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*Multiloop Calculations: towards R at Order α_s^4

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We discuss recent developments in multiloop calculations aiming eventually in computing the total cross section for e^+e^- annihilation into hadrons σ_{had} in order $\mathcal{O}(\alpha_s^4)$.

1. Introduction

Grand Unification, the merging of the three gauge theories based on $U(1), SU(2)$ and $SU(3)$ groups with their three independent coupling g_1, g_2 and g_3 into the unified framework of $SU(5)$ or, more probable, $SO(10)$ gauge theory, is one of the most attractive possibilities for physics beyond the Standard Model. As is well known, this can only be considered as attractive and natural option consistent with the present values of the coupling constants if supersymmetry is realized in Nature. Indeed, most of the models predict that at least some supersymmetric partners of quarks, leptons, gauge and Higgs bosons will be discovered and studied in detail at the next generation of colliders, LHS or ILC.

Once the masses of sufficiently many SUSY-partners are measured precisely, the detailed energy dependence of the coupling constants will be fixed and the test for the unification can proceed with significantly improved precision. In fact, it is conceivable that some hints about symmetry breaking mechanism responsible for the transition from, say, $SO(10)$ to $SU(1)\times SU(2)\times SU(3)$ could be inferred from such an analysis, in particular, if combined with a similar study for Yukawa unification.

However, for such a program to be pursued successfully, the initial values entering into the evolution equations—gauge couplings and masses as measured at low energies—have to be known with sufficient precision. Of particular interest is the gain in resolution power from a more precise determination of the strong coupling. A significant improvement is anticipated for the GIGA-Z option of the linear collider. Indeed, it has been argued in [1] that a sample of 10^9 Z decays might well lead to an experimental uncertainty of $\delta\alpha_s = 0.0008$, and even a value of $\delta\alpha_s = 0.0005$ has been quoted [2].

The determination of α_s from Z is either based on the measurement of the leptonic branching ratio

$$R_\ell = \Gamma_h/\Gamma_\ell = 20.767 \pm 0.025$$

which amounts to a simple counting of leptonic and hadronic final states, or it exploits the leptonic peak cross-section

$$\sigma_\ell \sim \frac{\Gamma_\ell^2}{\Gamma_{\text{tot}}^2} = 2.003 \pm 0.0027 \text{ nb}.$$

In the Standard Model fit

$$\alpha_s = 0.1183 \pm 0.0030$$

all this information is combined [3]. In all these cases the α_s dependence enters through the formula

$$\Gamma_{\text{had}} = \Gamma_0 \left(1 + \frac{\alpha_s}{\pi} + 1.409\,\frac{\alpha_s^2}{\pi^2} - 12.767\,\frac{\alpha_s^3}{\pi^3}\right) + \text{corrections} \quad (1)$$

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where the second factor, conventually denoted as $1/3R(s)$, gives the perturbative series valid for massless quarks. The "corrections" in the above formula originate from quark mass terms $\sim \frac{m_b^2}{M_Z^2}$, from singlet terms which start in order α_s^2, which present in the axial amplitude only and which originate from the imbalance between top and bottom quark loops and, finally, mixed, non-factorizeable QCD $\times$ electroweak corrections of order $\alpha_s\, \alpha_{\rm weak}$.

All these terms are discussed in [4], and, indeed, the "corrections" terms in eq. (1) are typically well under control. The dominant theory error one encounters in the extraction of α_s originates from estimates of the not yet available terms of order α_s^4. To estimate this uncertainty, different strategies have been advocated and we shall only list a small subset of the possible choices.

The most conservative estimate of the truncation error for an asymptotic series is identical to the last calculated term, presently of order α_s^3 and leads to an uncertainty of $\delta\alpha_s = 0.002$ corresponding to $\delta\alpha_s/\alpha_s = 1.8\%$. Alternatively one may vary the renormalization scale which appears as argument of the strong coupling μ in a plausible range around $\sqrt{s}$, say from $1/3\sqrt{s}$ to $3\sqrt{s}$. Using the correspondingly modified perturbative series to estimate $\alpha_s(\mu)$ (and evolving $\alpha_s(\mu)$ back to $\alpha_s(M_Z)$) one obtains a quite asymmetric variation $\delta\alpha_s = {}^{+0.002}_{-0.0002}$ of the same order as estimated above.

Various prescriptions have been used to estimate not yet calculated higher order terms. From the Principe of Minimal Sensitivity (PMS) [5] or Fastest Apparent Convergence (FAC) [6] the forth order term is predicted as $-97(\alpha_s/\pi)^4$ [7]. This then would lead to a shift in α_s of 0.0006, corresponding to $\delta\alpha_s/\alpha_s = 0.5\%$.

From this considerations it is evident that for a reliable determinations of α_s from GIGA-Z, fully exploiting the experimental precision, the evaluation of the α_s^4 term is a must. This arguments get even stronger in those cases, where low energy measurements are involved. The most extreme case in this direction is the determination of α_s from the semileptonic versus leptonic decay rate of the tau lepton [8]. Considering the enormous events rates at low energy electron-positron storage rings, like B- or charm- or Tau- factories, R measurements at 10.5 GeV or at 3.7 GeV, just below the threshold for open bottom or charm production, could also lead to fairly precise measurements of α_s. Indeed, to achieve the benchmark precision Indeed, to achieve the benchmark precision $\delta\alpha_s(M_Z) = 0.003$ would require $\delta\alpha_s = .007$ at 10.5 GeV and $\delta\alpha_s = .013$ at 3.7 GeV Evidently only systematic uncertainties are the limiting factors for an ultra precise determination of α_s .

2. The long march towards R in α_s^4

Let us now briefly describe the strategy to evaluate the α_s^4 term and the status of this calculation and a variety of physically and mathematically relevant results that are available at present.

Consider the correlator of two currents (all Lorentz and flavour indices are suppressed and quark masses are set to zero) $j = \bar\psi\Gamma\psi$ and $j^\dagger = \bar\psi\Gamma^\dagger\psi$

$$\Pi^{jj}(q^2) = i\int {\rm d}x e^{iqx}\langle 0|T[j(x)\,j^\dagger(0)]|0\rangle$$

which is related to the corresponding absorptive part R(s) through

$$R^{jj}(s) \approx \Im\,\Pi^{jj}(s - i\delta).$$

Renormalized and bare correlators are related through

$$\Pi^{jj} = Z^{jj} + Z^j\,\Pi_B^{jj}(q^2, \alpha_s^B)$$

and the independence of Π^B on the renormalization scale is reflected in the renormalization group equation of the form ($L = \ln\frac{\mu^2}{-q^2}$)

$$\left(\frac{\partial}{\partial L} - 2\gamma^j(\alpha_s) + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\Pi^{jj} = \gamma^{jj}(\alpha_s). \tag{2}$$

Here the anomalous dimension $\gamma^j(\alpha_s)$ is related to the corresponding renormalization constant as

$$\gamma^j = \mu\frac{\partial}{\partial\mu}\ln(Z^j)|_{\alpha_s^B} = \sum_{i>1}\gamma_i^j\left(\frac{\alpha_s}{\pi}\right)^i$$

while the QCD beta-function

$$\beta(\alpha_s) = \mu \frac{\partial}{\partial \mu} \alpha_s|_{\alpha_s^B} = -\alpha_s \sum_{i \geq 0} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{(i+1)} .$$

Equation (2) clearly demonstrates that in order to find the q^2 dependent part of Π at (N+1)-loop (corresponding to α_s^N order) one needs the (N+1)-loop anomalous dimension γ^{jj} and the N-loop approximation of Π, (i.e. of order α_s^{N-1}) including its constant part. Note that the operation $\alpha_s \frac{\partial}{\partial \alpha_s}$ raises the power of α_s by one unit as $\beta(\alpha_s)$ starts from α_s^2. The problem of finding of a (N+1)-loop anomalous dimension, however, can be reduced in a systematic automatized way to the evaluation a proper combination of N-loop massless propagator integrals [9,10].

The situation is significantly simplified whenever γ^{jj} happens to vanish. One particular and physically relevant case is the m_q^2/s term in the small mass expansion of the absorptive part of the vector correlator, where the corresponding four-loop $\mathcal{O}(\alpha_s^3)$ contribution to Π^{VV} is indeed sufficient to obtain the term of order $\mathcal{O}(\alpha_s^4 m_q^2/s)$ [11,12].

2.1. Reduction of massless propagators

Thus, to obtain the massless R-ratio in order α_s^4, a large number of four-loop massless propagators, including their finite parts, must be evaluated. The complications can be best judged by contrasting the cases of the three- and four-loop calculations: 11 (about 150) topologies with and 3 (11) topologies without insertions are involved in the three-(four-)loop case. The reduction has to be performed to 6 (28) master integrals. Most importantly, however, a set of recursion relations based on integration by parts identities is available for the 3-loop case. These recursion relations have been constructed manually [13] and implemented in the program MINCER [14].

A straightforward implementation of this concept to the four-loop case seems to be difficult at present. An alternative concept has been advocated in [15] which allows to perform the reduction of an arbitrary massless propagator to a sum of master integrals "mechanically" in the limit of large space-time dimension d. Consider the reduction of an amplitude $f(d)$ which depends on the topology, the power of the various propagators and the space-time dimension d to master integrals:

$$f(d) = \sum_{\alpha=\text{masters}} C^\alpha(d) \star M^\alpha(d),$$

where M^α, with $\alpha = 1 - 28$ stand for master integrals. The coefficients $C^{(\alpha}(d)$ are known to be rational functions of d; their determination is the central problem to be solved. In the limit of large d we have

$$C^\alpha(d) = \frac{P^n(d)}{Q^m(d)} \overline{\overline{_{d\to\infty}}} \sum_k C_k^\alpha \quad (1/d)^k.$$

The terms in the $1/d$ expansion can be expressed (with the use of the new representation for Feynman integrals developed in [16,17]) through simple Gaussian integrals—a task of purely algebraic nature. Obviously, given sufficiently many coefficients, the functions C^α can be reconstructed. However, the degrees m and n are strongly dependent on the power of the propagators in the integral under consideration with drastic effects on the effort involved in the evaluation. In the process of reduction a 4-loop diagram of a typical topology to Gaussian integrals one should handle a polynomial of 9 variables of degree k consisting of $\frac{(9+k)!}{9!k!}$ terms. For k=24 this corresponds to approximately $4 \cdot 10^7$ terms or 4 GB storage space, while k=40 leads to $2 \cdot 10^9$ terms, corresponding to 200 GB. Weeks, partly even months of runtime and hundreds of GB disk space are required for the evaluation. On the other hand, the purely algebraic manipulations required to perform the Gaussian integrals are ideally suited for treatment within the algebraic manipulation program FORM [18] and its parallel version PARFORM [19,20], which both are specially tuned for dealing with a huge volume of algebraic manipulations. The method of large d expansion has been successfully applied for solving a number of problems which will be briefly discussed in the next section.

2.2. Important dots

The reduction of the $(N + 1)$ loop anomalous dimension to the calculation of N-loop massless propagators discussed above works "natively" only for logarithmically divergent integrals while Π^{jj} is in general *quadratically* divergent. The only known way to apply the reduction in such cases is to use (double!) differentiation w.r.t. the external momentum q to decrease the dimension. This lead to "dots" $\equiv$ squared propagators which immensely complicate all calculations.

An important and non-trivial simplification exists for the scalar (SS) correlator due to the well-known Ward identity:

$$q_\mu q_\nu \Pi^{\mathrm{V/A}}_{\mu\nu,ij}(q) = (m_i \mp m_j)^2 \Pi^{\mathrm{S/P}}_{ij}(q) + \mathcal{O}(m_q^4).$$

Basically this means that the $\mathcal{O}(m_q^2)$ part of the longitudinal part of VV correlator is identical to the massless SS one. This allows to compute the $\mathcal{O}(m_q^2)$ part of the VV correlator instead of the massless SS which resulted in diagrams with **one squared propagator less** and saves a lot of work! Unfortunately, no such simplification is known for the case of the massless vector correlator.

3. Results

Although the R ratio to order α_s^4 for the vector correlator is not yet available a number of intermediate results of phenomenological relevance have been obtained recently.

- All 28 master integrals have been evaluated analytically [21].
- The terms of order $\alpha_s^4 n_f^3$ and $\alpha_s^4 n_f^2$ for the R ratio discussed in section 1 were obtained in [22, 23]) ($a_s = \alpha_s/\pi$, dots stand for not yet computed term of order n_f and n_f^0)

$$R(s) = 3\Big\{1 + a_s + a_s^2\,(1.986 - 0.1153\; n_f) + a_s^3\left(-6.6369 - 1.2001\; n_f - 0.00518\; n_f^2\right) + a_s^4\left(0.02152\; n_f^3 - 0.7974\; n_f^2 + \ldots\right)\Big\}.$$

- The full dependence of the τ-lepton decay rate on the strange quark mass m_s up to order α_s^3 has been obtained in [24].
- The quadratic term in the small quark mass expansion of the R-ratio

$$R(s) = 3\left\{r_0^V + \frac{m^2}{s} r_2^V\right\} + \ldots.$$

has been obtained in [12]. For the number of active quark flavours $n_f = 4$ the result reads

$$r_2^V/12 = a_s + 9.09722 a_s^2 + 52.913 a_s^3 + 128.499\, a_s^4, \qquad (3)$$

Let us mention that the a_s^4 term is predicted to 177 and 193 by the estimates based on PMS and FAC respectively.

- Most recently the five loop anomalous dimension of the scalar correlator was obtained [25]. In combination with the finite part of the four-loop scalar correlator this corresponds to the R-ratio of the scalar correlator (denoted with an extra tilde below) and has important applications for QCD sum rules and for the Higgs decay rate to b-quarks [25,26]. Written for brevity in numerical form only, the result reads:

$$\widetilde{R} = 1 + 5.6667 a_s + [35.94 - 1.359\, n_f]\, a_s^2 + a_s^3\left[164.1 - 25.77\, n_f + 0.259\, n_f^2\right] + a_s^4\left[39.34 - 220.9\, n_f + 9.685\, n_f^2 - 0.02046\, n_f^3\right]. \qquad (4)$$

In Table (1) the result for $\tilde{r}_4$ – the coefficient in front of a_s^4 term in (4) – is compared with predictions obtained in works [27,28]. Note that the Principle of Minimal Sensitivity or the Principle of Fastest Apparent Convergence used in [27] produce identical result at order α_s^4. The two predictions of FAC/PMS for $\tilde{r}_4$ correspond to either the consequence of the prediction for the corresponding Euclidean quantity (second line) or to the direct application of FAC/PMS to estimate $\tilde{r}_4$ (the third line). As a consequence of the large cancellations in $\tilde{r}_4$ the second prediction looks much better than the first, despite the fact that the estimation of the corresponding Euclidian coefficient is quite close (within 10%) to the exact result (for more details see [25]). The Asymptotic Padé-Approximant Method (APAM) estimation of $\tilde{r}_4$ constructed in [28] fails to reproduce even the sign of the exact result. Finally, predictions of the prescription proposed by Brodsky, Lepage and Mackenzie (BLM) [29] for the n_f dependent terms of order α_s^4 have been communicated to the authors [3]: $a_s^4(-260\, n_f + 13\, n_f^2 - 0.046\, n_f^3)$ and are also in reasonable agreement with the exact result of eq. (4).

[3] M. Binger and S. Brodsky, private communication.

Table 1
Comparison of the results for $\tilde{r}_4$ with earlier estimates based on PMS, FAC and APAM.

n_f	3	4	5
$\tilde{r}_4$ (exact)	-536.8	-690.7	-825.7
$\tilde{r}_4$ ([27], PMS, FAC)	-945	-1099	-1237
$\tilde{r}_4$ ([27], PMS, FAC)	-528	-749	-949
$\tilde{r}_4$ ([28], APAM)	252	147	64

4. Conclusion

The calculation of the $\widetilde{R}$-ratio of the scalar correlator at order α_s^4 has clearly demonstrated the enormous complexity inherent into any calculation of such multiloop level in QCD. The total CPU time consumption amounts (very roughly) $3 \cdot 10^8$ seconds (about 10 years) if normalized to the use of a stand-alone 1.5 GH PC. Due to the heavy use of the SGI cluster (of 32 parallel SMP CPU of 1.5 GH frequence each) the calculation took about 15 calendar months.

The corresponding calculations for the vector correlator will increase this demand by another factor of 3 (optimistically) or 10 (pessimistically). Clearly the combined experience of improved programs and better hardware will lead to the desired result within the next few years.

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