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QCD Decoupling at Four Loops

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Abstract

We present the matching condition for the strong coupling contant $\alpha_s^{(n_f)}$ at a heavy quark threshold to four loops in the modified minimal subtraction scheme. Our results lead to further decrease of the theoretical uncertainty of the evolution of the strong coupling constant through heavy quark thresholds. Using a low energy theorem we furthermore derive the effective coupling of the Higgs boson to gluons (induced by a virtual heavy quark) in four- and (partially) through five-loop approximation.

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1 Introduction

The masses of the known quark species differ vastly in their magnitude. As a result of this, in many QCD applications the mass of a heavy quark h is much larger than the characteristic momentum scale $\sqrt{s}$ which is intrinsic to the physical process. In such a situation there appear two interrelated problems when using an MS-like renormalization scheme.

First, given the two large but in general quite different mass scales, $\sqrt{s}$ and m, two different types of potentially dangerously large logarithms may arise. The standard trick of a proper choice of the renormalization scale μ is no longer effective; one can not set one parameter μ equal to two very different mass scales simultaneously.

Second, according to the Appelquist-Carazzone theorem [1] the effects due to heavy particles should eventually "decouple" from the low-energy physics[2]. However, a peculiarity of mass-independent renormalization schemes is that the decoupling theorem does not hold in its naive form for theories renormalized in this framework. The effective QCD action to appear will not be canonically normalized. Even worse, potentially large mass logarithms in general appear, when one calculates a physical observable.

The well-known way to bring the large mass logarithms under control is to construct an effective field theory by "integrating out" the heavy quark field h [2–6]. By construction, the resulting effective QCD Lagrangian will not include the heavy quark field. In order to be specific, let us consider QCD with $n_l = n_f - 1$ light quarks and one heavy quark h with mass m. The quark-gluon coupling constants in both theories, the full n_f-flavor QCD and the effective n_l-flavor one, $\alpha_s^{(n_f)}$ and $\alpha_s^{(n_l)}$ are related by the so-called matching condition of the form [2, 3]

$$\alpha_s^{(n_l)}(\mu) = \zeta_g^2(\mu, \alpha_s^{(n_f)}(\mu), m)\ \alpha_s^{(n_f)}(\mu). \tag{1}$$

Here $m = m(\mu)$ is the (running) mass of the heavy quark; the decoupling function ζ_g is equal to one at the leading (tree) level but receives nontrivial

[2]It should be stressed that the statement is literally valid only if power-suppressed corrections of order $(s/m^2)^n$ with $n > 0$ are neglected.

corrections in higher orders. The matching point μ in eq. (1) should be chosen in such a way to minimize the effects of logarithms of the heavy quark mass, e.g. $\mu = \mathcal{O}(m)$.

An important phenomenological application of eq. (1) appears in the determination of $\alpha_s(M_Z)$ at the Z-boson scale through evolution with the renormalization group equation, starting from the measured value $\alpha_s(m_\tau)$ at the τ-lepton scale. A careful analysis of the effects of four-loop running and three-loop matching in extracting $\alpha_s^{(5)}(M_Z)$ with 5 active quark flavors from $\alpha_s^{(3)}(m_\tau)$ with 3 active quark flavors has been recently performed in ref. [7]. We will demonstrate that the inclusion of the newly computed four-loop matching condition leads to further reduction of the theoretical error from the evolution.

The second aspect, mentioned above, is of importance for Higgs-boson production in hadronic collisions. The dominant subprocess for the production of the Standard-Model (SM) Higgs boson at the CERN Large Hadron Collider (LHC) will be the one via gluon fusion. Therefore, an important ingredient for the Higgs-boson search will be the effective coupling of the Higgs boson to gluons, usually called C_1. In ref. [8] a low-energy theorem, valid to all orders, was established, which relates the effective Higgs-boson-gluon coupling, induced by the virtual presence of a heavy quark, to the logarithmic derivative of ζ_g with respect to the heavy quark mass. In that paper the method was used to squeeze from the three-loop decoupling function ζ_g the analytical result for C_1 in three- and even in four-loop approximation (the latter in a indirect way through a sophisticated use of the RG evolution equations and the four-loop QCD β-function). With our new full four-loop result for ζ_g we are able to confirm the result of ref. [8] in a completely independent way, without using the four-loop contribution to the QCD β-function. In addition a five-loop prediction for C_1 can be obtained (modulo yet unknown contribution from the five-loop n_f-dependent term in the β-function).

The outline of the paper is as follows. In Section 2 we recall the main formulae from [8] which reduce the evaluation of the decoupling function ζ_g to the calculation of vacuum integrals. Then we discuss shortly the technique used to compute these integrals. Section 3 describes our four-loop results for the decoupling function. In Section 4 we make use of the low energy theorem to derive the effective coupling of the Higgs boson to gluons (due to the virtual presence of a heavy quark) through four and (partially) through five loops. Section 5 deals with phenomenological applications of our result which lead to a reduction of the theoretical error due to evolution of the coupling constant α_s through heavy quark thresholds. Summary and conclusions are presented in the final Section 6.

2 Formalism

2.1 Decoupling for the Gauge Coupling Constant

Let us consider QCD with $n_l = n_f - 1$ massless quarks $\psi = \{\psi_l | l = 1\text{-}n_l\}$ and one heavy quark h with mass m. The corresponding bare QCD Lagrangian reads

$$\mathcal{L}(g_0, m_0, \xi_0; \psi_0, G^{0,a}_\mu, c^{0,a}, h_0) = -\frac{1}{4}(G^{0,a}_{\mu\nu})^2 + i\overline{\psi}_0 \not{D}\psi_0 + \overline{h}_0(i\not{D}_0 - m_0)h_0$$
$$+ \text{ terms with ghost fields and the gauge-fixing term,} \tag{2}$$

where the index '0' marks bare quantities, $G^{0,a}_\mu$ and $c^{0,a}$ are the gluon and ghost field respectively. The relations between the bare and renormalized quantities read

$$\begin{aligned} g_s^0 &= \mu^\varepsilon Z_g g_s, \qquad m^0 = Z_m m, \qquad \xi^0 - 1 = Z_3(\xi - 1), \\ \psi_q^0 &= \sqrt{Z_2}\psi_q, \qquad G^{0,a}_\mu = \sqrt{Z_3}G^a_\mu, \qquad c^{0,a} = \sqrt{\tilde{Z}_3}c^a, \end{aligned} \tag{3}$$

where $g_s = \sqrt{4\pi\alpha_s}$ is the (renormalized) QCD gauge coupling, μ is the renormalization scale, $d = 4 - 2\varepsilon$ is the dimensionality of space-time. The gauge parameter ξ is defined through the gluon propagator in lowest order,

$$\frac{i}{q^2 + i\epsilon}\left(-g^{\mu\nu} + \xi\frac{q^\mu q^\nu}{q^2}\right). \tag{4}$$

All renormalization constants appearing in (3) are known by now through order $\mathcal{O}(\alpha_s^4)$ from [9–13].

Integrating out the heavy quark transforms the full QCD Lagrangian (2) into the one corresponding to the effective massless QCD with $n_f' =$

$n_f - 1 = n_l$ quark flavors (plus additional higher dimension interaction terms suppressed by powers of the heavy mass and neglected in what follows). Denoting the effective fields and parameters by an extra prime, we write the effective Lagrangian as follows:

$$\mathcal{L}' = \mathcal{L}\left(g_s^{0\prime}, \xi^{0\prime}; \psi_q^{0\prime}, G_\mu^{0\prime,a}, c^{0\prime,a}\right). \tag{5}$$

Here the primed quantities are related to non-primed ones through

$$\begin{aligned} g_s^{0\prime} &= \zeta_g^0 g_s^0, \qquad \xi^{0\prime} - 1 = \zeta_3^0(\xi^0 - 1), \qquad \psi_q^{0\prime} = \sqrt{\zeta_2^0}\psi_q^0, \\ G_\mu^{0\prime,a} &= \sqrt{\zeta_3^0} G_\mu^{0,a}, \qquad c^{0\prime,a} = \sqrt{\tilde{\zeta}_3^0} c^{0,a}, \end{aligned} \tag{6}$$

where the primes mark the quantities of the effective n_l-flavor theory.

As was first demonstrated in [8] the bare decoupling constants can all be expressed in terms of massive Feynman integrals without any external momenta (so-called massive tadpoles). The corresponding relation for ζ_g^0 reads:

$$\zeta_g^0 = \frac{\tilde{\zeta}_1^0}{\tilde{\zeta}_3^0\sqrt{\zeta_3^0}}, \tag{7}$$

where

$$\tilde{\zeta}_1^0 = 1 + \Gamma_{G\bar{c}c}^{0h}(0,0), \tag{8}$$

$$\zeta_3^0 = 1 + \Pi_G^{0h}(0), \tag{9}$$

$$\tilde{\zeta}_3^0 = 1 + \Pi_c^{0h}(0). \tag{10}$$

Here $\Pi_G(p^2)$ and $\Pi_c(p^2)$ are the gluon and ghost vacuum polarization functions, respectively. $\Pi_G(p^2)$ and $\Pi_c(p^2)$ are related to the gluon and ghost propagators through

$$\delta^{ab}\left\{\frac{g^{\mu\nu}}{p^2\left[1+\Pi_G^0(p^2)\right]} + \text{terms proportional to } p^\mu p^\nu\right\} \tag{11}$$

$$= i\int dx\, e^{ip\cdot x}\left\langle T G^{0,a\mu}(x) G^{0,b\nu}(0)\right\rangle,$$

$$-\frac{\delta^{ab}}{p^2\left[1+\Pi_c^0(p^2)\right]} = i\int dx\, e^{ip\cdot x}\left\langle T c^{0,a}(x)\bar{c}^{0,b}(0)\right\rangle, \tag{12}$$

respectively. The vertex function $\Gamma^0_{G\bar{c}c}(p,k)$ is defined through the one-particle-irreducible (1PI) part of the amputated $G\bar{c}c$ Green function as

$$
\begin{aligned}
&p^\mu g_s^0 \left\{-i f^{abc}\left[1+\Gamma^0_{G\bar{c}c}(p,k)\right] + \text{other color structures}\right\} \\
&= i^2 \int dx dy\, e^{i(p\cdot x + k\cdot y)} \left\langle T c^{0,a}(x) \bar{c}^{0,b}(0) G^{0,c\mu}(y)\right\rangle^{1\mathrm{PI}}, \qquad (13)
\end{aligned}
$$

where p and k are the outgoing four-momenta of c and G, respectively, and f^{abc} are the structure constants of the QCD gauge group.

Finally, in order to renormalize the bare decoupling constant ζ_g^0 we combine eq. (7) with eqs. (3) and arrive at ($\alpha_s = g_s^2/(4\pi), \alpha_s' = (g_s')^2/(4\pi)$)

$$
\alpha_s'(\mu) = \left(\frac{Z_g}{Z_g'}\zeta_g^0\right)^2 \alpha_s(\mu) = \zeta_g^2 \alpha_s(\mu). \qquad (14)
$$

Direct application of this equation is rather clumsy as the constant Z_g' on its right side depends on the *renormalized* effective coupling constant which we are looking for. Of course, within perturbation theory, one can always solve eq. (14) by iteration. A simpler way, which we have used, reduces to inverting first the series

$$
\alpha_s^0 = Z_\alpha(\alpha_s,\epsilon)\,\alpha_s, \quad Z_\alpha(\alpha_s,\epsilon) = Z_g^2 = 1 + \sum_{i\geq 1} Z_{\alpha,i}(\epsilon)\,\alpha_s^i \qquad (15)
$$

to express the *renormalized* coupling constant α_s in terms of the *bare one*:

$$
\alpha_s = Z_\alpha^0(\alpha_s^0,\epsilon)\,\alpha_s^0, \quad Z_\alpha^0(\alpha_s,\epsilon) = 1 + \sum_{i\geq 1} Z^0_{\alpha,i}(\epsilon)\,\left(\alpha_s^0\right)^i, \qquad (16)
$$

where, up to four loops,

$$
\begin{aligned}
Z^0_{\alpha,1} &= -Z_{\alpha,1}, \quad Z^0_{\alpha,2} = -Z_{\alpha,2} + 2\,Z^2_{\alpha,1}, \quad Z^0_{\alpha,3} = -Z_{\alpha,3} + 5\,Z_{\alpha,1}\,Z_{\alpha,2} - 5\,Z^3_{\alpha,1} \\
Z^0_{\alpha,4} &= -Z_{\alpha,4} + 6\,Z_{\alpha,1}\,Z_{\alpha,3} + 3\,Z^2_{\alpha,2} - 21\,Z^2_{\alpha,1}\,Z_{\alpha,2} + 14\,Z^4_{\alpha,1}\,. \qquad (17)
\end{aligned}
$$

With the use of eq. (17) one now could conveniently transform all the primed (that is effective quantities) in eq. (14) into the non-primed ones. After this the renormalization can be done directly.

2.2 Vacuum Integrals

At the end of the day we have to evaluate a host of four-loop massive tadpoles entering the definitions of the bare decoupling constants. The evaluation of these massive tadpoles in three-loop approximation has been pioneered in ref. [14] and automated in ref. [15].
Similar to the three-loop case, the analytical evaluation of four-loop tadpole integrals is based on the traditional Integration-By-Parts (IBP) method. In contrast to the three-loop case the manual construction of algorithms to reduce arbitrary diagrams to a small set of master integrals is replaced by Laporta's algorithm [16, 17]. In this context the IBP identities are generated with numerical values for the powers of the propagators and the irreducible scalar products. In the next step, the resulting system of linear equations is solved by expressing systematically complicated integrals in terms of simpler ones. The resulting solutions are then substituted into all the other equations.
This reduction has been implemented in an automated `FORM3` [18, 19] based program in which partially ideas described in ref. [17, 20, 21] have been implemented. The rational functions in the space-time dimension d, which arise in this procedure, are simplified with the program `FERMAT` [22]. The automated exploitation of *all* symmetries of the diagrams by reshuffling the powers of the propagators of a given topology in a unique way strongly reduces the number of equations which need to be solved.

In general, the tadpole diagrams contributing to the decoupling constants contain both massive and massless lines. In contrast, the computation of the four-loop β-function can be reduced to the evaluation of four-loop tadpoles composed of completely massive propagators. These special cases have been considered in [9, 13, 21].

All four-loop tadpole diagrams encountered during our calculations were expressed through the set of 13 master integrals shown in figures 1 and 2. The first four integrals of Fig. 1 have analytical solution in terms of Gamma functions for generic valuse of ϵ. The first nine terms of the ϵ-expansion of the fifth integral (T_{52}) can be obtained from results of Refs. [14, 23] (for more details see Ref. [28]). As for the less simple master integrals pictured in Fig. 2 one finds that for the case of ζ_g all necessary ingredients are known analytically ([24–29]) except for $T_{54,3}$, $T_{62,2}$ and $T_{91,0}$, where we have denoted

$$T_n \cdot \left(\epsilon \cdot \int \frac{\mathrm{d}^{4-2\epsilon}p}{\pi^{2-\epsilon}} \frac{1}{(p^2+1)^2}\right)^{(-4)} \underset{\epsilon\to 0}{==} \sum_{i\geq -4} \epsilon^i \, T_{n,i} \,. \qquad (18)$$

These three integrals are known numerically from [28, 29] and read

$$\begin{aligned} T_{54,3} &= -8445.8046390310298, && (19)\\ T_{62,2} &= -4553.4004372195263, && (20)\\ T_{91,0} &= 1.808879546208. && (21) \end{aligned}$$

In fact, in refs. [29, 30] an analytical result for $T_{91,0}$ (in terms of $T_{62,2}$ and $T_{54,3}$) has been also derived. Thus, the results of the next section will contain only *two* numerical constants, viz. $T_{62,2}$ and $T_{54,3}$.

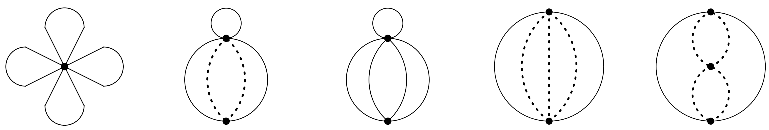

Figure 1: Analytically known master integrals

3 Decoupling at four loops: results

We have generated the relevant diagrams with the help of the program QGRAF [31]. The total number of diagrams at four loops amount to 6070, 765 and 9907 for ζ_3^0, $\tilde{\zeta}_3^0$ and ζ_1^0 respectively. All calculations were performed in a general covariant gauge with the gluon propagator as given in eq. (4). However, since extra squared propagators lead to significant calculational complications, only linear terms in ξ were kept. The bare results are rather lengthy and will be made available (in computer-readable form) in http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp05/ttp05-27.

After constructing the bare decoupling constant from eq. (7) followed by the renormalization as described in section 2 we arrive at the following gauge

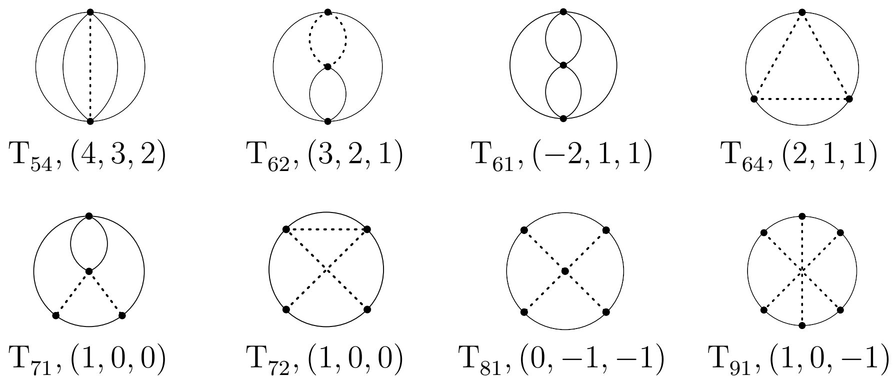

Figure 2: Master integrals where only a few terms of their ϵ-expansion are known analytically. The solid (dashed) lines denote massive (massless) propagators. The three numbers in brackets (n_1, n_2, n_3) are decoded as follows: n_1 is the maximal power of the spurious pole in ϵ which might appear in front of the integral, n_2 is the maximal power of the spurious pole in ϵ which happens to enter into the decomposition of the bare decoupling constant ζ_g^0 in terms of the master integrals, n_3 is the maximal analytically known power of the ϵ-expansion of the same integral as determined in [28] and confirmed in [29].

independent[3] result for the decoupling function ζ_g^{MS}:

$$(\zeta_g^{\mathrm{MS}})^2 = 1 + \sum_{i\geq 1} a_s^i(\mu)\, d_{\mathrm{MS,i}}, \tag{22}$$

where we use the notation

$$a_s(\mu) = \frac{\alpha_s^{(n_f)}(\mu)}{\pi}$$

and the coefficients $d_{\mathrm{MS,i}}$ read

$$d_{\mathrm{MS,1}} = -\frac{1}{6}\,\ell_{\mu m}\,, \tag{23}$$

[3] In fact, the gauge dependence on ξ disappears already for the bare decoupling constant as it should be.

$$d_{\rm MS,2} = \frac{11}{72} - \frac{11}{24}\,\ell_{\mu m} + \frac{1}{36}\,\ell_{\mu m}^2, \tag{24}$$

$$\begin{aligned} d_{\rm MS,3} &= \frac{564731}{124416} - \frac{82043}{27648}\,\zeta_3 - \frac{955}{576}\,\ell_{\mu m} + \frac{53}{576}\,\ell_{\mu m}^2 - \frac{1}{216}\,\ell_{\mu m}^3 \\ &+ n_l \left[-\frac{2633}{31104} + \frac{67}{576}\,\ell_{\mu m} - \frac{1}{36}\,\ell_{\mu m}^2 \right], \end{aligned} \tag{25}$$

$$\begin{aligned} d_{\rm MS,4} &= \Delta_{\rm MS,4} + \frac{7391699}{746496}\,\ell_{\mu m} - \frac{2529743}{165888}\,\zeta_3\,\ell_{\mu m} + \frac{2177}{3456}\,\ell_{\mu m}^2 - \frac{1883}{10368}\,\ell_{\mu m}^3 + \frac{1}{1296}\,\ell_{\mu m}^4 \\ &+ n_l \left[-\frac{110341}{373248}\,\ell_{\mu m} + \frac{110779}{82944}\,\zeta_3\,\ell_{\mu m} - \frac{1483}{10368}\,\ell_{\mu m}^2 - \frac{127}{5184}\,\ell_{\mu m}^3 \right] \\ &+ n_l^2 \left[\frac{6865}{186624}\,\ell_{\mu m} - \frac{77}{20736}\,\ell_{\mu m}^2 + \frac{1}{324}\,\ell_{\mu m}^3 \right], \end{aligned} \tag{26}$$

$$\begin{aligned} \Delta_{\rm MS,4} &= \\ &\left[\frac{134805853579559}{43342154956800} - \frac{254709337}{783820800}\,\pi^4 - \frac{151369}{30481920}\,\pi^6 - \frac{18233772727}{783820800}\,\zeta_3 + \frac{151369}{544320}\,\zeta_3^2 \right. \\ &\quad + \frac{4330717}{207360}\,\zeta_5 + \frac{9869857}{272160}\,a_4 - \frac{121}{36}\,a_5 - \frac{2057}{51840}\,\pi^4 \ln 2 - \frac{9869857}{6531840}\,\pi^2 \ln^2 2 \\ &\quad \left. - \frac{121}{2592}\,\pi^2 \ln^3 2 + \frac{9869857}{6531840} \ln^4 2 + \frac{121}{4320} \ln^5 2 + \frac{82037}{30965760}\,T_{54,3} - \frac{151369}{11612160}\,T_{62,2} \right] \\ &+ n_l \left[-\frac{4770941}{2239488} - \frac{541549}{14929920}\,\pi^4 + \frac{3645913}{995328}\,\zeta_3 + \frac{115}{576}\,\zeta_5 + \frac{685}{5184}\,a_4 \right. \\ &\quad \left. - \frac{685}{124416}\,\pi^2 \ln^2 2 + \frac{685}{124416} \ln^4 2 \right] \\ &+ n_l^2 \left[-\frac{271883}{4478976} + \frac{167}{5184}\,\zeta_3 \right]. \end{aligned} \tag{27}$$

In eqs. (23-27) $\ell_{\mu m} = \ln \frac{\mu^2}{m^2(\mu)}$, $m(\mu)$ is the (running) heavy quark mass in the $\overline{\rm MS}$-scheme and μ represents the renormalization scale. Furthermore, $\zeta_n = \zeta(n)$ is Riemann's zeta function and $a_n = {\rm Li}_n(1/2) = \sum_{i=1}^{\infty} 1/(2^i i^n)$.

For another convenient definition of the quark mass — the so-called scale-invariant mass, defined through the relation $\mu_h = m(\mu_h)$ — eqs. (22-27) are transformed to ($\ell_{\mu h} = \ln \frac{\mu^2}{\mu_h^2}$):

$$(\zeta_g^{\rm SI})^2 = 1 + \sum_{i \geq 1} a_s^i(\mu)\, d_{\rm SI,i}, \tag{28}$$

where

$$d_{\rm SI,1} = -\,\frac{1}{6}\,\ell_{\mu h}\,, \qquad (29)$$

$$d_{\rm SI,2} = \frac{11}{72} - \frac{19}{24}\,\ell_{\mu h} + \frac{1}{36}\,\ell_{\mu h}^2, \qquad (30)$$

$$\begin{aligned} d_{\rm SI,3} &= \frac{564731}{124416} - \frac{82043}{27648}\,\zeta_3 - \frac{6793}{1728}\,\ell_{\mu h} - \frac{131}{576}\,\ell_{\mu h}^2 - \frac{1}{216}\,\ell_{\mu h}^3 \\ &+ n_l\left[-\frac{2633}{31104} + \frac{281}{1728}\,\ell_{\mu h}\right], \end{aligned} \qquad (31)$$

$$\begin{aligned} d_{\rm SI,4} &= \Delta_{\rm MS,4} - \frac{2398621}{746496}\,\ell_{\mu h} - \frac{2483663}{165888}\,\zeta_3\,\ell_{\mu h} - \frac{14023}{3456}\,\ell_{\mu h}^2 - \frac{8371}{10368}\,\ell_{\mu h}^3 + \frac{1}{1296}\,\ell_{\mu h}^4 \\ &+ n_l\left[\frac{190283}{373248}\,\ell_{\mu h} + \frac{133819}{82944}\,\zeta_3\,\ell_{\mu h} + \frac{983}{3456}\,\ell_{\mu h}^2 + \frac{107}{1728}\,\ell_{\mu h}^3\right] \\ &+ n_l^2\left[\frac{8545}{186624}\,\ell_{\mu h} - \frac{79}{6912}\,\ell_{\mu h}^2\right]. \end{aligned} \qquad (32)$$

For practical applications also the inverted formulae are needed:

$$\frac{1}{(\zeta_g^{\rm MS})^2} = 1 + \sum_{i\ge1}\left(a'_s(\mu)\right)^i\, d'_{\rm MS,i}, \qquad (33)$$

and

$$\frac{1}{(\zeta_g^{\rm SI})^2} = 1 + \sum_{i\ge1}\left(a'_s(\mu)\right)^i\, d'_{\rm SI,i}, \qquad (34)$$

where we use the notation

$$a'_s(\mu) = \frac{\alpha_s^{(n_l)}(\mu)}{\pi}$$

and the coefficients $d'_{\rm MS,i}$, $d'_{\rm SI,i}$ read

$$d'_{\rm MS,1} = \frac{1}{6}\,\ell_{\mu m}\,, \qquad (35)$$

$$d'_{\rm MS,2} = -\frac{11}{72} + \frac{11}{24}\,\ell_{\mu m} + \frac{1}{36}\,\ell_{\mu m}^2, \qquad (36)$$

$$d'_{\mathrm{MS},3} = -\frac{564731}{124416} + \frac{82043}{27648}\zeta_3 + \frac{2645}{1728}\ell_{\mu m} + \frac{167}{576}\ell_{\mu m}^2 + \frac{1}{216}\ell_{\mu m}^3 + n_l\left[\frac{2633}{31104} - \frac{67}{576}\ell_{\mu m} + \frac{1}{36}\ell_{\mu m}^2\right], \quad (37)$$

$$d'_{\mathrm{MS},4} = \frac{121}{1728} - \Delta_{\mathrm{MS},4} - \frac{11093717}{746496}\ell_{\mu m} + \frac{3022001}{165888}\zeta_3\,\ell_{\mu m} + \frac{1837}{1152}\ell_{\mu m}^2 + \frac{2909}{10368}\ell_{\mu m}^3 + \frac{1}{1296}\ell_{\mu m}^4 + n_l\left[\frac{141937}{373248}\ell_{\mu m} - \frac{110779}{82944}\zeta_3\,\ell_{\mu m} + \frac{277}{10368}\ell_{\mu m}^2 + \frac{271}{5184}\ell_{\mu m}^3\right] + n_l^2\left[-\frac{6865}{186624}\ell_{\mu m} + \frac{77}{20736}\ell_{\mu m}^2 - \frac{1}{324}\ell_{\mu m}^3\right], \quad (38)$$

$$d'_{\mathrm{SI},1} = \frac{1}{6}\ell_{\mu h}\,, \quad (39)$$

$$d'_{\mathrm{SI},2} = -\frac{11}{72} + \frac{19}{24}\ell_{\mu h} + \frac{1}{36}\ell_{\mu h}^2, \quad (40)$$

$$d'_{\mathrm{SI},3} = -\frac{564731}{124416} + \frac{82043}{27648}\zeta_3 + \frac{2191}{576}\ell_{\mu h} + \frac{511}{576}\ell_{\mu h}^2 + \frac{1}{216}\ell_{\mu h}^3 + n_l\left[\frac{2633}{31104} - \frac{281}{1728}\ell_{\mu h}\right], \quad (41)$$

$$d'_{\mathrm{SI},4} = \frac{121}{1728} - \Delta_{\mathrm{MS},4} - \frac{1531493}{746496}\ell_{\mu h} + \frac{2975921}{165888}\zeta_3\,\ell_{\mu h} + \frac{33887}{3456}\ell_{\mu h}^2 + \frac{14149}{10368}\ell_{\mu h}^3 + \frac{1}{1296}\ell_{\mu h}^4 + n_l\left[-\frac{158687}{373248}\ell_{\mu h} - \frac{133819}{82944}\zeta_3\,\ell_{\mu h} - \frac{515}{1152}\ell_{\mu h}^2 - \frac{107}{1728}\ell_{\mu h}^3\right] + n_l^2\left[-\frac{8545}{186624}\ell_{\mu h} + \frac{79}{6912}\ell_{\mu h}^2\right]. \quad (42)$$

Numerically, eqs. (22) and (28) read

$$(\zeta_g^{\mathrm{MS}})^2 \underset{\mu=\mu_h}{=\!=} 1 + 0.152778\,a_s^2(\mu_h) + a_s^3(\mu_h)\,(0.972057 - 0.0846515\,n_l) + a_s^4(\mu_h)\left(5.17035 - 1.00993\,n_l - 0.0219784\,{n_l}^2\right), \quad (43)$$

$$\frac{1}{(\zeta_g^{\rm MS})^2} \;\overline{\overline{\mu=\mu_h}}\; 1 - 0.152778\, a_s'^2(\mu_h) + a_s'^3(\mu_h)\,(-0.972057 + 0.0846515\; n_l)$$
$$+\; a_s'^4(\mu_h)\left(-5.10032 + 1.00993\; n_l + 0.0219784\; {n_l}^2\right). \quad (44)$$

Using the three loop relation between the pole and $\overline{\rm MS}$ masses [32–34] one could easily express the decoupling relations in terms of the pole mass of the heavy quark. As the resulting expressions are rather lengthy they are relegated to the Appendix.

4 Coupling of the Higgs boson to gluons

Within the Standard Model (SM) the scalar Higgs boson is responsible for the mechanism of the mass generation. Its future (non-)discovery will be of primary importance for all the particle physics. The SM Higgs boson mass is constrained from below, $M_h > 114$GeV, by experiments at LEP and SLC [35, 36]. Indirect constraints coming from precision electroweak measurements [37] lead to an upper limit of about 200 GeV.

With the SM Higgs boson mass within this range, its coupling to a pair of gluons is mediated by virtual quarks [38] and plays a crucial rôle in Higgs phenomenology. Indeed, with Yukawa couplings of the Higgs boson to quarks being proportional to the respective quark masses, the ggH coupling of the SM is essentially generated by the top quark only. The ggH coupling strength becomes independent of the top quark mass M_t in the limit $M_H \ll 2M_t$.

In general, the theoretical description of such interactions is very complicated because there are two different mass scales involved, M_H and M_t. However, in the limit $M_H \ll 2M_t$, the situation may be greatly simplified by integrating out the top quark, i.e. by constructing a heavy-top-quark effective Lagrangian [39, 40].

This Lagrangian is a linear combination of certain dimension-four operators acting in QCD with five quark flavors, while all M_t dependence is contained in the coefficient functions. The final renormalized version of $\mathcal{L}_{\rm eff}$ is (G_F is Fermi's constant)

$$\mathcal{L}_{\rm eff} = -2^{1/4} G_F^{1/2} H C_1\,[O_1']\,. \quad (45)$$

Here, $[O_1']$ is the renormalized counterpart of the bare operator $O_1' = G^{0\prime}_{a\mu\nu} G^{0\prime\mu\nu}_{a}$, where $G_{a\mu\nu}$ is the color field strength, the superscript 0 denotes bare fields, and primed objects refer to the five-flavor effective theory. C_1 is the corresponding renormalized coefficient function, which carries all M_t dependence.

In ref. [8] a low-energy theorem was established, valid to all orders, which relates the effective coupling of the Higgs boson to gluons, induced by a presence of heavy quark h, to the logarithmic derivative of ζ_g w.r.t. m. The theorem states that

$$C_1 \; = \; -\frac{1}{2}\, m^2 \, \frac{\partial}{\partial\, m^2} \ln \zeta_g^2. \tag{46}$$

Another, equivalent, but more convenient form of (46) was also derived in [8] by exploiting evolution equations in full and effective theories. It reads

$$C_1 = \frac{\pi}{2\,[1 - 2\gamma_m(a_s)]} \left[\hat{\beta}'(a_s') - \hat{\beta}(a_s) - \hat{\beta}(a_s)\, a_s \frac{\partial}{\partial a_s} \ln \zeta_g^2 \right], \tag{47}$$

where γ_m is the quark mass anomalous dimension, $\zeta_g^2 = \zeta_g^2(\mu, a_s, m)$ and the β-function is defined as follows:

$$\mu^2 \frac{d}{d\,\mu^2}\, a_s^{(n_f)} \; = \; \beta^{(n_f)} \left(a_s^{(n_f)} \right) = - \sum_{N=1}^{\infty} \beta_{N-1}^{(n_f)} a_s^{N+1}, \tag{48}$$

$$a_s = a_s^{(n_f)}, \quad a_s' = a_s^{(n_l)}, \quad a_s\, \hat{\beta}(a_s) = \beta^{(n_f)}(a_s), \quad a_s'\, \hat{\beta}'(a_s') = \beta^{(n_l)}(a_s').$$

An important feature of the low energy theorem (47) is the fact that the decoupling function ζ_g appears there multiplied by at least one power of a_s. It means that in order to compute C_1 , say, at four loops one should know the QCD β-function to *four loops* (only n_f dependent part) but the quark anomalous dimension γ_m and the decoupling function ζ_g only to *three loops*. It is this fact which allowed to find C_1 at four loops in [8] long before the *four-loop* result for the decoupling function would be available.

Now, armed with the newly computed four-loop term in ζ_g, we could easily check that the old result for C_1 of [8] by a direct use of eq. (46). We have done this simple exercise and found full agreement. In fact, one could even use eq. (47) in order to construct C_1 at five loops in terms of the known four-loop QCD decoupling function, the quark anomalous dimension γ_m and the β-function and the only yet unknown parameter: the n_f-dependent piece of the five-loop contribution to the β-function. The result reads

$$
\begin{aligned}
C_1 = & -\frac{1}{12}\,a_s(\mu_h)\Biggl\{1+\frac{11}{4}\,a_s(\mu_h)+a_s^2(\mu_h)\left[\frac{2821}{288}+n_l\left(-\frac{67}{96}\right)\right]\\
+ & \;a_s^3(\mu_h)\left[-\frac{4004351}{62208}+\frac{1305893}{13824}\zeta(3)+n_l\left(\frac{115607}{62208}-\frac{110779}{13824}\zeta(3)\right)+n_l^2\left(-\frac{6865}{31104}\right)\right]\\
+ & \;a_s^4(\mu_h)\Biggl[-\frac{13546105}{41472}-\frac{91}{2}\Delta_{\mathrm{MS},4}-\frac{31}{432}\,\pi^4-\frac{853091}{6912}\,\zeta_3+\frac{475}{9}\,\zeta_5\\
+ & \;n_l\left(\frac{28598581}{497664}+3\Delta_{\mathrm{MS},4}-\frac{29}{432}\,\pi^4+\frac{3843215}{110592}\,\zeta_3-\frac{575}{36}\,\zeta_5\right)\\
+ & \;n_l^2\left(-\frac{3503}{62208}+\frac{1}{216}\,\pi^4-\frac{3}{4}\,\zeta_3\right)+n_l^3\left(\frac{83}{7776}-\frac{1}{54}\,\zeta_3\right)+6\,\Delta\beta_4\Biggr]\Biggr\}\\
\approx & \;-\frac{1}{12}\,a_s(\mu_h)\Bigl[1+2.7500\,a_s(\mu_h)+(9.7951-0.6979\,n_l)\,a_s^2(\mu_h)\\
& +\left(49.1827-7.7743\,n_l-0.2207\,n_l^2\right)a_s^3(\mu_h)\\
& +\left(-662.507+137.601\,n_l-2.53666\,n_l^2-0.077522\,n_l^3+6\,\Delta\beta_4\right)a_s^4(\mu_h)\Bigr],
\end{aligned}
\tag{49}
$$

where

$$
\Delta\beta_4=\beta_4^{(n_f-1)}-\beta_4^{(n_f)}.
$$

To save space we have written eq. (49) for the value of $\mu=\mu_h$. Unfortunately, simple estimations show that the β-function dependent contribution to eq. (49) could be numerically important.

5 Application: $\alpha_s(M_Z)$ from $\alpha_s(M_\tau)$

A central feature of QCD is the possibility to describe measurements performed at very different energy scales with the help of only one coupling constant. The customary choice is the ($\overline{\mathrm{MS}}$) coupling constant $\alpha_s(M_Z)$ which can be measured precisely in Z-boson decays. (For a review see, e.g. [41].)

On the other hand the dependence of the τ-decay rate on the strong coupling α_s has been used for a determination of α_s at lower energies, with the most recent results of $0.340 \pm 0.005_{\rm exp} \pm 0.014_{\rm th}$ and $0.348 \pm 0.009_{\rm exp} \pm 0.019_{\rm th}$ by the ALEPH [42] and OPAL [43] collaborations.

After evolution up to higher energies (taking into account the threshold effects due to the c- and b-quarks) these results agree remarkably well with determinations based on the hadronic Z decay rate, which provides a genuine test of asymptotic freedom of QCD.

Very recently an analysis of the evolution has been performed in [7]. They start from an updated determination of $\alpha_s(m_\tau)$

$$\alpha_s(m_\tau) = 0.345 \pm 0.004_{\rm exp} \pm 0.008_{\rm th} \tag{50}$$

based on most recent experimental results of the ALEPH collaboration [42]. Their result for the evolution of $\alpha_s(m_\tau)$ given in eq. (50), based on the use of the four-loop running and and three-loop quark-flavor matching reads

$$\begin{aligned} a_s(M_Z) &= 0.1215 \pm 0.0004_{\rm exp} \pm 0.0010_{\rm th} \pm 0.0005_{\rm evol} \ , \\ &= 0.1215 \pm 0.0012 \ , \end{aligned} \tag{51}$$

where the first two errors originate from the $\alpha_s(m_\tau)$ determination given in Eq. (50), and the last error stands for the ambiguities in the evolution due to uncertainties in the matching scales of the quark thresholds. That evolution error received contributions from the uncertainties in the c-quark mass (0.00020, $\mu_c = 1.31 \pm 0.1$ GeV) and the b-quark mass (0.00005, $\mu_b = 4.13 \pm 0.1$ GeV), the matching scale (0.00023, μ varied between $0.7\,\mu_q$ and $3.0\,\mu_q$), the three-loop truncation in the matching expansion (0.00026) and the four-loop truncation in the RGE equation (0.00031). (For the last two errors the size of the shift due the highest known perturbative term was treated as systematic uncertainty.) The errors had been added in quadrature.

We have repeated this analysis including the newly computed four-loop approximation for the matching.[4] As a result the value of $a_s(M_Z)$ has been marginally increased (by 0.0001) and both errors from the choice of the matching scale and from the four-loop truncation in the matching equation have been halved. The updated version of (51) now reads

$$\begin{aligned} a_s(M_Z) &= 0.1216 \pm 0.0004_{\rm exp} \pm 0.0010_{\rm th} \pm 0.0004_{\rm evol} \ , \\ &= 0.1216 \pm 0.0012 \ . \end{aligned} \tag{52}$$

[4] We have used the package RunDec [44].

6 Summary and Conclusions

We have computed the decoupling relation for the QCD quark gluon coupling constant in four-loop approximation. The new contribution leads to a decrease of the matching related uncertainties in the process of the evolution of the $\alpha_s(m_\tau)$ to $\alpha_s(M_Z)$ by a factor of two.

As a by-product we have directly confirmed the long available result for C_1, the effective coupling of the Higgs boson to gluons at four loops, and (partially) extended it to five loops. It remains only to find the QCD β-function at *five* loops to get the full result for C_1. In the light of recent significant progress in the multiloop technology [45, 46] the completely analytical evaluation of the former seems to be possible in not too far distant future.

We would like to mention that the result for the decoupling function at four loops as well as for the Higgs boson to gluons effective coupling at five loops have been obtained by a completely independent calculation in [30]. The results are in full mutual agreement.

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A ζ_g for the heavy quark mass in the on-shell scheme

The relevant formulas for the case of the heavy quark mass renormalized in the on-shell scheme (denoted as M) are:

$$(\zeta_g^{\rm OS})^2 = 1 + \sum_{i\geq 1} a_s^i(\mu)\, d_{\rm OS,i} \tag{53}$$

and

$$\frac{1}{(\zeta_g^{\rm OS})^2} = 1 + \sum_{i\geq 1} a_s'^i(\mu)\, d'_{\rm OS,i}, \tag{54}$$

where $\ell_{\mu M} = \ln \frac{\mu^2}{M^2}$ and

$$d_{\rm OS,1} = -\frac{1}{6}\,\ell_{\mu M}\,, \tag{55}$$

$$d_{\rm OS,2} = -\frac{7}{24} - \frac{19}{24}\,\ell_{\mu M} + \frac{1}{36}\,\ell_{\mu M}^2, \tag{56}$$

$$\begin{aligned} d_{\rm OS,3} &= -\frac{58933}{124416} - \frac{1}{9}\,\pi^2 - \frac{80507}{27648}\,\zeta_3 - \frac{1}{27}\,\pi^2 \ln 2 - \frac{8521}{1728}\,\ell_{\mu M} \\ &- \frac{131}{576}\,\ell_{\mu M}^2 - \frac{1}{216}\,\ell_{\mu M}^3 + n_l\left[\frac{2479}{31104} + \frac{1}{54}\,\pi^2 + \frac{409}{1728}\,\ell_{\mu M}\right], \end{aligned} \tag{57}$$

$$\begin{aligned} d_{\rm OS,4} &= \Delta_{\rm OS,4} - \frac{19696909}{746496}\,\ell_{\mu M} - \frac{29}{54}\,\pi^2\,\ell_{\mu M} - \frac{2439119}{165888}\,\zeta_3\,\ell_{\mu M} - \frac{29}{162}\,\pi^2 \ln 2\,\ell_{\mu M} \\ &- \frac{7693}{1152}\,\ell_{\mu M}^2 - \frac{8371}{10368}\,\ell_{\mu M}^3 + \frac{1}{1296}\,\ell_{\mu M}^4 \\ &+ n_l\left[\frac{1110443}{373248}\,\ell_{\mu M} + \frac{41}{324}\,\pi^2\,\ell_{\mu M} + \frac{132283}{82944}\,\zeta_3\,\ell_{\mu M} + \frac{1}{81}\,\pi^2 \ln 2\,\ell_{\mu M} + \frac{6661}{10368}\,\ell_{\mu M}^2 \right. \\ &\qquad \left. + \frac{107}{1728}\,\ell_{\mu M}^3\right] \\ &+ n_l^2\left[-\frac{1679}{186624}\,\ell_{\mu M} - \frac{1}{162}\,\pi^2\,\ell_{\mu M} - \frac{493}{20736}\,\ell_{\mu M}^2\right], \end{aligned} \tag{58}$$

$$
\begin{aligned}
\Delta_{\mathrm{OS},4} = & \\
- & \frac{2180918653146841}{43342154956800} - \frac{697121}{116640}\,\pi^2 - \frac{231357337}{783820800}\,\pi^4 - \frac{151369}{30481920}\,\pi^6 - \frac{18646246327}{783820800}\,\zeta_3 \\
& + \frac{1439}{1296}\,\pi^2\,\zeta_3 + \frac{151369}{544320}\,\zeta_3^2 + \frac{3698717}{207360}\,\zeta_5 + \frac{10609057}{272160}\,a_4 - \frac{121}{36}\,a_5 \\
& + \frac{1027}{972}\,\pi^2 \ln 2 - \frac{2057}{51840}\,\pi^4 \ln 2 - \frac{9278497}{6531840}\,\pi^2 \ln^2 2 - \frac{121}{2592}\,\pi^2 \ln^3 2 + \frac{10609057}{6531840} \ln^4 2 \\
& + \frac{121}{4320} \ln^5 2 + \frac{82037}{30965760}\,T_{54,3} - \frac{151369}{11612160}\,T_{62,2} \\
+ & \; n_l \Big[\frac{1773073}{746496} + \frac{557}{972}\,\pi^2 - \frac{697709}{14929920}\,\pi^4 + \frac{4756441}{995328}\,\zeta_3 + \frac{115}{576}\,\zeta_5 \\
& \quad + \frac{173}{5184}\,a_4 + \frac{11}{243}\,\pi^2 \ln 2 - \frac{1709}{124416}\,\pi^2 \ln^2 2 + \frac{173}{124416} \ln^4 2 \Big] \\
+ & \; n_l^2 \Big[-\frac{140825}{1492992} - \frac{13}{972}\,\pi^2 - \frac{19}{1728}\,\zeta_3 \Big] , \qquad (59)
\end{aligned}
$$

$$
d'_{\mathrm{OS},1} = \frac{1}{6}\,\ell_{\mu M} \,, \qquad (60)
$$

$$
d'_{\mathrm{OS},2} = \frac{7}{24} + \frac{19}{24}\,\ell_{\mu M} + \frac{1}{36}\,\ell_{\mu M}^2 , \qquad (61)
$$

$$
\begin{aligned}
d'_{\mathrm{OS},3} = \quad & \frac{58933}{124416} + \frac{1}{9}\,\pi^2 + \frac{80507}{27648}\,\zeta_3 + \frac{1}{27}\,\pi^2 \ln 2 + \frac{8941}{1728}\,\ell_{\mu M} \\
+ \quad & \frac{511}{576}\,\ell_{\mu M}^2 + \frac{1}{216}\,\ell_{\mu M}^3 \\
+ \quad & n_l \Big[-\frac{2479}{31104} - \frac{1}{54}\,\pi^2 - \frac{409}{1728}\,\ell_{\mu M} \Big] , \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
d'_{\mathrm{OS},4} = \quad & \frac{49}{192} - \Delta_{\mathrm{OS},4} + \frac{21084715}{746496}\,\ell_{\mu M} + \frac{35}{54}\,\pi^2\,\ell_{\mu M} + \frac{2922161}{165888}\,\zeta_3\,\ell_{\mu M} \\
& + \frac{35}{162}\,\pi^2 \ln 2\,\ell_{\mu M} + \frac{47039}{3456}\,\ell_{\mu M}^2 + \frac{14149}{10368}\,\ell_{\mu M}^3 + \frac{1}{1296}\,\ell_{\mu M}^4 \\
+ \quad & n_l \Big[-\frac{1140191}{373248}\,\ell_{\mu M} - \frac{47}{324}\,\pi^2\,\ell_{\mu M} - \frac{132283}{82944}\,\zeta_3\,\ell_{\mu M} - \frac{1}{81}\,\pi^2 \ln 2\,\ell_{\mu M} - \frac{9115}{10368}\,\ell_{\mu M}^2 \\
& \quad - \frac{107}{1728}\,\ell_{\mu M}^3 \Big] \\
+ \quad & n_l^2 \Big[\frac{1679}{186624}\,\ell_{\mu M} + \frac{1}{162}\,\pi^2\,\ell_{\mu M} + \frac{493}{20736}\,\ell_{\mu M}^2 \Big] , \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
(\zeta_g^{\mathrm{OS}})^2 \underset{\mu=M}{=\!=} 1 &- 0.291667\, a_s^2(M) + a_s^3(M)\ (-5.32389 + 0.262471\ n_l) \\
&+ a_s^4(M)\ \Big(-85.875 + 9.69229\ n_l - 0.239542\ {n_l}^2\Big)\,, \qquad (64)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(\zeta_g^{\mathrm{OS}})^2} \underset{\mu=M}{=\!=} 1 &+ 0.291667\, {a'_s}^2(M) + {a'_s}^3(M)\ (5.32389 - 0.262471\ n_l) \\
&+ {a'_s}^4(M)\ \Big(86.1302 - 9.69229\ n_l + 0.239542\ {n_l}^2\Big)\,. \qquad (65)
\end{aligned}
$$

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