_r}, the beta functions corresponding to the couplings η_r, contribute to $\beta_s^{\overline{\rm DR}}$ only at the four-loop order. Furthermore, only the approximation $\beta_{\eta_r} = -\epsilon\frac{\eta_r}{\pi}$ is needed for the two-loop calculation of β_e.

In analogy to Eqs. (9) and (10) we introduce the anomalous mass dimensions which are given by

$$\begin{aligned}
\gamma_m^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}}) &= \frac{\mu^2}{m^{\overline{\rm MS}}}\frac{{\rm d}}{{\rm d}\mu^2}m^{\overline{\rm MS}} = -\pi\beta^{\overline{\rm MS}}\frac{\partial\ln Z_m^{\overline{\rm MS}}}{\partial\alpha_s^{\overline{\rm MS}}}\,,\\
\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) &= \frac{\mu^2}{m^{\overline{\rm DR}}}\frac{{\rm d}}{{\rm d}\mu^2}m^{\overline{\rm DR}}\\
&= -\pi\beta_s^{\overline{\rm DR}}\frac{\partial\ln Z_m^{\overline{\rm DR}}}{\partial\alpha_s^{\overline{\rm DR}}} - \pi\beta_e\frac{\partial\ln Z_m^{\overline{\rm DR}}}{\partial\alpha_e} - \pi\sum_r\beta_{\eta_r}\frac{\partial\ln Z_m^{\overline{\rm DR}}}{\partial\eta_r}\,. \qquad (11)
\end{aligned}$$

As in the case of the β function, $\gamma_m^{\overline{\rm DR}}$ also gets additional terms due to the dependence of Z_m on the evanescent coupling g_e and on the quartic ε-scalar couplings λ_r . The four-loop result for $\gamma_m^{\overline{\rm MS}}$ can be found in Refs. [17, 18].

Let us add a few remarks concerning the meaning of the evanescent coupling α_e at this point. In a non-supersymmetric theory, α_e can be set to an arbitrary value $\hat{\alpha}_e$ at an arbitrary, fixed scale $\hat{\mu}$, $\alpha_e(\hat{\mu}) \equiv \hat{\alpha}_e$ This corresponds to a choice of scheme and in turn determines the value of $\alpha_s^{\overline{\rm DR}}$ through, say, an experimental measurement. At any scale μ, both $\alpha_s^{\overline{\rm DR}}$ and α_e are then determined by the renormalization group equations (10). One particular scheme choice would be to set $\alpha_e(\hat{\mu}) = \alpha_s^{\overline{\rm DR}}(\hat{\mu})$. Note, however, that already at one-loop level one will have $\alpha_e(\mu) \neq \alpha_s^{\overline{\rm DR}}(\mu)$ for any $\mu \neq \hat{\mu}$ due to the difference in the renormalization group functions β_s and β_e.

In a supersymmetric theory, on the other hand, one necessarily has $\beta_s^{\overline{\rm DR}} = \beta_e$ and $\alpha_s^{\overline{\rm DR}} = \alpha_e$ at all scales. Thus, if one assumes that QCD is a low energy effective theory of SUSY-QCD, α_e is no longer a free parameter. Rather, α_e and $\alpha_s^{\overline{\rm DR}}$ are both related to the unique SUSY-QCD gauge coupling by matching relations (see, e.g., Ref. [19]) and renormalization group equations.

These considerations show that the choice $\alpha_e = \alpha_s^{\overline{\rm DR}}$ as it has been assumed in phenomenological analyses (see, e.g., Ref. [11]) is not compatible with the renormalization group evolution of these couplings unless all SUSY particles are taken into account in the running. In fact, it cannot be assumed at *any* scale as soon as one or more SUSY particles are integrated out. An example where this is relevant already at one-loop level is the $m^{\overline{\rm DR}} \leftrightarrow m^{\overline{\rm MS}}$ relation as will be pointed out in connection with Eq. (21) below. An analogous discussion holds also for the evanescent couplings η_r.

3 β function to three-loop order

Within the framework of DRED outlined in the previous section we have computed Z_1, Z_2, Z_3, $\tilde{Z}_1$ and $\tilde{Z}_3$ to three-loop order. They are obtained from the two- and three-point functions according to Eq. (4) (see, e.g., Ref. [20] for explicit formulae). Thus, according to Eq. (6), Z_s is computed in two different ways and complete agreement is found. Furthermore, we compute Z_1^ε and Z_3^ε to two-loop order and hence obtain Z_e to the same approximation.

Since only the divergent parts enter the renormalization constants, we can set all particle masses to zero and choose one proper external momentum in order to avoid infrared problems. For the generation of the about 11,000 diagrams we use `QGRAF` [21] and process the diagrams with `q2e` and `exp` [22, 23] in order to map them to `MINCER` [24] which can compute massless one-, two- and three-loop propagator-type diagrams.

The n-loop calculation leads to counter terms for g_s, g_e, and the gauge parameter ξ, which are then inserted into the $(n+1)$-loop calculation in order to subtract the sub-divergences. We remark that ε-scalars are treated just like physical particles in this procedure.

For the β function, to a large extent it is possible to avoid the calculation with ε-scalars and evaluate the Feynman diagrams by applying only slight modifications as compared to DREG. For that, after the projectors have been applied and the traces have been taken in $D = 4 - 2\epsilon$ dimensions, one sets $\epsilon = 0$. The evaluation of the momentum integrals, however, proceeds in D dimensions, just as for DREG. During the calculation it is necessary to keep track of the $q\bar{q}g$ vertices since the difference between DREG and DRED in the results of the corresponding diagrams effectively accounts for the contributions from the $q\bar{q}\varepsilon$ vertex. Thus, the renormalization constant Z_e has to be used for this contribution.

We refrain from listing explicit results for Z_s but instead present the results obtained from Eq. (10). Although β_e is only needed to one-loop order for the three-loop calculation of $\beta_s^{\overline{\rm DR}}$ we present the two-loop expression which enters the three-loop calculation of $\gamma_m^{\overline{\rm DR}}$. Writing

$$\begin{aligned}
\beta_s^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}}, \alpha_e, \{\eta_r\}) &= -\epsilon\frac{\alpha_s^{\overline{\rm DR}}}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{\overline{\rm DR}} \left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i \left(\frac{\alpha_e}{\pi}\right)^j \left(\frac{\eta_1}{\pi}\right)^k \left(\frac{\eta_2}{\pi}\right)^l \left(\frac{\eta_3}{\pi}\right)^m , \\
\beta_e(\alpha_s^{\overline{\rm DR}}, \alpha_e, \{\eta_r\}) &= -\epsilon\frac{\alpha_e}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{e} \left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i \left(\frac{\alpha_e}{\pi}\right)^j \left(\frac{\eta_1}{\pi}\right)^k \left(\frac{\eta_2}{\pi}\right)^l \left(\frac{\eta_3}{\pi}\right)^m ,
\end{aligned} \tag{12}$$

we find for the non-vanishing coefficients up to three respectively two loops:

$$
\begin{aligned}
\beta_{20}^{\overline{\rm DR}} &= \frac{11}{12}C_A - \frac{1}{3}Tn_f\,,\\
\beta_{30}^{\overline{\rm DR}} &= \frac{17}{24}C_A^2 - \frac{5}{12}C_ATn_f - \frac{1}{4}C_FTn_f\,,\\
\beta_{40}^{\overline{\rm DR}} &= \frac{3115}{3456}C_A^3 - \frac{1439}{1728}C_A^2Tn_f - \frac{193}{576}C_AC_FTn_f\\
&\quad + \frac{1}{32}C_F^2Tn_f + \frac{79}{864}C_AT^2n_f^2 + \frac{11}{144}C_FT^2n_f^2\,,\\
\beta_{31}^{\overline{\rm DR}} &= -\frac{3}{16}C_F^2Tn_f\,,\\
\beta_{22}^{\overline{\rm DR}} &= -C_FTn_f\left(\frac{1}{16}C_A - \frac{1}{8}C_F - \frac{1}{16}Tn_f\right)\,,\\
\beta_{02}^{e} &= -C_F - \frac{1}{2}Tn_f + \frac{1}{2}C_A\,,\\
\beta_{11}^{e} &= \frac{3}{2}C_F\,,\\
\beta_{03}^{e} &= \frac{3}{8}C_A^2 - \frac{5}{4}C_AC_F + C_F^2 - \frac{3}{8}C_ATn_f + \frac{3}{4}C_FTn_f\,,\\
\beta_{21}^{e} &= -\frac{7}{64}C_A^2 + \frac{55}{48}C_AC_F + \frac{3}{16}C_F^2 + \frac{1}{8}C_ATn_f - \frac{5}{12}C_FTn_f\,,\\
\beta_{12}^{e} &= -\frac{3}{8}C_A^2 + \frac{5}{2}C_AC_F - \frac{11}{4}C_F^2 - \frac{5}{8}C_FTn_f\,,\\
\beta_{02100}^{e} &= -\frac{9}{8}\,,\qquad \beta_{02010}^{e} = \frac{5}{4}\,,\qquad \beta_{02001}^{e} = \frac{3}{4}\,,\qquad \beta_{01200}^{e} = \frac{27}{64}\,,\\
\beta_{01101}^{e} &= -\frac{9}{16}\,,\qquad \beta_{01020}^{e} = -\frac{15}{4}\,,\qquad \beta_{01002}^{e} = \frac{21}{32}\,,
\end{aligned}
\tag{13}
$$

where

$$
C_F = \frac{N_c^2-1}{2N_c}\,,\qquad C_A = N_c\,,\qquad T = \frac{1}{2} \tag{14}
$$

are the usual colour factors of QCD, and n_f is the number of active quark flavours. In Eq. (12) we introduced five indices for the coefficients of $\beta_s^{\overline{\rm DR}}$ and β_e. However, we drop the last three indices whenever there is no dependence on η_r. In particular, β_s depends on the η_r only starting from four-loop order. Note that those terms involving η_r are only valid for $N_c = 3$, whereas the remaining ones hold for a general $SU(N_c)$ group.

As a first check on the results given in Eq. (13), we specialize them to the supersymmetric Yang-Mills theory containing one Majorana fermion in the adjoint representation, by setting $C_A = C_F = 2T$, $n_f = 1$, and $\alpha_s^{\overline{\rm DR}} = \alpha_e = \eta_1$ and $\eta_2 = \eta_3 = 0$. Accordingly, we obtain for the non-vanishing coefficients of the $\beta_s^{\overline{\rm DR}}$

$$
\beta_{20}^{\overline{\rm DR}} = \frac{3}{4}C_A\,,\quad \beta_{30}^{\overline{\rm DR}} = \frac{3}{8}C_A^2\,,\quad \beta_{40}^{\overline{\rm DR}} = \frac{21}{64}C_A^3\,, \tag{15}
$$

in agreement with Ref. [25]. Moreover, comparing these coefficients for pure QCD with the literature, one finds that the two-loop result for $\beta_s^{\overline{\rm DR}}$ and the one-loop result for β_e agree with Ref. [8]. Actually, up to this order, the result for the first two perturbative coefficients of β_s is the same in the $\overline{\rm DR}$ and the $\overline{\rm MS}$ scheme which is a well-known consequence of mass-independent renormalization schemes. However, our three-loop result for $\beta_s^{\overline{\rm DR}}$ differs in the terms proportional to $C_F^2 T n_f$, $C_A C_F T n_f$ and $C_F T^2 n_f^2$ from the one that can be found in Ref. [10].

In order to explain this difference, let us have a closer look at the method used in Ref. [10]. The function $\beta_s^{\overline{\rm DR}}$ was derived from the known result for $\beta_s^{\overline{\rm MS}}$ by inserting the relation between $\alpha_s^{\overline{\rm DR}}$ and $\alpha_s^{\overline{\rm MS}}$. The couplings α_e and $\alpha_s^{\overline{\rm DR}}$, as well as their β-functions β_e and $\beta_s^{\overline{\rm DR}}$ were identified throughout the calculation. But as we will show shortly, this identification makes it impossible to obtain consistent higher order results. Keeping the couplings different, on the other hand, the relation between $\alpha_s^{\overline{\rm DR}}$ and $\alpha_s^{\overline{\rm MS}}$ reads

$$\alpha_s^{\overline{\rm DR}} = \alpha_s^{\overline{\rm MS}}\left[1+\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{C_A}{12}+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2\frac{11}{72}C_A^2-\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{\alpha_e}{\pi}\frac{1}{8}C_F T n_f+\ldots\right], \qquad (16)$$

where the dots denote higher orders in $\alpha_s^{\overline{\rm MS}}$, α_e, and η_r.

This equation has to be inserted into

$$\beta_s^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) = \mu^2\frac{{\rm d}}{{\rm d}\mu^2}\frac{\alpha_s^{\overline{\rm DR}}}{\pi} = \beta_s^{\overline{\rm MS}}(\alpha_s^{\overline{\rm MS}})\frac{\partial\alpha_s^{\overline{\rm DR}}}{\partial\alpha_s^{\overline{\rm MS}}}+\beta_e(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\})\frac{\partial\alpha_s^{\overline{\rm DR}}}{\partial\alpha_e}+\ldots, \qquad (17)$$

where the first equality is due to the definition of $\beta_s^{\overline{\rm DR}}$ and the second one is a consequence of the chain rule, with terms arising through the η_r represented by dots. Using the three-loop expression for $\beta_s^{\overline{\rm MS}}$ (see Refs. [15, 16] and references therein), we obtain the same result as in Eq. (13) which not only provides a powerful check on the various steps of the calculation, but also confirms the equivalence of DREG and DRED at this order [26].

Let us stress that even if one sets $\alpha_e = \alpha_s^{\overline{\rm DR}}$ in the final result (cf. Eq. (13)), one does not arrive at the expression for $\beta_s^{\overline{\rm DR}}$ provided in Ref. [10].

Indeed, a way to see that the identification of $\alpha_s^{\overline{\rm DR}}$ and α_e at intermediate steps leads to inconsistent results is as follows. Whereas in the case of the β function the error is a finite, gauge parameter independent term, it leads to a much more obvious problem for the quark mass renormalization: Z_m will contain non-local terms at three-loop order if $g_e = g_s$ is assumed throughout the calculation, and γ_m as evaluated from Eq. (11) will not be finite.

4 Mass anomalous dimension to three loops

In this section we use the framework of Section 2 in order to obtain the anomalous dimension of the quark masses within DRED as defined in Eq. (11). The result will be derived both by a direct calculation of the relevant Feynman diagrams in DRED, as well as indirectly by using the result from DREG and the $\overline{\rm MS}$–$\overline{\rm DR}$ relation between the strong coupling and quark mass.

The evaluation of Z_m to three-loop order proceeds along the same lines as for the renormalization constants of the previous section. However, in contrast to Z_s, the coupling α_e already appears at one-loop order. Thus the two-loop expression for Z_e is required which can be obtained from Eq. (13). At one-loop order we find complete agreement with the result given in Ref. [8]; the two-loop term is — to our knowledge — new.

From the three-loop result for Z_m we obtain the anomalous dimension

$$\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) = -\sum_{i,j,k,l,m}\gamma_{ijklm}^{\overline{\rm DR}}\left(\frac{\alpha_s^{\overline{\rm DR}}}{\pi}\right)^i\left(\frac{\alpha_e}{\pi}\right)^j\left(\frac{\eta_1}{\pi}\right)^k\left(\frac{\eta_2}{\pi}\right)^l\left(\frac{\eta_3}{\pi}\right)^m\,, \tag{18}$$

with

$$\begin{aligned}
\gamma_{10}^{\overline{\rm DR}} &= \frac{3}{4}C_F\,,\\
\gamma_{20}^{\overline{\rm DR}} &= \frac{3}{32}C_F^2+\frac{91}{96}C_AC_F-\frac{5}{24}C_FTn_f\,,\\
\gamma_{11}^{\overline{\rm DR}} &= -\frac{3}{8}C_F^2\,,\\
\gamma_{02}^{\overline{\rm DR}} &= \frac{1}{4}C_F^2-\frac{1}{8}C_AC_F+\frac{1}{8}C_FTn_f\,,\\
\gamma_{30}^{\overline{\rm DR}} &= \frac{129}{128}C_F^3-\frac{133}{256}C_F^2C_A+\frac{10255}{6912}C_FC_A^2+\frac{-23+24\zeta_3}{32}C_F^2Tn_f\\
&\quad-\left(\frac{281}{864}+\frac{3}{4}\zeta_3\right)C_AC_FTn_f-\frac{35}{432}C_FT^2n_f^2\,,\\
\gamma_{21}^{\overline{\rm DR}} &= -\frac{27}{64}C_F^3-\frac{21}{32}C_F^2C_A-\frac{15}{256}C_FC_A^2+\frac{9}{32}C_F^2Tn_f\,,\\
\gamma_{12}^{\overline{\rm DR}} &= \frac{9}{8}C_F^3-\frac{21}{32}C_F^2C_A+\frac{3}{64}C_FC_A^2+\frac{3}{64}C_AC_FTn_f+\frac{3}{8}C_F^2Tn_f\,,\\
\gamma_{03}^{\overline{\rm DR}} &= -\frac{3}{8}C_F^3+\frac{3}{8}C_F^2C_A-\frac{3}{32}C_FC_A^2+\frac{1}{8}C_AC_FTn_f-\frac{5}{16}C_F^2Tn_f-\frac{1}{32}C_FT^2n_f^2\,,\\
\gamma_{02100}^{\overline{\rm DR}} &= \frac{3}{8}\,,\qquad \gamma_{02010}^{\overline{\rm DR}} = -\frac{5}{12}\,,\qquad \gamma_{02001}^{\overline{\rm DR}} = -\frac{1}{4}\,,\qquad \gamma_{01200}^{\overline{\rm DR}} = -\frac{9}{64}\,,\\
\gamma_{01020}^{\overline{\rm DR}} &= \frac{5}{4}\,,\qquad \gamma_{01101}^{\overline{\rm DR}} = \frac{3}{16}\,,\qquad \gamma_{01002}^{\overline{\rm DR}} = -\frac{7}{32}\,.
\end{aligned}\tag{19}$$

Again the last three indices are suppressed whenever there is no dependence on η_r. Furthermore, those terms involving η_r are only valid for $N_c=3$, whereas the remaining ones hold for a general $SU(N_c)$ group.

On the other hand, $\gamma_m^{\overline{\rm DR}}$ can be derived indirectly from the $\overline{\rm MS}$ result obtained within DREG. The analogous equation to (17) is given by

$$\gamma_m^{\overline{\rm DR}}(\alpha_s^{\overline{\rm DR}},\alpha_e,\{\eta_r\}) = \gamma_m^{\overline{\rm MS}}\frac{\partial\ln m^{\overline{\rm DR}}}{\partial\ln m^{\overline{\rm MS}}}+\frac{\pi\beta_s^{\overline{\rm MS}}}{m^{\overline{\rm DR}}}\frac{\partial m^{\overline{\rm DR}}}{\partial\alpha_s^{\overline{\rm MS}}}+\frac{\pi\beta_e}{m^{\overline{\rm DR}}}\frac{\partial m^{\overline{\rm DR}}}{\partial\alpha_e}+\ldots\,, \tag{20}$$

which requires the two-loop relation between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$ in order to obtain $\gamma_m^{\overline{\rm DR}}$ to three loops. We compute the two-loop relation between $m^{\overline{\rm DR}}$ and $m^{\overline{\rm MS}}$, keeping not only the divergent but also the finite parts in the calculation of the fermion propagator. Our result reads

$$
\begin{aligned}
m^{\overline{\rm DR}} &= m^{\overline{\rm MS}}\Bigg[1-\frac{\alpha_e}{\pi}\frac{1}{4}C_F+\left(\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\right)^2\frac{11}{192}C_AC_F-\frac{\alpha_s^{\overline{\rm MS}}}{\pi}\frac{\alpha_e}{\pi}\left(\frac{1}{4}C_F^2+\frac{3}{32}C_AC_F\right)\\
&+\left(\frac{\alpha_e}{\pi}\right)^2\left(\frac{3}{32}C_F^2+\frac{1}{32}C_FTn_f\right)+\ldots\Bigg], \qquad (21)
\end{aligned}
$$

where the dots denote higher orders in $\alpha_s^{\overline{\rm MS}}$, α_e, and η_r. The one- and two-loop terms of Eq. (21) agree with Ref. [27] in the limit $\alpha_e=\alpha_s^{\overline{\rm DR}}$. Let us remark that in order to get Eq. (21), also the one-loop relation between the $\overline{\rm DR}$ and $\overline{\rm MS}$ version of the gauge parameter is a necessary ingredient unless one works in Landau gauge ($\xi=1$). We performed the calculation for general covariant gauge and use the cancellation of the gauge parameter in the final result as a welcome check.

As a further ingredient we need $\gamma_m^{\overline{\rm MS}}$ which can be found in Refs. [17, 18]. Inserting this result and Eq. (21) into (20) leads to Eq. (19). Again, this is a powerful check on our calculation and shows the equivalence of DRED and DREG at this order. Note that in the indirect approach the η_r enter only through the factor β_e in Eq. (20).

The two-loop result of $\gamma_m^{\overline{\rm DR}}$ can also be found in Ref. [27] and we agree for $\alpha_e=\alpha_s$. The three-loop result for $\gamma_m^{\overline{\rm DR}}$ is new.

Note that Ref. [11] quotes Eq. (21) with $\alpha_s^{\overline{\rm DR}}=\alpha_e$ as a formula to convert the $\overline{\rm DR}$ to the $\overline{\rm MS}$ quark mass. However, as we pointed out at the end of Section 2, this identification is inconsistent. In fact, we can calculate the numerical effect induced by this inconsistency if we require that $\alpha_s^{\overline{\rm DR}}(\mu)=\alpha_e(\mu)$ at scales where all SUSY particles contribute to the running of the coupling, then integrate out the SUSY particles at $\mu=M_Z$, and evolve α_e and $\alpha_s^{\overline{\rm DR}}$ separately down to, say, m_b. Using $\alpha_s^{\overline{\rm DR}}=0.120$ as input, we arrive at[1] $\alpha_s^{\overline{\rm DR}}(m_b)=0.218$ and $\alpha_e(m_b)=0.167$, and therefore a shift in $m_b^{\overline{\rm DR}}$ of roughly 30 MeV compared to the value obtained by using $\alpha_s^{\overline{\rm DR}}=\alpha_e$.

5 Conclusions

In many cases, DRED poses an attractive alternative to DREG — not only for supersymmetric theories. We computed the QCD renormalization group function of the strong coupling constant (β) and of the quark masses (γ_m) to three-loop order in this scheme using two different methods. The agreement of the results obtained in both ways confirms the equivalence of the $\overline{\rm DR}$ and the $\overline{\rm MS}$ renormalization scheme at this order, in the sense that they are related by an analytic redefinition of the couplings and masses [26]. Furthermore, we find that the three-loop β-function found in the literature differs from

[1] Only one-loop running of α_s and α_e is applied here.

ours. We trace this difference to the fact that the evanescent coupling of the $q\bar{q}\varepsilon$ vertex had been identified wrongly with α_s in Ref. [10].

Let us stress that higher order calculations within the framework of DRED should also be useful in the context of the Minimal Supersymmetric Standard Model where precision calculations will be important in order to be prepared for measurements at the CERN Large Hadron Collider and other future high energy experiments.

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References

[1] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972) 189.

[2] C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12** (1972) 20.

[3] W. Siegel, Phys. Lett. B **84** (1979) 193.

[4] W. Siegel, Phys. Lett. B **94** (1980) 37.

[5] L. V. Avdeev, G. A. Chochia and A. A. Vladimirov, Phys. Lett. B **105** (1981) 272.

[6] D. Stöckinger, JHEP **0503** (2005) 076 [hep-ph/0503129].

[7] W. Hollik and D. Stöckinger, Phys. Lett. B **634** (2006) 63 [hep-ph/0509298].

[8] I. Jack, D. R. T. Jones and K. L. Roberts, Z. Phys. C **62** (1994) 161 [hep-ph/9310301].

[9] Z. Kunszt, A. Signer and Z. Trocsanyi, Nucl. Phys. B **411** (1994) 397 [hep-ph/9305239].

[10] Z. Bern, A. De Freitas, L. J. Dixon and H. L. Wong, Phys. Rev. D **66** (2002) 085002 [hep-ph/0202271].

[11] J. A. Aguilar-Saavedra *et al.*, [hep-ph/0511344].

[12] D. M. Capper, D. R. T. Jones and P. van Nieuwenhuizen, Nucl. Phys. B **167** (1980) 479.

[13] A. Bednyakov, A. Onishchenko, V. Velizhanin and O. Veretin, Eur. Phys. J. C **29** (2003) 87 [hep-ph/0210258].

[14] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, Phys. Rev. D **18** (1978) 3998.

[15] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B **400** (1997) 379 [hep-ph/9701390].

[16] M. Czakon, Nucl. Phys. B **710** (2005) 485 [hep-ph/0411261].

[17] K. G. Chetyrkin, Phys. Lett. B **404** (1997) 161 [hep-ph/9703278].

[18] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, Phys. Lett. B **405** (1997) 327 [hep-ph/9703284].

[19] R. Harlander, L. Mihaila and M. Steinhauser, Phys. Rev. D **72** (2005) 095009 [hep-ph/0509048].

[20] M. Steinhauser, Phys. Rev. D **59** (1999) 054005 [hep-ph/9809507].

[21] P. Nogueira, J. Comput. Phys. **105** (1993) 279.

[22] T. Seidensticker, [hep-ph/9905298].

[23] R. Harlander, T. Seidensticker and M. Steinhauser, Phys. Lett. B **426** (1998) 125 [hep-ph/9712228].

[24] S. A. Larin, F. V. Tkachov and J. A. M. Vermaseren, NIKHEF-H-91-18

[25] L. V. Avdeev and O. V. Tarasov, Phys. Lett. B **112** (1982) 356.

[26] I. Jack, D. R. T. Jones and K. L. Roberts, Z. Phys. C **63** (1994) 151 [hep-ph/9401349].

[27] L. V. Avdeev and M. Y. Kalmykov, Nucl. Phys. B **502** (1997) 419 [hep-ph/9701308].