SFB/CPP-06-01
TTP06-02
hep-ph/0601165

ϵ-Finite Basis of Master Integrals for the Integration-By-Parts Method

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Abstract

It is shown that for every problem within dimensional regularization, using the Integration-By-Parts method, one is able to construct a set of master integrals such that each corresponding coefficient function is *finite* in the limit of dimension equal to four. We argue that the use of such a basis simplifies and stabilizes the numerical evaluation of the master integrals. As an example we explicitly construct the ϵ-finite basis for the set of all QED-like four-loop massive tadpoles. Using a semi-numerical approach based on Padé approximations we evaluate analytically the divergent and numerically the finite part of this set of master integrals.

The calculations confirm the recent results of Schröder and Vuorinen. All the contributions found there by fitting the high precision numerical results have been confirmed by direct analytical calculation without using any numerical input.

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1 Introduction

The evaluation of higher order perturbative corrections plays an important role in testing the Standard Model on a high precision level as well as in the search for new physics beyond the established theory. Especially given the increasing resolution of modern experiments, it is mandatory to push ahead the efforts on the theory side. Many reviews, see e.g. [1, 2], deal with the application of such calculations. The present work focuses on the actual evaluation of the integrals appearing when considering higher order corrections.

Let us begin by noting that basically every analytical method applicable for higher order calculations can effectively be put into action only within the "calculation friendly" environment provided by dimensional regularization [3–5]. Of course, computing e.g. the 1-loop β function one could use a variety of approaches. However already at 2-loop level it is almost impossible to avoid the use of dimensional regularization unless one is considering a simple scalar toy model.

It should also be stressed that from a purely calculational point of view the most valuable feature of dimensionally regularized integrals is their similarity with the usual absolutely convergent integrals. As a result almost all standard operations are mathematically justified.

In particular, the traditional Integration By Parts (IBP) method is based on the possibility to discard the surface terms within this framework[3] and allows for generating exact identities between different integrals. In a next step the resulting equations are used to reduce every Feynman diagram to a limited set of mutually independent so-called *master integrals*.

This reduction procedure is a highly non-trivial problem by itself as the growing number and complexity of IBP identities defies any attempt of a straightforward solution starting from some number of loops and external legs. Therefore the classical way of "manually" examining the IBP identities seems to be unsuitable for such problems. A prominent exception is the crack-down of the long-standing problem of computing the generic moments of the structure functions in deep-inelastic scattering at three-loop level [9] which has been completed only recently. At the moment there are a few approaches trying to automatize the procedure of reduction to master integrals [10–22]. Once the reduction has been achieved one is left with the evaluation of the resulting master integrals. A complete up-to-date review about various methods of calculating Feynman diagrams can be found in [23].

[3] The possibility is by no means obvious for a general case of integrals containing *both* UV and IR divergences and requires a dedicated proof. The latter, in turn, is not possible without a proper definition of the very dimensional regularization for such integrals. To the best of our knowledge the elaborated mathematically meaningful definition of dimensional regularization and its main properties was presented in [6, 7] and reviewed in [8].

As is well-known within the method of dimensional regularization, the usual UV divergences manifest themselves as poles in $\epsilon = (4 - d)/2$, with d being the spacetime dimension. The calculation of the master integrals is complicated by appearance of unphysical, so-called spurious, poles in the coefficient functions which are often induced during the reduction procedure. Through the appearance of these spurious poles one is forced to compute extra terms in the ϵ-expansion of the master integrals. In general the evaluation of every next term in the ϵ-expansion is challenging, especially if the application of numerical methods is necessary (see, e.g. [24–29]).

On the other hand, the choice of master integrals is by no means unique.

In the present work we discuss a convenient way of finding a new basis of master integrals. In particular we show in the next two sections that for every problem reducible with the help of the IBP method to a set of master integrals the latter can be always chosen in such a form that *all* the corresponding coefficient functions are *finite* in the limit $\epsilon \to 0$. We call this basis the ϵ-finite basis. In section 4 we explicitly construct such an ϵ-finite basis for the set of all QED-like four-loop massive tadpoles appearing in the first moment of the low energy expansion of the vacuum polarization in QCD. Using a semi-numerical approach based on Padé approximations we evaluate numerically all the masters of this ϵ-finite set and confirm the recent results of Schröder and Vuorinen [26]. Then we focus on analytical results which can be easily derived following this method. Our general comments are presented in section 5.

2 Generalities and Notation

Let us consider a quantity P which we want to compute up to L-loop level. The sum of all bare contributions P_B is given by

$$P_B = \sum_{l=0}^{L} h^l P_B^{(l)}. \tag{1}$$

Here we have introduced a formal parameter h counting the number of loops. The renormalized $L-$loop contribution $h^L\, P_R^{(L)}$ is expressible through

$$h^L\, P_R^{(L)} = h^L\, P_B^{(L)} + \sum_{l=1}^{L} h^l Z^{(l)}\, h^{L-l} P_B^{(L-l)} \tag{2}$$

as a linear combination of the bare quantities $P_B^{(l)}$ and the corresponding renormalization constants, here summarized in $h^l\, Z^{(l)}$ for the l-loop contribution. We assume that the calculation of the integrals up to h^{L-1} order is well under control. This includes the sum in the second term of equation (2). Therefore we concentrate on the bare L-loop contribution $P_B^{(L)}$.

After the evaluation of all the traces in d dimensions each l-loop contribution can be written as a linear combination of d-dimensional integrals $F_i^{(l)}(\epsilon)$ and associated coefficient functions $f_i^{(l)}(\epsilon)$

$$P_B^{(l)} = \sum_i f_i^{(l)}(\epsilon) F_i^{(l)}(\epsilon). \tag{3}$$

Here possible divergencies in ϵ can only result from the evaluation of the various integrals $F_i^{(l)}(\epsilon)$ whereas the functions $f_i^{(l)}(\epsilon)$ remain finite in limit $\epsilon \to 0$. The resulting number of integrals $\{F_i^{(l)}(\epsilon)\}$ is in general extremely high and therefore reduced by use of IBP relations to a small set of N master integrals $\{M_i^{(l)}(\epsilon)\}$ of l-loop order. In this framework the contribution $P_B^{(L)}$ can be written as

$$P_B^{(l)} = \sum_{i=1}^{N} a_i^{(l)}(\epsilon) M_i^{(l)}(\epsilon). \tag{4}$$

Through the dependence on the spacetime dimension d of the IBP relations the coefficient functions $a_i^{(l)}(\epsilon)$ in equation (4) can contain divergencies for $\epsilon \to 0$. In this case the expansion in ϵ of the corresponding master integrals $M_i^{(l)}(\epsilon)$ has to be evaluated deeper than to the finite contribution in order to compensate for spurious poles. In the next section we will construct a different set of N master integrals $\{\tilde{M}_i^{(l)}(\epsilon)\}$ such that the expansion

$$P_B^{(l)} = \sum_{i=1}^{N} c_i^{(l)}(\epsilon) \tilde{M}_i^{(l)}(\epsilon) \tag{5}$$

leads to coefficient functions $c_i^{(l)}(\epsilon)$ being finite in the limit $\epsilon \to 0$. This new set of master integrals is called ϵ-finite basis.

3 Construction of the ϵ-finite set

The comparison of equations (3) and (4)

$$\sum_i f_i^{(l)}(\epsilon) F_i^{(l)}(\epsilon) = \sum_{i=1}^{N} a_i^{(l)}(\epsilon) M_i^{(l)}(\epsilon) \tag{6}$$

suggests that the ϵ-finite basis can be found in the set of integrals $\{F_i^{(l)}(\epsilon)\}$ as the coefficients $f_i^{(l)}(\epsilon)$ already expose the desired behavior. By definition all the initial integrals can be expressed in terms of master integrals

$$F_i^{(l)}(\epsilon) = \sum_{j=1}^{N} a_{ij}^{(l)}(\epsilon) M_j^{(l)}(\epsilon). \tag{7}$$

Among this in general huge set of equations only the subset $\{F_{\rm div}^{(l)}\}$ exhibiting a divergent coefficient function $a_{ij}^{(l)}(\epsilon)$ is interesting. Let $a_{mn}^{(l)}(\epsilon)$ be one of the functions showing the $1/\epsilon^{n_{\rm max}}$-divergency with the highest power $n_{\rm max}$. Starting from the relation for $F_m^{(l)}(\epsilon)$ involving $a_{mn}^{(l)}(\epsilon)$ we are able to construct a new basis of master integrals by solving equation (7) for the master integral associated with $a_{mn}^{(l)}(\epsilon)$

$$M_n^{(l)}(\epsilon) = \frac{1}{a_{mn}^{(l)}(\epsilon)} \left(F_m^{(l)}(\epsilon) - \sum_{\substack{j=1 \\ j\neq n}}^{N} a_{mj}^{(l)}(\epsilon) M_j^{(l)}(\epsilon) \right). \tag{8}$$

As $a_{mn}^{(l)}(\epsilon)$ was divergent for $\epsilon \to 0$ the reciprocal function is finite. By means of this replacement $F_m^{(l)}(\epsilon)$ is now a member of the new set of master integrals $\{\hat{M}_i^{(l)}(\epsilon)\}$ replacing $M_n^{(l)}(\epsilon)$ in the following way:

$$\{\hat{M}_i^{(l)}(\epsilon)\} = \{M_1^{(l)}(\epsilon), \ldots, M_{n-1}^{(l)}(\epsilon), F_m^{(l)}(\epsilon), M_{n+1}^{(l)}(\epsilon), \ldots, M_N^{(l)}(\epsilon)\}. \tag{9}$$

In a next step the replacement (8) is performed for all equations in the set $\{F_{\rm div}^{(l)}\}$.

This procedure is iterated until all coefficient functions in $\{F_{\rm div}^{(l)}\}$ are finite. The thus obtained final set of master integrals $\{\hat{M}_i^{(l)}(\epsilon)\}$ fulfills the relation

$$P_B^{(l)} = \sum_i f_i^{(l)}(\epsilon) F_i^{(l)}(\epsilon) = \sum_{i,j} f_i^{(l)}(\epsilon) a_{ij}^{(l)}(\epsilon) \hat{M}_j^{(l)}(\epsilon), \tag{10}$$

where each appearing coefficient function is by construction finite. If we now identify

$$c_j^{(l)}(\epsilon) = \sum_i f_i^{(l)}(\epsilon) a_{ij}^{(l)}(\epsilon), \tag{11}$$

$$\tilde{M}_j^{(l)}(\epsilon) = \hat{M}_j^{(l)}(\epsilon), \tag{12}$$

we arrive at the desired form (5) and call the set of integrals $\{\tilde{M}_i^{(l)}(\epsilon)\}$ the ϵ-finite basis. We again stress the fact that all the integrals in this ϵ-finite basis need to be evaluated only up to the finite part in their ϵ-expansion, as by construction no spurious poles arise.

4 Applications to four-loop tadpoles

The concept of the ϵ-finite basis is now applied to the set of master integrals appearing in the calculation of QED-like four-loop tadpole diagrams. We concentrate on the subset of master integrals shown in figures 1 and 2 needed for the calculation of the low energy expansion of the four-loop vacuum polarization function, completing the work in [33]. We denote each of the master integrals

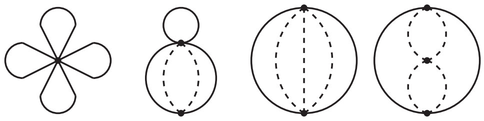


Figure 1: Subset of master integrals fully analytically known and therefore not considered in the construction of the ϵ-finite basis.

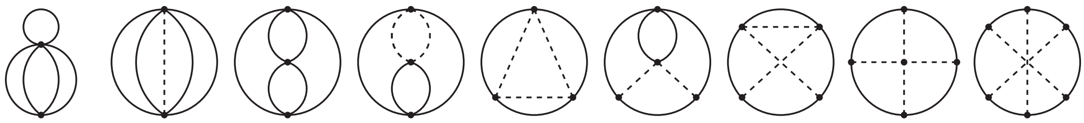


Figure 2: Subset of master integrals for which the ϵ-finite basis is constructed. The two numbers in brackets (n_1, n_2) are decoded as follows: n_1 is the maximal power of a spurious pole $(\frac{1}{\epsilon})^{n_1}$ which could appear in front of the diagrams pictured above; n_2 is the maximal analytically known power ϵ^{n_2} of the ϵ-expansion of the same integral as determined in [30, 31] (for T_{52}) and in [26, 32] for all the rest.

according to the following rule: after a capital letter "T" we write the number of lines in the given diagram. The second number enumerates the different topologies exhibiting the same number of lines. The diagrams of the first subset shown in figure 1 are analytically fully known and expressible in terms of Γ-functions. We thus construct the ϵ-finite basis only for the second subset of the most complicated master integrals shown in figure 2. This is done by treating the coefficient functions of the "simple" diagrams in the set of equations $\{F^{(4)}_{\rm div}\}$ as finite.

4.1 Construction of the ϵ-finite set

Following the description in section 3 we find the diagrams shown in figure 3 as the ϵ-finite basis. Here the name of the ϵ-finite master integrals is deduced by using the name of the original master, which had been replaced, and the additional superscript letter "f". Note that except for T^f_{54} all members of the ϵ-finite basis pictured in figure 3 are just scalar Feynman integrals. The integrand corresponding to T^f_{54} contains a scalar product of two line momenta in its numerator. According to equation (7) the integrals of the ϵ-finite basis can be expressed in terms of the original master integrals of figure 1 and 2. This has been achieved by using the traditional IBP method in combination with the Laporta Algorithm. In this context the IBP identities have been generated with numerical values for the powers of the propagators and the irreducible scalar products. In the next

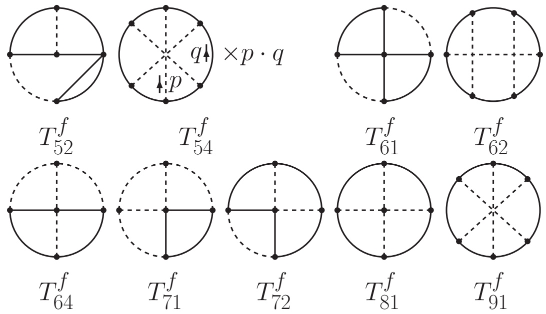


Figure 3: Diagrams of the ϵ-finite basis.

step, the resulting system of linear equations has been solved, by expressing systematically complicated integrals in terms of simpler ones and substituting the solutions into all the other equations. For this reduction to the original master integrals an automated program, which is based on `FORM3` [34, 35] has been developed in which also partially ideas described in ref. [11, 17, 36, 37] have been implemented. The coefficient functions of the integrals in the linear system of equations depend on the spacetime dimension d and need to be simplified during the solving procedure. This has been done with the program `FERMAT` [38]. Having expressed the integrals of figure 3 in terms of the original basis and following than equation (8), one obtains the relations which replace the original master integrals by the ϵ-finite ones. As these equations are rather lengthy, they are summarized in appendix A.

4.2 Calculation of the ϵ-finite basis

In order to calculate the results for the ϵ-finite basis we follow the method based on the ideas in ref. [39] and already described in [40]. One line of the four-loop tadpole under consideration is cut, thus obtaining a three-loop self-energy diagram. Since there is no other four-momentum involved, the self-energy diagram only depends on the momentum flowing through the applied cut. Therefore we are able to immediately perform the angular part of the remaining loop integration. Following this procedure, the calculation of the four-loop tadpole is transformed to the problem of evaluating a three-loop self-energy and a one-dimensional numerical integration over the modulus of the momentum through the cut.

The resulting three-loop diagrams are treated with the help of asymptotic expansions. Using the program packages `EXP` [41, 42], `MATAD` [43] and `MINCER` [44] one is able to compute the low and high energy expansion of the self-energy diagram. In a final step we combine the obtained high and low energy expansions in a Padé approximation as described in [45]. The resulting function is integrated numerically up to a cutoff Λ as the integration up to infinity is in general diverging.

The remaining integration region from Λ to infinity can be treated by direct analytical integration of the high energy expansion. Therefore one is able to compute the structure of the ϵ-poles analytically. We checked that the result for the finite part of the master integrals is cutoff-independent. In addition we tested that for finite tadpole diagrams like T^f_{91} the difference between the introduction of Λ and the numerical integration up to infinity is below the error of the numerical integration. A more detailed description of the procedure can be found in [46]. For diagram T^f_{52} we explicitly show the steps in the next subsection.

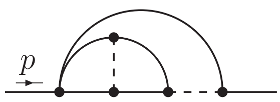
Note that in general we have a vast freedom in choosing the members of the ϵ-finite basis. Thus we give some additional remarks about selecting appropriate diagrams. For the applicability of the Padé method in the described way the self-energy diagram resulting after cutting one line should not contain any massless cuts (see [46]). Nevertheless, for each integral shown in figure 3 at least two possible cuts leading to two different self-energy diagrams could easily be found. This also provides us with a additional possibility to check the results.

The self-energy integrals appearing after cutting one line might contain UV and/or IR divergences. In general this is not a problem for the presented method as one is always able to subtract suitable diagrams containing the same sub-divergencies but of lower loop order, thus ending up with a self-energy function finite in the limit $\epsilon \to 0$. In appendix B we demonstrate this procedure explicitly for one diagram. However, a simple power counting applied to the diagrams of figure 3 demonstrates that excluding diagram T^f_{52} every possible cut leads to *finite* self-energies. In fact, the minimal presence of any divergencies was one of the main principles used by us in choosing suitable candidates for the ϵ-finite basis.

4.3 Treatment of the basis integral T^f_{52}

This diagram has two possible equivalent cuts which lead to the same finite self-energy function $\Sigma^{T^f_{52}}(p)$ related to the diagram

$$\Sigma^{T^f_{52}}(p) = \qquad (13)$$



resulting from the cut procedure. In general one might be left with a diagram which does not allow for a cut resulting in a finite self-energy function. In appendix B we present an alternative proceeding for diagram T^f_{52} involving a cut leading to a divergent self-energy function. As outlined in the previous section we calculate the high and low energy expansion using `EXP`, `MATAD` and `MINCER` before combining these results in a Padé approximation

$$f_{[n/m]}(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_1 z + \cdots + b_m z^m}. \qquad (14)$$

Given $n_{\rm low}$ terms of the low energy and $n_{\rm high}$ terms of the high energy expansion we are able to evaluate a Padé approximant $\Sigma^{T^f_{52}}_{[n/m]}(p^2)$ of the self-energy function in equation (13) for all values of n and m obeying the relation

$$n_{\rm low} + n_{\rm high} = n + m. \qquad (15)$$

This function we are then able to integrate numerically up to a cutoff Λ according to

$$T^f_{52[n/m]}(\Lambda) = \int_\Lambda \frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2+m^2_{\rm cut}}\Sigma^{T^f_{52}}_{[n/m]}(p^2) \qquad (16)$$

$$= \frac{2\pi^{d/2}}{(2\pi)^d}\int_0^\Lambda dp\,\frac{p^{d-1}}{p^2+m^2_{\rm cut}}\Sigma^{T^f_{52}}_{[n/m]}(p^2) \qquad (17)$$

where $m_{\rm cut} = m$ if we cut a massive and $m_{\rm cut} = 0$ if we cut a massless line.

The remaining integration from the cutoff Λ to infinity is performed by explicit analytical integration over the high energy expansion $\Sigma^{T^f_{52}}_{q\to\infty}(p^2)$ of the self-energy (13). In contrast to (17) here one just has to replace the Padé approximant by the high energy expansion

$$T^f_{52,\,q\to\infty}(\Lambda) = \frac{2\pi^{d/2}}{(2\pi)^d}\int_\Lambda^\infty dp\,\frac{p^{d-1}}{p^2+m^2_{\rm cut}}\Sigma^{T^f_{52}}_{q\to\infty}(p^2). \qquad (18)$$

For $m_{\rm cut} = 0$ or if we expand the denominator of the integrand in case of $m_{\rm cut} = m$ this integral has the simple form

$$T^f_{52,\,q\to\infty}(\Lambda) = \sum_{i=0}^{i_{\rm max}}\int_\Lambda^\infty dp\, a_i(\epsilon)\, p^{-n_i(\epsilon)} \qquad (19)$$

with noninteger positive exponents $n_i(\epsilon)$, at most linearly dependent on ϵ, and can be evaluated analytically

$$T^f_{52,\,q\to\infty}(\Lambda) = \sum_{i=0}^{i_{\rm max}}\frac{a_i(\epsilon)}{-n_i(\epsilon)+1}\,(\Lambda)^{-n_i(\epsilon)+1}. \qquad (20)$$

As some of the exponents $n_i(\epsilon)$ are linear dependent on ϵ, one has to evaluate the high-energy expansion $\Sigma^{T^f_{52}}_{q\to\infty}(p^2)$ up to the order linear in ϵ. Following this way the divergent part can be fully recovered through this analytical integration as we have chosen the diagrams and the cut in such a way that the self-energy is finite for $\epsilon \to 0$. The final result for the diagram T^f_{52} is then given by the sum

$$T^f_{52[n/m]} = T^f_{52[n/m]}(\Lambda) + T^f_{52,\,q\to\infty}(\Lambda) \qquad (21)$$

of the two contributions (17) and (18). Here numerical evaluations only enter in the calculation of the first term resulting from the Padé approximation. Therefore

$T^f_{52 n_{\text{low}}, n_{\text{high}}}$		n_{high}		
		7	8	9
n_{low}	4	27.30068104(1290)	27.30068323(70)	27.30068178(198)
	5	27.30067759(59)	27.30067456(1180)	27.30068017(86)
	6	27.30068220(998)	27.30068156(146)	27.30068059(36)
	7	27.30067826(492)	27.30068069(34)	27.30068067(5)

Table 1: Results for diagram T^f_{52} depending on different number of low and high energy input terms. In brackets we give the error of the procedure deduced from the standard deviation.

we obtain the full pole part of the diagrams analytically. In addition, we stress again, that the evaluation of any higher term in the ϵ-expansion is not needed.

According to equation (15) we are left with some freedom how to choose the Padé parameters n and m for a given value of n_{low} and n_{high}. Thus we average for each combination of n_{low} and n_{high} the results of all possible n, m-values. The average value $T^f_{52 n_{\text{low}}, n_{\text{high}}}$ is what we give as the final numerical result for a given set of high and low energy terms. The resulting standard deviation is a measure of the accuracy of this numerical approach. The values obtained for diagram T^f_{52} depending on different values of n_{low} and n_{high} are given in table 1. The result is nicely converging for increasing number of input terms.

4.4 Results for the ϵ-finite basis

Following this method we evaluated all the diagrams of the ϵ-finite basis shown in figure 3 up to the finite part. The numerical value of the constant term is obtained by using again nine terms in the high and seven terms in the low energy expansion of the self energy function arising after the cut. For each integration we use for the euclidian integrals in analogy to [33]

$$\frac{\mu^{4-d}}{N} \int \frac{d^d l}{(2\pi)^d} \cdots \tag{22}$$

with the normalization factor

$$N = \epsilon \mu^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2+m^2)^2} = \frac{1}{16\pi^2} \Gamma(1+\epsilon) \left(\frac{m^2}{4\pi\mu^2} \right)^{-\epsilon}. \tag{23}$$

In the following we consider $m = 1$. The results for the ϵ-finite basis then read:

$$
\begin{aligned}
T^f_{52} =& \frac{3\,\zeta_3}{2\,\epsilon^2} + \frac{15\,\zeta_3}{2\,\epsilon} - \frac{91\,\pi^4}{360\,\epsilon} - \frac{4\,\pi^2\,\log^2 2}{3\,\epsilon} + \frac{4\log^4 2}{3\,\epsilon} \\
&+ \frac{32\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{\epsilon} + 27.30068067(5), && (24)\\
T^f_{54} =& \frac{5\,\zeta_5}{4\,\epsilon} - 6.762240238547(2), && (25)\\
T^f_{61} =& \frac{5\,\zeta_5}{\epsilon} - 29.703462427815(3), && (26)\\
T^f_{62} =& 2.4403458233350757(6), && (27)\\
T^f_{64} =& \frac{5\,\zeta_5}{\epsilon} - 18.026245978729184(4), && (28)\\
T^f_{71} =& \frac{5\,\zeta_5}{\epsilon} - 5.19831391(6), && (29)\\
T^f_{72} =& \frac{5\,\zeta_5}{\epsilon} - 26.4964794044474738(5), && (30)\\
T^f_{81} =& \frac{5\,\zeta_5}{\epsilon} - 22.53760348(3), && (31)\\
T^f_{91} =& 1.808879546207(2). && (32)
\end{aligned}
$$

We emphasize that all divergent parts of these integrals are obtained analytically without any numerical input and that higher orders in the ϵ-expansion are not needed. A further investigation on the analytic results presented here is given in the next section. By applying the relations from appendix A the results for the ϵ-finite basis can be related to the master integrals in figure 2. We find full agreement with the values found in [26] as it is shown in table 2.

4.5 Analytical results for the master integrals

In subsection 4.3 we discussed a method which can be used to find the pole part of any massive four-loop tadpole analytically. More generally, this procedure is formulated in the following statement: the pole part of any $(L+1)$-loop tadpole is obtained analytically in terms of pole and finite parts of properly constructed L-loop tadpoles and L-loop massless integrals. An explicit construction of such an algorithm for the particular case of three-loop completely massive tadpoles can be found in [47].

It is instructive to investigate in some detail the internal mechanism of the cancellation of the spurious poles after the "old" basis set is expanded in terms of the new ϵ-finite basis. As we will see, the very existence of such cancellations leads to additional analytical results for master integrals.

With s_n we denote the maximal power of spurious poles which could appear in the coefficient function of an "old" master integral T_n. The values of s_n for

diagram	Padé method	[26]
T^f_{52}	27.30068067(5)	27.30068068
T^f_{54}	-6.762240238547(2)	-6.762240238546
T^f_{61}	-29.703462427815(3)	-29.703462427816
T^f_{62}	2.440345823350757(6)	2.440345823350754
T^f_{64}	-18.026245978729184(4)	-18.026245978729184
T^f_{71}	-5.19831391(6)	-5.19831389
T^f_{72}	-26.4964794044474738(5)	-26.4964794044474733
T^f_{81}	-22.53760348(3)	-22.53760347
T^f_{91}	1.808879546207(2)	1.808879546208

Table 2: Comparison of the constant part of the ϵ-finite basis with results extracted from [26]. Note that in the last column we only keep the accuracy needed for this comparison, whereas in [26] higher precision results are available.

all non-trivial masters is given in figure 2. Let us start with integral T_{71}. As we find a value of $s_{71} = 1$ one expects that all coefficient functions should be at least proportional to ϵ when expressing T_{71} in terms of ϵ-finite basis. Here we, of course, exclude terms proportional to the analytically known integrals T_{41}, T_{51}, T_{53} and T_{63} which had by construction been excluded from the ϵ-finite basis. The explicit result reads

$$T_{71} = \epsilon\,(4\,T^f_{64} + 6\,T^f_{71} + 2\,T^f_{72}) + O(\epsilon^2)\,T^f + \ldots, \tag{33}$$

in full agreement with our expectations. Here we denote by $O(\epsilon^2)\,T^f$ terms with coefficient functions suppressed by at least as ϵ^2 at $\epsilon \to 0$ and $\ldots$ represents contributions coming from analytically known master integrals. Equation (34) also carries important information about the *finite* part of T_{71}. Indeed, as the pole parts of the new master integrals are known analytically, one could easily use equation(34) to get again analytically the pole and the finite part of T_{71} leading to the result

$$T_{71} = -\frac{1}{6\epsilon^4} - \frac{3}{2\epsilon^3} - \frac{26}{3\epsilon^2} - \frac{\zeta_3}{\epsilon^2} - \frac{41}{\epsilon} - \frac{10\zeta_3}{3\epsilon} - \frac{\pi^4}{60\epsilon} - \frac{7\pi^4}{30} + 3\zeta_3 - \frac{1039}{6} + 53\zeta_5 + O(\epsilon). \tag{34}$$

Repeating the same procedure for the integrals $T_{52}, T_{54}, T_{61}, T_{62}, T_{64}$ and T_{72} we obtain the following results:

$$
\begin{aligned}
T_{52} =& -8\epsilon^2\, T^f_{52} + O(\epsilon^3)\, T^f + \dots & (35)\\
=& -\frac{2}{\epsilon^4} - \frac{29}{3\,\epsilon^3} - \frac{163}{6\,\epsilon^2} - \frac{601}{12\,\epsilon} - \frac{635}{24} - \frac{112\,\zeta_3}{3} + \epsilon\left(\frac{4549}{16} + \frac{136\,\pi^4}{45}\right.\\
&\left. + \frac{32\,\pi^2\,\log^2 2}{3} - \frac{32\,\log^4 2}{3} - 256\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{952\,\zeta_3}{3}\right) + O(\epsilon^2), & (36)
\end{aligned}
$$

$$
\begin{aligned}
T_{54} =& \epsilon^3\,\frac{256}{3}\left\{T^f_{52} + T^f_{62} + \frac{1}{3}\,T^f_{91}\right\} + O(\epsilon^4)\, T^f + \dots & (37)\\
=& -\frac{1}{\epsilon^4} - \frac{9}{2\,\epsilon^3} - \frac{415}{36\,\epsilon^2} - \frac{4991}{216\,\epsilon} - \frac{89383}{1296} + \epsilon\left(-\frac{3367679}{7776} + \frac{1792\,\zeta_3}{9}\right)\\
&+ \epsilon^2\left(-\frac{137735095}{46656} + \frac{68992\,\zeta_3}{27} - \frac{4352\,\pi^4}{135} - \frac{1024\,\pi^2\,\log^2 2}{9}\right.\\
&\left. + \frac{1024\,\log^4 2}{9} + \frac{8192\,\mathrm{Li}_4\left(\frac{1}{2}\right)}{3}\right) + O(\epsilon^3), & (38)
\end{aligned}
$$

$$
\begin{aligned}
T_{61} =& \epsilon^2\,\left\{24\,T^f_{52} + 6\,T^f_{61} + 12\,T^f_{64} + 18\,T^f_{71} + 6\,T^f_{72}\right\} + O(\epsilon^3)\, T^f + \dots & (39)\\
=& \frac{3}{2\,\epsilon^4} + \frac{19}{2\,\epsilon^3} + \frac{67}{2\,\epsilon^2} + \frac{127}{2\,\epsilon} - \frac{3\,\zeta_3}{\epsilon} - \frac{237}{2} + 97\,\zeta_3 - \frac{\pi^4}{20} + \epsilon\left(-\frac{3969}{2}\right.\\
&- \frac{559\,\pi^4}{60} - 32\,\pi^2\,\log^2 2 + 32\,\log^4 2 + 768\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) + 1261\,\zeta_3\\
&\left. + 189\,\zeta_5\right) + O(\epsilon^2), & (40)
\end{aligned}
$$

$$
\begin{aligned}
T_{62} =& \epsilon^2\,\left\{16\,T^f_{52} + 16\,T^f_{62} + \frac{32}{3}\,T^f_{91}\right\} + O(\epsilon^3)\, T^f + \dots & (41)\\
=& \frac{2}{3\,\epsilon^4} + \frac{4}{\epsilon^3} + \frac{38}{3\,\epsilon^2} + \frac{44}{3\,\epsilon} + \frac{16\,\zeta_3}{3\,\epsilon} - 118 + 88\,\zeta_3 - \frac{4\,\pi^4}{15} + \epsilon\left(-1156\right.\\
&- \frac{374\,\pi^4}{45} - \frac{64\,\pi^2\,\log^2 2}{3} + \frac{64\,\log^4 2}{3} + 512\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \frac{2152\,\zeta_3}{3}\\
&\left. + 96\,\zeta_5\right) + O(\epsilon^2), & (42)
\end{aligned}
$$

$$
\begin{aligned}
T_{64} =& 9\epsilon^2\, T^f_{71} + O(\epsilon^3)\, T^f + \dots & (43)\\
=& \frac{1}{4\,\epsilon^4} + \frac{3}{2\,\epsilon^3} + \frac{9}{2\,\epsilon^2} + \frac{2}{\epsilon} + \frac{13\,\zeta_3}{2\,\epsilon} - \frac{147}{2} + 30\,\zeta_3 - \frac{5\,\pi^4}{8} + \epsilon\left(-624\right.\\
&\left. - \frac{27\,\pi^4}{10} + 45\,\zeta_3 + \frac{693\,\zeta_5}{2}\right) + O(\epsilon^2), & (44)
\end{aligned}
$$

$$T_{72} = \epsilon \left\{ 2\, T^f_{64} + 3\, T^f_{71} \right\} + O(\epsilon^2)\, T^f + \ldots \tag{45}$$

$$= -\frac{1}{6\,\epsilon^4} - \frac{3}{2\,\epsilon^3} - \frac{26}{3\,\epsilon^2} - \frac{\zeta_3}{2\,\epsilon^2} - \frac{41}{\epsilon} + \frac{\zeta_3}{6\,\epsilon} - \frac{\pi^4}{120\,\epsilon} - \frac{1039}{6} - \frac{29\,\pi^4}{120} + \frac{17\,\zeta_3}{2} + \frac{43\,\zeta_5}{2} + O(\epsilon). \tag{46}$$

It is remarkable, that depth of the expansion in ϵ of these analytical results for "old" masters $T_{54}, T_{61}, T_{62}, T_{64}, T_{71}$ and T_{72} listed above are in full agreement with the depth obtained in [26] by fitting the high precision numerical results.

It is even more remarkable that in mostly all of the cases the highest power ϵ^{d_n} of the analytically accessible part of the expansion is correlated with the maximal power of the corresponding spurious pole s_n introduced before, such that

$$d_n = s_n - 1. \tag{47}$$

Thus, we observe some rationale as for how many terms of the ϵ-expansion of a master integral can be computed analytically.

However, equation (47) seems to be violated in case of the integrals T_{54}, T_{62} and T_{91}. Here we observe the relation

$$d_i = s_i - 2. \tag{48}$$

At first glance equation (48) is in apparent contradiction to the very concept of the ϵ-finite basis. Indeed, consider, for example, the integral T_{54}. Given the value $s_{54} = 4$ from figure 2, it is known that this integral sometimes appears accompanied by a spurious pole $\frac{1}{\epsilon^4}$. A naive application of equation (37) tells us that after expanding $\frac{1}{\epsilon^4} T_{54}$ in terms of the ϵ-finite basis the integrals T^f_{52}, T^f_{62} and T^f_{91} would have *non*-ϵ-finite coefficients! The problem is solved by noting that these $\frac{1}{\epsilon^4}$ poles always appear in one and the same combination, namely

$$T = \frac{1}{128} \frac{1}{\epsilon^4} \left\{ 9\, T_{54} - 48\, \epsilon\, T_{62} + 256\, \epsilon^3\, T_{91} \right\}. \tag{49}$$

As expected the result of expanding T in the ϵ-finite basis has finite coefficients in front of non-trivial masters:

$$T = 11\, T^f_{52} - 8\, T^f_{54} - 8\, T^f_{61} + 11\, T^f_{62} + 6\, T^f_{64} + 18\, T^f_{71} + 10\, T^f_{72} - \frac{73}{3}\, T_{91} + O(\epsilon)\, T^f + \ldots \tag{50}$$

Hence, this relation should lead to an extra analytical constraint on T_{52}, T_{62} and T_{91}. Indeed, equations (24-32) allows us to compute analytically the pole part of the right hand side of (50) and, simultaneously, the same pole part could expressed through Taylor expansion the right hand side of (49) at $\epsilon \to 0$. As a result we get

$$\frac{1}{4}\, T_{91,0} + \frac{9}{1024}\, T_{54,3} - \frac{3}{64}\, T_{62,2} = \frac{5511907345}{31850496} + \frac{89\,\pi^4}{1440} + \frac{\pi^6}{56} - \frac{11\,\pi^2\,\log^2 2}{6} + \frac{11\,\log^4 2}{6} + 44\,\mathrm{Li}_4\left(\tfrac{1}{2}\right) - \frac{4103\,\zeta_3}{144} - {\zeta_3}^2 - \frac{273\,\zeta_5}{8}, \quad (51)$$

where we have used the notation:

$$T_n = \sum_{i=n_{\min}}^{\infty} \epsilon^i\, T_{n,i}. \quad (52)$$

The integrals T_{54} and T_{62}, being of a "sunset"-like type, are certainly much more suitable for a numeric evaluation than T_{91} (for an exhaustive review see, e.g. [48]). Moreover, the integral T_{91} seems to be rather complicated for a numerical treatment as illustrated by the fact that in [26] it was computed with significantly less precision than all others. Thus, it is natural to take (51) as an equation for the calculation of $T_{91,0}$. Substituting $T_{62,2}$ and $T_{54,3}$ as found in [26] one obtains immediately without any further calculation

$$T_{91,0} = 1.80887954620833474142636459508695209 0 \quad (53)$$

which should be compared to the original result given in [26] (for further discussion see the next section)

$$T_{91,0} = 1.80887954620833474. \quad (54)$$

5 General Comments

There is a significant amount of freedom in the choice of specific members of the ϵ-finite basis. As we have seen in subsection 4.2 this freedom can be used in order to greatly simplify the evaluation of the master integrals. In fact, the very idea of the ϵ-finite basis appeared after facing problems in trying to evaluate the "old" master integrals with the help of the Padé method.

An important feature of the ϵ-finite basis is extremely simple structure of the UV and IR divergences of its members. Indeed, except for T_{81} and T_{91} all members of the "old" basis set do suffer from overlapping UV divergences. Whereas all new masters but one are either finite or possess only mild superficial UV divergences. The integral T_{52}^f is not primitively divergent but its only UV divergent subgraph could be very easily isolated as demonstrated in appendix B.

The fact that the IBP identities sometimes provide some nontrivial constraints on the *values* of master integrals is, of course, not entirely new. For example, in

[49] the divergent parts of some of the irreducible four-loop completely massive tadpoles, appearing in the calculation of the four-loop QCD β function, have been obtained from the finiteness of the same integral with higher powers of denominators. However, we have already discussed in section 4.5 that the evaluation of the pole part of a given four-loop tadpole is not a problem at all. We are only aware of few cases when IBP identities were actually used to evaluate the finite term or terms proportional to a nonnegative integer power of ϵ in the ϵ-expansion of nontrivial four-loop master integrals.

In [32] the order ϵ coefficient of the master integral T_{54} (there denoted as BB4) could be deduced, as it had been the only analytically unknown term entering the cancellation of the divergent contributions during the renormalization procedure when calculating the so-called ρ parameter. Furthermore, the coefficient of order ϵ^2 was subsequently obtained by combining the numerically known value with the basis of transcendentals from [50]. Within our approach *all* the terms up to the ϵ^2 contribution are directly accessible by the use of ϵ-finite basis without any need for either a high-precision numerical evaluation or the basis of transcendentals which could appear.

Schematically, the relations which were used to fix $T_{54,1}$ in [32] can be written as

$$\rho_B - (\text{known UV sub-divergencies}) = \mathcal{O}(\epsilon^0) \tag{55}$$

and

$$\rho_B - (\text{known UV sub-divergencies}) = \alpha_s^3\, T_{54,1}/\epsilon + \text{known terms} \tag{56}$$

with known UV sub-divergencies. Equations (55) and (56) directly leads to the value of $T_{54,1}$. Nevertheless relations like (55) involving the cancellation of divergencies can not, obviously, provide any new information in addition to what is obtainable from the use of the ϵ-finite basis. Furthermore there is no systematic way to find such relations, not to mention the requirement that all renormalization constants have to be known analytically.

The fact that such relations can sometimes be found by chance is also well illustrated in the next example. In section 4.5 equation (51) was derived in a regular way as natural by-product of the properties of the constructed ϵ-finite basis. Recently a paper has appeared where the same relation (51) has been independently obtained in a similar way like the result for $T_{54,1}$ in [32]. The authors of reference [51] have computed the so-called decoupling function ζ_g for the QCD quark gluon coupling constant in four-loop approximation. The bare function ζ_g^0 is essentially composed of three pieces:

$$\zeta_g^0 = \frac{\tilde{\zeta}_1^0}{\tilde{\zeta}_3^0 \sqrt{\zeta_3^0}}, \tag{57}$$

where the three auxiliary bare functions on the right side $\tilde{\zeta}_1^0$, $\tilde{\zeta}_3^0$ and ζ_3^0 are related to the heavy quark containing contributions to the ghost-ghost-gluon vertex, the gluon propagator and the ghost propagator respectively (for more details see [32, 52, 53]). Once the three bare functions are determined and the physically relevant quantity ζ_g^0 is constructed one can renormalize ζ_g^0 directly with the help of (long-available from [54–56]) renormalization constants for α_s and the quark mass and *without* any necessity to renormalize the three auxiliary functions themselves. The latter is certainly possible but requires knowledge of three extra renormalization constants, namely those for the ghost-ghost-gluon vertex, the gluon and ghost wave functions. The corresponding calculation is certainly possible but is *not* a small enterprise and the results for these constants appeared only recently [49, 57]. The remarkable fact is that the combination T (defined in eq. (49)) enters into the renormalization of the auxiliary functions as follows (b is a known constant)

$$\tilde{\zeta}_1^0 - \text{(known UV sub-divergencies) does not contain } T, \tag{58}$$

$$\tilde{\zeta}_3^0 - \text{(known UV sub-divergencies)} = 1 + b\,\alpha_s^3\,T + \text{known terms}, \tag{59}$$

and

$$\zeta_3^0 - \text{(known UV sub-divergencies)} = 1 - 2\,b\,\alpha_s^3\,T + \text{known terms}. \tag{60}$$

We observe that

(i) the relation (51) can be found in the process of renormalizing one of eqs. (59 - 60) but *cannot* if one deals only with the physical quantity ζ_g.

(ii) our way of finding (51) is completely self contained and does not require any knowledge of extra non-trivial renormalization constants, while the same result obtained in [51] does depend on such knowledge.

(iii) full agreement between relation (53) obtained in section 5 and eq. (3.2) of [51] provides a non-trivial check of the results in [49, 57].

An interesting question is whether the ϵ-finiteness of a basis is preserved if one considers a different physical problem whose solution is expressible in terms of the "old" master integrals. On general grounds we can not exclude that for an "old" master, say, T_i the maximal power of the spurious pole would e.g. exceed the one displayed on figure 2. In such a case the basis given in section 4.1 does not have coefficient functions finite in the limit $\epsilon \to 0$ and we would have to construct a new, more general ϵ-finite basis following the prescription of section 3. However, we do think that such a situation is unlikely and that therefore the in section 4.1 constructed ϵ-finite basis is a universal one for all four-loop problems reducible to the evaluation of QED like massive tadpoles. The following observations support our point of view:

- As stated in section 4 we have only used the reduction and integrals appearing in the low energy expansion of the four-loop vacuum polarization function (see [33]) for constructing the ϵ-finite basis. In fact, the evaluation of decoupling function ζ_g for the QCD quark gluon coupling constant in four-loop approximation requires the reduction of additional more complicated integrals to masters. We have checked that the basis given in figure 3 remains ϵ-finite for all these extra integrals.
- If our hypothesis on the universality property of the constructed ϵ-finite basis is wrong it also implies that using the method described in section 4.3 one should be able to find more terms in the ϵ-expansions of the "old" and of the ϵ-finite master integrals analytically. This extension of the equations (24-32,34-45,51) without using more sophisticated techniques seems not very probable to us.

Conclusions

As a conclusion we want to point out that the very idea of the ϵ-finite basis, namely the procedure of its systematic construction as well as the application to finding *direct* analytic information of *ϵ-finite* contributions to *non-ϵ-finite* master integrals are all new. We believe that choosing the ϵ-finite basis should be especially useful and convenient in all cases when numerical evaluation of master integrals is necessary.

Our results are also available in computer readable form under the URL `http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp06-02`.

Acknowledgments

The authors are grateful to R.V. Harlander, J.H. Kühn, Y. Schröder and M. Steinhauser for useful discussions. K.Ch. thanks M. Kalmykov for useful advice about the integral T_{52}.

The work was supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics".

The work of C.S. was also partially supported by MIUR under contract 2001023713_006.

A Old masters in terms of ϵ-finite basis

$$
\begin{aligned}
& T_{52} = \\
+ \quad & \frac{2\,(d-2)^2\,(1298-953\,d+233\,d^2-19\,d^3)}{(d-3)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T_{41} \\
- \quad & \frac{4\,(d-5)\,(d-4)\,(4\,d-17)}{(2\,d-7)\,(73-34\,d+4\,d^2)}\,T_{51} \\
+ \quad & \frac{16\,(2\,d-7)\,(2\,d-5)\,(209-96\,d+11\,d^2)}{3\,(d-2)\,(3\,d-11)\,(73-34\,d+4\,d^2)}\,T_{53} \\
+ \quad & \frac{32\,(d-4)\,(d-3)^2\,(2\,d-7)\,(149-69\,d+8\,d^2)}{(d-2)\,(3\,d-11)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T_{63} \\
+ \quad & \frac{32\,(d-5)\,(d-4)^2\,(d-3)}{(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{52} \\
- \quad & \frac{256\,(d-5)\,(d-4)^4}{3\,(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{54} \\
- \quad & \frac{16\,(d-4)^3\,(767-351\,d+40\,d^2)}{3\,(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{61} \\
- \quad & \frac{128\,(d-5)^2\,(d-4)^3}{(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{62} \\
- \quad & \frac{32\,(d-4)^3\,(109-51\,d+6\,d^2)}{(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{64} \\
- \quad & \frac{48\,(d-4)^3\,(149-69\,d+8\,d^2)}{(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{71} \\
+ \quad & \frac{16\,(d-4)^3\,(673-297\,d+32\,d^2)}{3\,(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T^f_{72} \\
- \quad & \frac{128\,(d-5)\,(d-4)^3\,(3\,d-13)}{3\,(d-2)\,(3\,d-10)\,(3\,d-8)\,(73-34\,d+4\,d^2)}\,T_{91}\,,
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
&T_{54} = && (62)\\
&- \frac{(d-2)^3\,(21458 - 20723\,d + 7488\,d^2 - 1200\,d^3 + 72\,d^4)}{2\,(d-3)\,(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T_{41}\\
&+ \frac{8\,(d-4)^2\,(d-2)\,(25 - 22\,d + 4\,d^2)}{(2\,d-7)^2\,(2\,d-5)\,(73 - 34\,d + 4\,d^2)}\,T_{51}\\
&- \frac{32\,(d-4)\,(10\,d-33)}{3\,(3\,d-11)\,(73 - 34\,d + 4\,d^2)}\,T_{53}\\
&- \frac{256\,(d-4)^2\,(d-3)^3}{(2\,d-5)\,(3\,d-11)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T_{63}\\
&+ \frac{256\,(d-5)\,(d-4)^3\,(d-3)}{(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{52}\\
&+ \frac{512\,(d-4)^4}{3\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{54}\\
&+ \frac{128\,(d-4)^4\,(11\,d-37)}{3\,(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{61}\\
&+ \frac{256\,(d-5)\,(d-4)^3}{(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{62}\\
&+ \frac{128\,(d-4)^4}{(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{64}\\
&+ \frac{384\,(d-4)^4\,(d-3)}{(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{71}\\
&- \frac{128\,(d-4)^4\,(25\,d-89)}{3\,(2\,d-7)\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T^f_{72}\\
&+ \frac{256\,(d-4)^3\,(3\,d-13)}{3\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(73 - 34\,d + 4\,d^2)}\,T_{91}\,,
\end{aligned}
$$

$$
\begin{aligned}
&T_{64} = && (63)\\
&+ \frac{3\,(d-4)\,(d-2)\,(3\,d-8)\,(82 - 48\,d + 7\,d^2)}{64\,(d-3)^3\,(2\,d-7)^2}\,T_{51}\\
&- \frac{(2\,d-5)\,(3\,d-10)^2\,(3\,d-8)}{4\,(d-4)\,(d-3)^3}\,T_{53}\\
&- \frac{3\,(3\,d-10)}{4\,(d-3)}\,T_{63}\\
&+ \frac{9\,(d-4)^2\,(3\,d-11)\,(3\,d-10)}{8\,(d-3)^3\,(2\,d-7)}\,T^f_{71}\,,
\end{aligned}
$$

$$
\begin{aligned}
& T_{61} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (64)\\
+ \quad & \frac{3\,(d-2)^3\,(11278 - 11551\,d + 4411\,d^2 - 745\,d^3 + 47\,d^4)}{16\,(d-3)^3\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T_{41}\\
+ \quad & \frac{3\,(d-5)\,(d-4)\,(d-2)\,(3\,d-8)\,(4\,d-17)\,(5\,d-16)}{8\,(d-3)^2\,(2\,d-7)^2\,(73 - 34\,d + 4\,d^2)}\,T_{51}\\
- \quad & \frac{(2\,d-5)\,(3\,d-8)\,(21406 - 22007\,d + 8444\,d^2 - 1434\,d^3 + 91\,d^4)}{2\,(d-4)\,(d-3)^2\,(3\,d-11)\,(73 - 34\,d + 4\,d^2)}\,T_{53}\\
- \quad & \frac{3\,(17566 - 18119\,d + 6980\,d^2 - 1191\,d^3 + 76\,d^4)}{(3\,d-11)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T_{63}\\
- \quad & \frac{3\,(d-5)\,(d-4)^2\,(5\,d-16)}{(d-3)\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{52}\\
+ \quad & \frac{8\,(d-5)\,(d-4)^4\,(5\,d-16)}{(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{54}\\
+ \quad & \frac{(d-4)^2\,(73178 - 75093\,d + 28748\,d^2 - 4869\,d^3 + 308\,d^4)}{2\,(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{61}\\
+ \quad & \frac{12\,(d-5)^2\,(d-4)^3\,(5\,d-16)}{(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{62}\\
+ \quad & \frac{3\,(d-4)^2\,(15006 - 15527\,d + 6004\,d^2 - 1029\,d^3 + 66\,d^4)}{(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{64}\\
+ \quad & \frac{9\,(d-4)^2\,(17566 - 18119\,d + 6980\,d^2 - 1191\,d^3 + 76\,d^4)}{2\,(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{71}\\
- \quad & \frac{(d-4)^2\,(18982 - 18219\,d + 6388\,d^2 - 963\,d^3 + 52\,d^4)}{2\,(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T^f_{72}\\
+ \quad & \frac{4\,(d-5)\,(d-4)^3\,(3\,d-13)\,(5\,d-16)}{(d-3)^2\,(2\,d-7)\,(3\,d-10)\,(73 - 34\,d + 4\,d^2)}\,T_{91}\,,
\end{aligned}
$$

$$
\begin{aligned}
& T_{62} = \qquad\qquad (65)\\
- & \frac{(d-4)^2\,(d-2)^3\,(d-1)}{4\,(d-3)^3\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T_{41}\\
+ & \frac{(d-2)\,(3\,d-8)\,(499-409\,d+111\,d^2-10\,d^3)}{4\,(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T_{51}\\
- & \frac{2\,(2\,d-5)\,(3\,d-10)\,(3\,d-8)\,(63-28\,d+3\,d^2)}{3\,(d-3)^2\,(3\,d-11)\,(73-34\,d+4\,d^2)}\,T_{53}\\
- & \frac{4\,(d-4)^2\,(4\,d-19)}{(3\,d-11)\,(73-34\,d+4\,d^2)}\,T_{63}\\
- & \frac{4\,(d-5)\,(d-4)^2}{(d-3)\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{52}\\
- & \frac{16\,(d-4)^3\,(33-16\,d+2\,d^2)}{3\,(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{54}\\
- & \frac{2\,(d-4)^3\,(255-125\,d+16\,d^2)}{3\,(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{61}\\
- & \frac{4\,(d-5)\,(d-4)^2}{(d-3)^2\,(73-34\,d+4\,d^2)}\,T^f_{62}\\
+ & \frac{4\,(d-4)^4\,(2\,d-9)}{(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{64}\\
+ & \frac{6\,(d-4)^4\,(4\,d-19)}{(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{71}\\
+ & \frac{2\,(d-4)^3\,(276-145\,d+20\,d^2)}{3\,(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T^f_{72}\\
- & \frac{8\,(d-4)^2\,(3\,d-13)\,(33-16\,d+2\,d^2)}{3\,(d-3)^2\,(2\,d-7)\,(73-34\,d+4\,d^2)}\,T_{91}\,,
\end{aligned}
$$

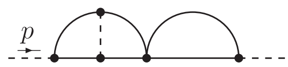
$$
\begin{aligned}
&T_{72} = \qquad (66)\\
&- \frac{(d-2)^3\,(1789-2111\,d+912\,d^2-172\,d^3+12\,d^4)}{8\,(d-3)^2\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{41}\\
&- \frac{(d-4)\,(d-2)\,(3\,d-8)\,(914-525\,d+90\,d^2-4\,d^3)}{16\,(d-3)\,(2\,d-7)^2\,(3\,d-10)\,(73-34\,d+4\,d^2)}\,T_{51}\\
&+ \frac{(2\,d-5)\,(3\,d-8)\,(9772-7470\,d+1165\,d^2+399\,d^3-142\,d^4+12\,d^5)}{6\,(d-4)^2\,(d-3)\,(3\,d-11)\,(3\,d-10)\,(73-34\,d+4\,d^2)}\,T_{53}\\
&+ \frac{(d-3)^2\,(12628-13328\,d+5255\,d^2-918\,d^3+60\,d^4)}{(d-4)\,(3\,d-11)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{63}\\
&- \frac{8\,(d-5)\,(d-4)^3}{(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{52}\\
&- \frac{16\,(d-4)^4}{3\,(d-3)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{54}\\
&- \frac{4\,(d-4)^4\,(11\,d-37)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{61}\\
&- \frac{8\,(d-5)\,(d-4)^3}{(d-3)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{62}\\
&+ \frac{2\,(d-4)\,(35898-47927\,d+25554\,d^2-6802\,d^3+904\,d^4-48\,d^5)}{(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{64}\\
&- \frac{3\,(d-4)\,(12628-13328\,d+5255\,d^2-918\,d^3+60\,d^4)}{2\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{71}\\
&+ \frac{4\,(d-4)^4\,(25\,d-89)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{72}\\
&- \frac{8\,(d-4)^3\,(3\,d-13)}{3\,(d-3)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{91}\,,
\end{aligned}
$$

$$
\begin{aligned}
&T_{71} = \qquad (67)\\
&- \frac{(d-2)^3\,(5925-6293\,d+2499\,d^2-440\,d^3+29\,d^4)}{4\,(d-3)^2\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{41}\\
&- \frac{(d-2)\,(3\,d-8)\,(9053-9587\,d+3799\,d^2-668\,d^3+44\,d^4)}{4\,(d-3)\,(2\,d-7)^2\,(3\,d-10)\,(73-34\,d+4\,d^2)}\,T_{51}\\
&- \frac{2\,(2\,d-5)\,(3\,d-8)\,(26546-35605\,d+19114\,d^2-5132\,d^3+689\,d^4-37\,d^5)}{3\,(d-4)^2\,(d-3)\,(3\,d-11)\,(3\,d-10)\,(73-34\,d+4\,d^2)}\,T_{53}\\
&- \frac{4\,(d-3)\,(19090-25909\,d+14064\,d^2-3815\,d^3+517\,d^4-28\,d^5)}{(d-4)\,(3\,d-11)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{63}\\
&- \frac{4\,(d-5)\,(d-4)^2\,(5\,d-19)}{(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{52}\\
&+ \frac{16\,(d-4)^3\,(554-435\,d+114\,d^2-10\,d^3)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{54}\\
&+ \frac{2\,(d-4)^3\,(7327-5682\,d+1473\,d^2-128\,d^3)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{61}\\
&- \frac{4\,(d-5)\,(d-4)^2\,(54-29\,d+4\,d^2)}{(d-3)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{62}\\
&+ \frac{4\,(d-4)\,(21906-29301\,d+15680\,d^2-4195\,d^3+561\,d^4-30\,d^5)}{(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{64}\\
&+ \frac{6\,(d-4)\,(19090-25909\,d+14064\,d^2-3815\,d^3+517\,d^4-28\,d^5)}{(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{71}\\
&- \frac{2\,(d-4)\,(50762-63841\,d+32144\,d^2-8107\,d^3+1025\,d^4-52\,d^5)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T^f_{72}\\
&+ \frac{8\,(d-4)^2\,(3\,d-13)\,(554-435\,d+114\,d^2-10\,d^3)}{3\,(d-3)\,(2\,d-7)\,(3\,d-10)^2\,(73-34\,d+4\,d^2)}\,T_{91}\,.
\end{aligned}
$$

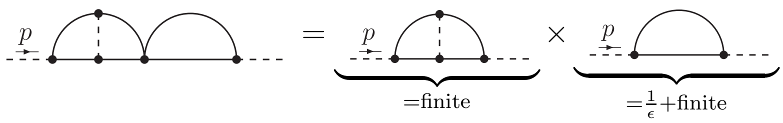
B Proceeding in case of a "divergent" cut

In this section we show an alternative approach of calculating diagram T^f_{52}. If we perform the cut of the diagram in the following way

$$\Sigma^{T^f_{52}}(p) = \qquad (68)$$

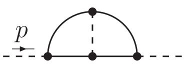
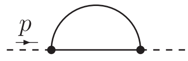
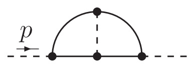


one obtains a divergent self energy function $\Sigma^{T^f_{52}}(p)$. This results from the fact that the diagram in equation (68) consists of



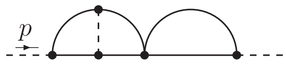
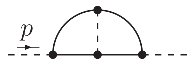
(69)

a finite two-loop multiplied by a divergent one-loop diagram. If one identifies such a sub-divergency one is able to construct the combination

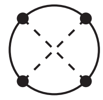
$\Sigma^{T^f_{52}}(p) =$  $\times \Big($  $- \frac{1}{\epsilon} \Big) + \frac{1}{\epsilon}$  (70)

$\equiv \tilde{\Sigma}^{T^f_{52}}(p) = \text{finite}$

of a finite self energy function $\tilde{\Sigma}^{T^f_{52}}(p)$ and a diagram of lower loop order which can be calculated analytically. In general one might have to treat several such sub-divergencies in the similar way. In a next step $\tilde{\Sigma}^{T^f_{52}}(p)$

$\tilde{\Sigma}^{T^f_{52}}(p) =$  $- \frac{1}{\epsilon}$  (71)

is evaluated to obtain $\tilde{T}^f_{52\,[n/m]}$ according to the proceeding described in section 4.2. Thus we are again able to extract the pole structure of $\tilde{T}^f_{52\,[n/m]}$ analytically and the constant part of the ϵ-expansion numerically. Finally one obtains the expansion for the diagram T^f_{52}

$T^f_{52\,[n/m]} = \tilde{T}^f_{52\,[n/m]} + \frac{1}{\epsilon}$  (72)

by adding the analytical result of the three-loop tadpole diagram. We find full agreement between this approach and the results discussed in section 4.2.

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