SFB/CPP-05-78
TTP05-26

# Two-Loop $\mathcal{O}(\alpha\alpha_s)$ Corrections to the On-Shell Fermion Propagator in the Standard Model

D. Eiras, M. Steinhauser

*Institut für Theoretische Teilchenphysik, Universität Karlsruhe*

*76128 Karlsruhe, Germany*

**Abstract**

In this paper we consider mixed two-loop electroweak corrections to the top quark propagator in the Standard Model. In particular, we compute the on-shell renormalization constant for the mass and wave function, which constitute building blocks for many physical processes. The results are expressed in terms of master integrals. For the latter practical approximations are derived. In the case of the mass renormalization constant we find agreement with the results in the literature.

PACS numbers: 14.65.Ha 12.38.Bx

# 1 Introduction

The outstanding precision reached at the CERN LEP and SLAC SLC triggered many higher order calculations in the Standard Model (SM) of particle physics. In particular, it happened for the first time that the experimental results were sensitive to the weak part of the SM. Since at LEP and SLC real top quarks could not be produced, the main emphasis of the theoretical investigations was put on processes with light quarks as external particles. Heavy particles like the top quark only appeared virtually in intermediate states. In many applications the masses of the light quarks can be neglected as compared to the other mass scales, which results in a significant simplification of the resulting mathematical expressions.

In a future International Linear Collider (ILC) [1] the center-of-mass energy is high enough to produce top quarks. The expected experimental precision requires on the theoretical side the inclusion of higher order corrections — both for the QCD and the electroweak sector of the SM. This is particularly true for the threshold production of top quark pairs, where the theoretical uncertainties of the second order QCD corrections are still significant [2]. Thus, next to third-order QCD calculations (see, e.g., Refs. [3–6]), also electroweak corrections have to be investigated.

In this paper we focus on the two-loop mixed electroweak/QCD corrections to the on-shell top quark propagator. Some sample diagrams are shown in Fig. 1. The top quark mass renormalization constant has been evaluated in Ref. [7]. The contribution from the scalar bosons has been considered in [8, 9]. In this paper we confirm the results of Ref. [7]. Furthermore, we compute the wave function renormalization constant to order $\alpha\alpha_s$. Practical approximations are derived for the diagrams with internal $W$ and $Z$ bosons[1] and both the exact expression and handy approximations are computed for the Higgs boson mass dependence.

The inverse fermion propagator can be decomposed as

$$S^{-1}(q) = Z_2^L\,(\not{q} - Z_m m + \not{q}\,\Sigma_L(q) + m\Sigma_S)\,L + Z_2^R\,(\not{q} - Z_m m + \not{q}\,\Sigma_R(q) + m\Sigma_S)\,R\,, \quad (1)$$

with $R = (1+\gamma_5)/2$ and $L = (1-\gamma_5)/2$. $m$ is the quark mass and $Z_m$ and $Z_2^{L,R}$ are the mass and (left/right) wave function renormalization constants, respectively. The functions $\Sigma_L, \Sigma_R$ and $\Sigma_S$ in Eq. (1) result from a convenient decomposition of the self energy $\Sigma(q)$ given by

$$\Sigma(q) \;=\; \not{q}\,(R\,\Sigma_R(q^2) + L\,\Sigma_L(q^2)) + m\,\Sigma_S(q^2)\,. \quad (2)$$

In the on-shell scheme one requires that $S^{-1}(q)$ vanishes for $q^2 = m^2$ which leads to the following condition for the on-shell mass renormalization constant

$$Z_m^{\rm OS} \;=\; 1 + \left(\Sigma_S(m^2) + \frac{1}{2}\left(\Sigma_L(m^2) + \Sigma_R(m^2)\right)\right)\,. \quad (3)$$

[1]Here and in the following we consider the $W$ and $Z$ boson always in combination with the corresponding Goldstone bosons.

Requiring furthermore that the residuum is $-1$, provides a condition for $Z_2^{L,\mathrm{OS}}$ and $Z_2^{R,\mathrm{OS}}$

$$
\begin{aligned}
Z_2^{L,\mathrm{OS}} &= -\Sigma_L(m^2) - 2m^2\left[\Sigma_S'(m^2) + \frac{1}{2}\left(\Sigma_L'(m^2) + \Sigma_R'(m^2)\right)\right] \,,\\
Z_2^{R,\mathrm{OS}} &= -\Sigma_R(m^2) - 2m^2\left[\Sigma_S'(m^2) + \frac{1}{2}\left(\Sigma_L'(m^2) + \Sigma_R'(m^2)\right)\right] \,.
\end{aligned}
\tag{4}
$$

In these formulae it is understood that only the real part of the self energy functions is taken.

In the practical calculation it is convenient to apply projectors in order to arrive at scalar momentum integrals. This is achieved via

$$
\begin{aligned}
Z_m^{\mathrm{OS}} &= \mathrm{Tr}\left(P_m\Sigma(q)\right)\Big|_{q^2=m^2} \,,\\
Z_2^{L/R,\mathrm{OS}} &= \mathrm{Tr}\left(P_2^{L/R}\Sigma(q)\right)\Big|_{q^2=m^2} \,,
\end{aligned}
\tag{5}
$$

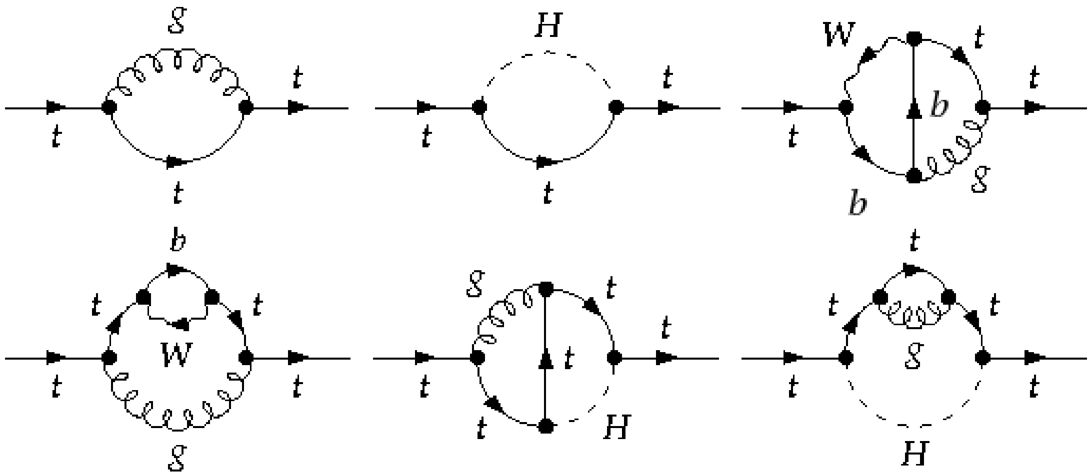



Figure 1: Sample diagrams contributing to the top quark propagator up to order $\alpha\alpha_s$. Next to the Higgs boson also the gauge bosons $W$ and $Z$ and the corresponding Goldstone bosons can be exchanged.

with

$$
\begin{aligned}
P_m &= \frac{\not{q}}{4q^2} + \frac{\mathbf{1}}{4m}\,, \\
P_2^L &= -\frac{\not{q}}{2q^2}L - 2m^2\frac{\partial}{\partial q^2}\left(\frac{\not{q}}{4q^2} + \frac{\mathbf{1}}{4m}\right)\,, \\
P_2^R &= -\frac{\not{q}}{2q^2}R - 2m^2\frac{\partial}{\partial q^2}\left(\frac{\not{q}}{4q^2} + \frac{\mathbf{1}}{4m}\right)\,.
\end{aligned}
\tag{6}
$$

In these formulae it is understood that the lower-order results are expressed in terms of the bare parameters. In what follows we compute $Z_m^{\mathrm{OS}}$ and $Z_2^{L/R,\mathrm{OS}}$ for the top quark neglecting the masses of the light quarks. Furthermore, we take the Cabibbo-Kobayashi-Maskawa matrix to be diagonal.

The outline of the paper is as follows: In the next Section we consider the classes of Feynman diagrams which are relevant for our calculation and discuss the reduction to a basic set of master integrals. The results for the master integrals appearing in our calculation are discussed in Appendix C. In Section 3 we present the renormalization constant for the top quark on-shell mass. In particular we consider the relation between the $\overline{\mathrm{MS}}$ and pole mass and compare our results with the literature. In Section 4 we move on to the wave function renormalization constants. Finally, our findings are summarized in Section 5, which also contains the conclusions.

# 2 Reduction to master integrals

At order $\alpha\alpha_s$ one has to consider about 35 Feynman diagrams contributing to the fermion propagator (cf. Fig. 1 for some sample diagrams). After the application of the projectors the external momentum is set on the mass shell of the heavy quark, which leads to integrals containing two scales, the quark mass and the boson mass. In this Section they are denoted by $m$ and $M$, respectively.

We generate all one-particle irreducible Feynman diagrams contributing to the fermion propagator with `QGRAF` [10]. The application of `q2e` and `exp` [11, 12] identifies the topology of the individual diagrams, adopts the notation and transforms the expressions into `FORM` [13] notation.

It is convenient to map the `QGRAF` output for each diagram with the help of `q2e` and

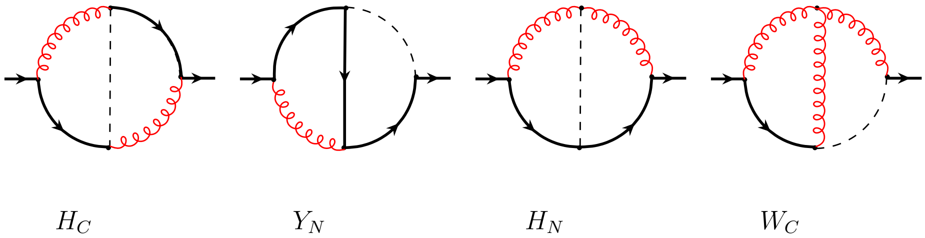



Figure 2: Graphical representation of the integral classes defined in Eq. (7). Solid and dashed lines carry mass $m$ and $M$, respectively. Curly lines are massless.

`exp` to one of the following four classes of integrals[2]

$$
\begin{aligned}
&H_N^+(n_1,n_2,n_3,n_4,n_5) = \\
&\quad \frac{e^{2\epsilon\gamma_E}}{(i\pi^{d/2})^2}\int \frac{\mathrm{d}^dk\mathrm{d}^dl}{(k^2+2kq)^{n_1}(l^2+2lq)^{n_2}(k^2)^{n_3}((k-l)^2-M^2)^{n_4}(l^2)^{n_5}}\,, \\
&Y_N^-(n_1,n_2,n_3,n_4,n_5) = \\
&\quad \frac{e^{2\epsilon\gamma_E}}{(i\pi^{d/2})^2}\int \frac{\mathrm{d}^dk\mathrm{d}^dl}{(k^2+2kq)^{n_1}(l^2-2lq)^{n_2}((k-l)^2+2q(k-l))^{n_3}(k^2)^{n_4}(l^2-M^2)^{n_5}}\,, \\
&H_C^+(n_1,n_2,n_3,n_4,n_5) = \\
&\quad \frac{e^{2\epsilon\gamma_E}}{(i\pi^{d/2})^2}\int \frac{\mathrm{d}^dk\mathrm{d}^dl}{(k^2+2kq)^{n_1}((l+q)^2)^{n_2}(k^2)^{n_3}((k-l)^2-M^2)^{n_4}(l^2+m^2)^{n_5}}\,, \\
&W_C^-(n_1,n_2,n_3,n_4,n_5) = \\
&\quad \frac{e^{2\epsilon\gamma_E}}{(i\pi^{d/2})^2}\int \frac{\mathrm{d}^dk\mathrm{d}^dl}{(k^2+2kq)^{n_1}((l-q)^2)^{n_2}((k-l+q)^2)^{n_3}(k^2)^{n_4}(l^2-M^2)^{n_5}}\,,
\end{aligned}
\tag{7}
$$

where $d=4-2\epsilon$ is the space-time dimension. The corresponding graphical representation can be found in Fig. 2. The integrals are defined in Minkowskian space and the $i\varepsilon$ prescription is understood.

Within `FORM` we apply the projectors, identify the external momentum with the top quark mass and decompose the numerators in terms of the denominators. This leads to a large number of scalar integrals which differ from each other by the power of the individual propagators.

A conventional way to reduce an arbitrary integral of a certain kind to a small set of so-called master integrals is based on integration-by-parts [14], which provides relations between several integrals of different complexity. The proper combination of these relations leads to new ones, so that the iteration of this procedure can be used for a systematic reduction of an arbitrary integral to a small set of master integrals.

[2]For the application at hand it is possible to work with a smaller set of integral types. E.g., in the case of $H_C^+$ we have $n_5 \leq 0$ for the self energies considered in this paper. However, in view of a future application to on-shell vertices it is advantageous to consider a more general set-up.

In Ref. [15] an algorithm has been formulated that performs automatically the aforementioned reduction for a given set of recurrence relations. Currently several implementations of this algorithm exist. However, to our knowledge, only one program is publicly available, `AIR` [16]. `AIR` is written in `MAPLE`, which certainly constitutes a serious restriction for large-scale problems like, e.g., four-loop vacuum integrals [17–20]. Nevertheless, for the problem at hand `AIR` is well suited to perform the reduction. It is straightforward to compose an interface which, for a given class of Feynman diagrams (cf. Eq. (7)), produces the corresponding integration-by-parts relations, passes them to `AIR` and transforms the output into a table that can be read into `FORM`. These tables can be used to express each integral occuring in our expression in terms of a few master integrals.

At this point a comment concerning `AIR` is in order. Our experience with `AIR` shows that there can be situations where the set of master integrals is not minimal, although the complete set of recurrence relations has been provided. Consider, e.g., the integrals $Y_N^-$ with $M = 0$. It is well-known that only three master are needed for the computation of the two-loop pure QCD corrections to $Z_m^{\rm OS}$. However, the naive application of `AIR` leads to four master integrals. A straightforward inspection of the involved integrals makes it possible to relate the additional master to the known ones. The same is also true for our types of integrals.

For the diagrams where a neutral boson is exchanged one requires altogether nine master integrals. Six of them either contain only one mass scale, are vacuum diagrams, or consist of a product of two one-loop integrals. They read

$$
\begin{aligned}
H_1 &= H_N^+(1,1,0,0,0)\,, & H_2 &= H_N^+(0,0,1,1,1)\,, & H_3 &= H_N^+(1,1,0,1,0)\,,\\
Y_1 &= Y_N^-(1,1,1,0,0)\,, & Y_2 &= Y_N^-(1,0,0,0,1)\,, & Y_3 &= Y_N^-(1,1,0,0,1)\,.
\end{aligned}
\tag{8}
$$

Explicit analytic results are given in Appendix C. The remaining three master integrals

$$
H_4 = H_N^+(1,0,0,1,1)\,, \qquad H_5 = H_N^+(2,0,0,1,1)\,, \qquad Y_4 = Y_N^-(1,1,1,0,1)\,, \tag{9}
$$

are less trivial. Analytic expressions for $H_4$, $H_5$ and $Y_4$ can be found in Ref. [7]. More precisely one has

$$
\begin{aligned}
H_4 &\leftrightarrow J_{012}(1,1,1,m^2,M^2)\,,\\
H_5 &\leftrightarrow J_{012}(1,2,1,m^2,M^2)\,,\\
Y_4 &\leftrightarrow V_{mmmM}(1,1,1,1)\,,
\end{aligned}
\tag{10}
$$

where the integrals $J_{012}(1,1,1,m^2,M^2)$ and $J_{012}(1,2,1,m^2,M^2)$ are given in Eq. (3.20) and $V_{mmmM}(1,1,1,1)$ in Eq. (3.26) of Ref. [7]. We have checked all master integrals by considering their evaluation in an asymptotic expansion around the three kinematical regions $m \ll M$, $m \approx M$, and $m \gg M$. Altogether we computed up to 16 expansion terms and found complete agreement with the results in the literature. As can be deduced from the results of Appendix C, where more details are provided, the inclusion of about five expansion terms in each region provides jointly a good approximation over almost the whole range in $m/M$.

In the case of charged boson exchange one gets in addition two simple master integrals

$$W_1 = W_C^-(1,1,1,0,0)\,, \qquad W_2 = W_C^-(1,1,0,0,1)\,, \tag{11}$$

and five two-scale integrals

$$\begin{aligned} H_6 &= H_C^+(0,1,1,1,0)\,, \quad H_7 = H_C^+(0,1,1,1,-1)\,, \quad H_8 = H_C^+(1,1,0,1,0)\,, \\ W_3 &= W_C^-(0,-1,1,1,1)\,, \quad W_4 = W_C^-(1,1,1,0,1)\,. \end{aligned} \tag{12}$$

As we will see in Sections 3 and 4, for the physical applications of this paper an expansion for $m \gg M$ of the integrals in Eq. (12) is sufficient to obtain final results which in the physical region are equivalent to the exact expressions.

# 3 On-shell mass renormalization constant

In this Section we discuss the results for the on-shell mass renormalization counterterm. The QCD corrections up to three loops can be found in Refs. [21–25]. The one-loop electroweak and two-loop mixed corrections for light quarks can be found in Refs. [26] and [27], respectively. In this case it is sufficient to evaluate the limiting behaviour for $m_q^2 \ll M^2$ (where $M$ represents a boson mass). The corrections of order $\alpha\alpha_s$ for the top quark have been considered in Ref. [7–9]. In a recent paper [28] the two-loop relation between a minimal subtracted and the on-shell mass has been considered in a more general framework. However, the masses of the vector bosons have been neglected.

The relation between the bare mass, $m_t^0$, and the one defined in the $\overline{\rm MS}$ and on-shell scheme, $\overline{m}_t$ and $m_t$, is given by

$$m_t^0 \;=\; Z_m^{\overline{\rm MS}}\overline{m}_t \;=\; Z_m^{\rm OS} m_t\,. \tag{13}$$

In order to discuss the result for the mass renormalization constant it is convenient to consider the finite ratio

$$z_m \;=\; \frac{m_t}{\overline{m}_t} \;=\; \frac{Z_m^{\overline{\rm MS}}}{Z_m^{\rm OS}} \;=\; 1 + \frac{\alpha_s}{\pi}C_F z_m^{\rm QCD} + \frac{\alpha}{\pi s_W^2} z_m^{\rm ew} + \frac{\alpha\alpha_s}{\pi^2 s_W^2} C_F z_m^{\rm mix}\,, \tag{14}$$

where $s_W \equiv \sin\theta_W$ is the sine of the Weinberg angle, $C_F = (N_c^2-1)/(2N_c)$ with $N_c = 3$ for SU(3) and

$$\begin{aligned} z_m^{\rm ew} &= z_m^{(1),\rm ew}\overline{L}_{\mu t} + z_m^{(0),\rm ew}\,, \\ z_m^{(0),\rm ew} &= z_m^{H,\rm ew}(\overline{y}_H) + z_m^{W,\rm ew}(\overline{y}_W) + z_m^{Z,\rm ew}(\overline{y}_Z) + z_m^{A,\rm ew} + z_m^{\rm tad,ew}\,, \\ z_m^{\rm mix} &= z_m^{(2),\rm mix}\overline{L}_{\mu t}^2 + z_m^{(1),\rm mix}\overline{L}_{\mu t} + z_m^{(0),\rm mix}\,, \\ z_m^{(0),\rm mix} &= z_m^{H,\rm mix}(\overline{y}_H) + z_m^{W,\rm mix}(\overline{y}_W) + z_m^{Z,\rm mix}(\overline{y}_Z) + z_m^{A,\rm mix} + z_m^{\rm tad,mix}\,, \end{aligned} \tag{15}$$

with an analog separation for $z_m^{(1),\text{mix}}$ and $z_m^{(2),\text{mix}}$. We furthermore introduce the notation

$$
\begin{aligned}
\overline{y}_H &= \frac{\overline{m}_t}{M_H}\,, \quad \overline{y}_W = \frac{\overline{m}_t}{M_W}\,, \quad \overline{y}_Z = \frac{\overline{m}_t}{M_Z}\,, \\
\overline{L}_H &= -\ln(\overline{y}_H^2)\,, \quad \overline{L}_W = -\ln(\overline{y}_W^2)\,, \quad \overline{L}_Z = -\ln(\overline{y}_Z^2)\,, \quad \overline{L}_{\mu t} = -\ln\left(\frac{\mu^2}{\overline{m}_t^2}\right)\,.
\end{aligned}
\tag{16}
$$

The renormalization constant in the $\overline{\text{MS}}$ scheme is given by (see, e.g., Ref. [7])

$$
\begin{aligned}
Z_m^{\overline{\text{MS}}} &= 1 - \frac{\alpha_s}{\pi} C_F \frac{3}{4\epsilon} + \frac{\alpha}{4\pi s_W^2}\frac{1}{\epsilon}\left(\frac{1}{4} + \frac{5}{4}a_t^2 s_W^2 - \frac{3}{4}v_t^2 s_W^2 - \frac{4}{3}s_W^2 + \frac{3}{8}\frac{\overline{m}_t^2}{M_W^2}\right. \\
&\quad \left. -\frac{1}{4} - 3a_t^2 s_W^2 \frac{M_Z^2}{M_H^2} - \frac{1}{2}a_t^2 s_W^2 - \frac{3}{8}\frac{M_H^2}{M_W^2} - \frac{3}{2}\frac{M_W^2}{M_H^2} + N_c \frac{\overline{m}_t^4}{M_W^2 M_H^2}\right) \\
&\quad + \frac{\alpha\alpha_s}{4\pi^2 s_W^2} C_F \left[\frac{1}{\epsilon^2}\left(-\frac{3}{16} - \frac{15}{16}a_t^2 s_W^2 + \frac{9}{16}v_t^2 s_W^2 + s_W^2 - \frac{9}{16}\frac{\overline{m}_t^2}{M_W^2}\right)\right. \\
&\quad + \frac{1}{\epsilon}\left(\frac{9}{32} + \frac{21}{32}a_t^2 s_W^2 - \frac{3}{32}v_t^2 s_W^2 - \frac{1}{6}s_W^2 + \frac{3}{8}\frac{\overline{m}_t^2}{M_W^2}\right) + \frac{N_c}{2\epsilon}\frac{\overline{m}_t^4}{M_W^2 M_H^2} \\
&\quad \left. + \frac{1}{\epsilon^2}\left(\frac{3}{16} + \frac{9}{4}a_t^2 s_W^2 \frac{M_Z^2}{M_H^2} + \frac{3}{8}a_t^2 s_W^2 + \frac{9}{32}\frac{M_H^2}{M_W^2} + \frac{9}{8}\frac{M_W^2}{M_H^2} - \frac{9N_c}{4}\frac{\overline{m}_t^4}{M_W^2 M_H^2}\right)\right],
\end{aligned}
\tag{17}
$$

with $a_t = 1/(2s_W c_W)$, $v_t = (1/2 - 4s_W^2/3)/(2s_W c_W)$ and $c_W = \sqrt{1 - s_W^2} = M_W/M_Z$. For convenience, in Eq. (17) the contribution from the tadpole diagrams is displayed separately in the second ($\mathcal{O}(\alpha)$) and last line ($\mathcal{O}(\alpha\alpha_s)$). Furthermore, all term proportional to $N_c$ originate from tadpole diagrams.

At this point a comment concerning the various gauge parameters is in order. In the electroweak sector we adopt Feynman gauge for the $W$ and $Z$ boson, however, we allow for a general gauge parameter $\xi$ for QCD defined via the gluon propagator

$$
D_g^{\mu\nu}(q) = -i\,\frac{g^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}}{q^2 + i\varepsilon}\,. \tag{18}
$$

On general grounds the on-shell mass and also $z_m$ has to be independent of $\xi$ which serves as a welcome check for our calculation.

In Eq. (15) the two-loop expression for $z_m$ is split into contributions induced by the Higgs, $W$ and $Z$ boson (including the corresponding Goldstone parts), the photon ($A$) and the tadpole diagrams. In the following we present analytical results for the individual contributions. As we will see in the discussion below it is sufficient to consider the limit $y_W \to \infty$ and $y_Z \to \infty$ in order to obtain agreement with the exact result below the percent level. For this reason we show only the corresponding analytical expressions. Since the Higgs boson mass is still unknown we present the exact result, but also handy expansions in the three limits $y_H \to 0$, $y_H \to 1$ and $y_H \to \infty$. Furthermore, for completeness also the tadpole result $z_m^{\rm tad}$ is listed, which can be extracted from Refs. [8, 27, 29].

Compact expressions for the Higgs boson contribution to $z_m^{H,{\rm ew}}$ and $z_m^{H,{\rm mix}}$ expressed in terms of (known) master integrals are given in Eqs. (37) and (43) in Appendix A, where also the results for the $\mu$-dependent terms $z_m^{(1),H,{\rm mix}}$ and $z_m^{(2),H,{\rm mix}}$ can be found. The expansions in the physically interesting regions read for the one-loop results

$$
\begin{aligned}
z_m^{\rm QCD} &= 1\,,\\
z_{m,0}^{H,{\rm ew}} &= \overline{y}_W^2\Bigg[-\frac{5}{64}+\frac{3\overline{L}_H}{32}+\left(-\frac{1}{96}+\frac{\overline{L}_H}{16}\right)\overline{y}_H^2+\left(-\frac{7}{128}+\frac{3\overline{L}_H}{32}\right)\overline{y}_H^4+\left(-\frac{47}{320}+\frac{3\overline{L}_H}{16}\right)\overline{y}_H^6\\
&\quad+\left(-\frac{379}{960}+\frac{7\overline{L}_H}{16}\right)\overline{y}_H^8+\mathcal{O}(\overline{y}_H^{10})\Bigg]\,,\\
z_{m,1}^{H,{\rm ew}} &= \overline{y}_W^2\Bigg[-\frac{3}{16}+\frac{\pi\sqrt{3}}{32}-\frac{1}{16}\overline{y}_{H,1}+\left(\frac{1}{32}-\frac{\pi\sqrt{3}}{96}\right)\overline{y}_{H,1}^2+\left(\frac{1}{96}-\frac{\pi\sqrt{3}}{216}\right)\overline{y}_{H,1}^3\\
&\quad+\left(\frac{1}{384}-\frac{\pi\sqrt{3}}{432}\right)\overline{y}_{H,1}^4+\left(\frac{1}{960}-\frac{\pi\sqrt{3}}{648}\right)\overline{y}_{H,1}^5+\mathcal{O}(\overline{y}_{H,1}^6)\Bigg]\,,\\
z_{m,\infty}^{H,{\rm ew}} &= \overline{y}_W^2\Bigg[-\frac{7}{32}+\frac{\pi}{8}\frac{1}{\overline{y}_H}+\left(-\frac{3}{32}+\frac{3\overline{L}_H}{32}\right)\frac{1}{\overline{y}_H^2}-\frac{3\pi}{64}\frac{1}{\overline{y}_H^3}+\left(\frac{1}{24}-\frac{\overline{L}_H}{64}\right)\frac{1}{\overline{y}_H^4}+\frac{3\pi}{1024}\frac{1}{\overline{y}_H^5}\\
&\quad-\frac{1}{640}\frac{1}{\overline{y}_H^6}+\mathcal{O}\left(\frac{1}{\overline{y}_H^7}\right)\Bigg]\,,\\
z_{m,\infty}^{W,{\rm ew}} &= \overline{y}_W^2\Bigg[-\frac{1}{16}+\left(-\frac{1}{16}+\frac{\overline{L}_W}{16}\right)\frac{1}{\overline{y}_W^2}+\left(\frac{3}{64}+\frac{3\overline{L}_W}{32}\right)\frac{1}{\overline{y}_W^4}+\left(\frac{1}{12}-\frac{\overline{L}_W}{16}\right)\frac{1}{\overline{y}_W^6}+\mathcal{O}\left(\frac{1}{\overline{y}_W^8}\right)\Bigg]\,,\\
z_{m,\infty}^{Z,{\rm ew}} &= \overline{y}_W^2\Bigg[\frac{1}{32}+\left(\frac{1}{32}+\frac{\overline{L}_Z}{32}\right)\frac{1}{\overline{y}_Z^2}-\frac{\pi}{32}\frac{1}{\overline{y}_Z^3}+\left(\frac{1}{32}-\frac{\overline{L}_Z}{64}\right)\frac{1}{\overline{y}_Z^4}+\frac{\pi}{256}\frac{1}{\overline{y}_Z^5}-\frac{1}{384}\frac{1}{\overline{y}_Z^6}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^7}\right)\Bigg]\\
&\quad+a_t^2s_W^2\Bigg[-\frac{1}{2}+\frac{3\pi}{8}\frac{1}{\overline{y}_Z}+\left(-\frac{5}{16}+\frac{\overline{L}_Z}{4}\right)\frac{1}{\overline{y}_Z^2}-\frac{7\pi}{64}\frac{1}{\overline{y}_Z^3}+\left(\frac{3}{32}-\frac{\overline{L}_Z}{32}\right)\frac{1}{\overline{y}_Z^4}+\frac{5\pi}{1024}\frac{1}{\overline{y}_Z^5}\\
&\quad-\frac{1}{480}\frac{1}{\overline{y}_Z^6}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^7}\right)\Bigg]+v_t^2s_W^2\Bigg[\frac{1}{4}-\frac{\pi}{8}\frac{1}{\overline{y}_Z}+\frac{3}{16}\frac{1}{\overline{y}_Z^2}-\frac{3\pi}{64}\frac{1}{\overline{y}_Z^3}+\left(\frac{5}{96}-\frac{\overline{L}_Z}{32}\right)\frac{1}{\overline{y}_Z^4}\\
&\quad+\frac{9\pi}{1024}\frac{1}{\overline{y}_Z^5}-\frac{1}{160}\frac{1}{\overline{y}_Z^6}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^7}\right)\Bigg]\,,\\
z_m^{A,{\rm ew}} &= \frac{4}{9}s_W^2\,,\\
z_m^{tad,{\rm ew}} &= \overline{y}_W^2\Bigg[\left(\frac{3}{32}-\frac{3\overline{L}_H}{32}\right)\frac{1}{\overline{y}_H^2}+\left(\frac{1}{8}-\frac{3\overline{L}_W}{8}\right)\frac{\overline{y}_H^2}{\overline{y}_W^4}+\left(\frac{1}{16}-\frac{\overline{L}_W}{16}\right)\frac{1}{\overline{y}_W^2}\Bigg]\\
&\quad+a_t^2s_W^2\Bigg[\left(\frac{1}{4}-\frac{3\overline{L}_Z}{4}\right)\frac{\overline{y}_H^2}{\overline{y}_Z^2}+\left(\frac{1}{8}-\frac{\overline{L}_Z}{8}\right)\Bigg]-\frac{N_c}{4}\overline{y}_W^2\overline{y}_H^2\,,
\end{aligned}
\tag{19}
$$

with $\overline{y}_{H,1} = 1 - 1/\overline{y}_H^2$. The subscripts 0, 1 and $\infty$ indicate the cases $\overline{m}_t \ll M_H$, $\overline{m}_t \approx M_H$, and $\overline{m}_t \gg M_{H,W,Z}$, respectively. The coefficients in front of $\overline{L}_{\mu t}$ are given by

$$
\begin{aligned}
z_m^{(1),\mathrm{QCD}} &= -\frac{3}{4}\,, \qquad z_m^{(1),H,\mathrm{ew}} = \frac{3\overline{y}_W^2}{32}\,, \qquad z_m^{(1),W,\mathrm{ew}} = \overline{y}_W^2\left(\frac{1}{32} + \frac{1}{16\overline{y}_W^2}\right)\,,\\
z_m^{(1),Z,\mathrm{ew}} &= -\frac{\overline{y}_W^2}{32} + \frac{5a_t^2 s_W^2}{16} - \frac{3v_t^2 s_W^2}{16}\,, \qquad z_m^{(1),A,\mathrm{ew}} = -\frac{s_W^2}{3}\,,\\
z_m^{(1),tad,\mathrm{ew}} &= \overline{y}_W^2\left(-\frac{3}{32\overline{y}_H^2} - \frac{3\overline{y}_H^2}{8\overline{y}_W^4} - \frac{1}{16\overline{y}_W^2}\right) + a_t^2 s_W^2\left(-\frac{3\overline{y}_H^2}{4\overline{y}_Z^2} - \frac{1}{8}\right) + \frac{N_c}{4}\overline{y}_W^2\overline{y}_H^2\,.
\end{aligned}
\tag{20}
$$

The corresponding two-loop expressions are slightly more lengthy but still rather compact. They read

$$
\begin{aligned}
z_{m,0}^{H,\mathrm{mix}} &= \overline{y}_W^2\Bigg[-\frac{9}{32} + \frac{33\overline{L}_H}{128} - \frac{9\overline{L}_H^2}{128} - \frac{3\pi^2}{128} + \left(-\frac{49}{1152} + \frac{185\overline{L}_H}{288} + \frac{13\overline{L}_H^2}{192} - \frac{\pi^2}{192}\right)\overline{y}_H^2\\
&+ \left(-\frac{21097}{18432} + \frac{665\overline{L}_H}{768} + \frac{45\overline{L}_H^2}{256} + \frac{15\pi^2}{256}\right)\overline{y}_H^4 + \left(-\frac{4145731}{1152000} + \frac{45727\overline{L}_H}{28800} + \frac{881\overline{L}_H^2}{1920} + \frac{379\pi^2}{1920}\right)\overline{y}_H^6\\
&+ \left(-\frac{4652609}{432000} + \frac{51199\overline{L}_H}{14400} + \frac{313\overline{L}_H^2}{240} + \frac{49\pi^2}{80}\right)\overline{y}_H^8 + \mathcal{O}(\overline{y}_H^{10})\Bigg]\,,\\
z_{m,1}^{H,\mathrm{mix}} &= \overline{y}_W^2\Bigg\{-\frac{133}{256} + \frac{Y^\epsilon}{32} + \left(\frac{\ln 3}{48} - \frac{35}{768}\right)\pi^2 + \left(\frac{9\ln 3}{32} + \frac{531}{256}\right)S_2 - \frac{\sqrt{3}}{16}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{7\sqrt{3}}{1152}\pi^3\\
&+ \left(-\frac{\ln 3}{24} - \frac{\ln^2 3}{192} + \frac{3S_2}{32} + \frac{53}{384}\right)\pi\sqrt{3} - \frac{\zeta(3)}{6} + \Bigg[\frac{41}{96} + \frac{Y^\epsilon}{24} + \left(\frac{\ln 3}{36} + \frac{1}{144}\right)\pi^2 + \left(\frac{3\ln 3}{8} - \frac{93}{64}\right)S_2\\
&- \frac{\sqrt{3}}{12}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{7\sqrt{3}}{864}\pi^3 + \left(\frac{\ln 3}{144} - \frac{\ln^2 3}{144} + \frac{S_2}{8} - \frac{13}{288}\right)\pi\sqrt{3} - \frac{2\zeta(3)}{9}\Bigg]\overline{y}_{H,1}\\
&+ \Bigg[\frac{1}{12} + \frac{Y^\epsilon}{96} + \left(\frac{\ln 3}{144} + \frac{7}{216}\right)\pi^2 + \left(\frac{3\ln 3}{32} - \frac{123}{128}\right)S_2 - \frac{\sqrt{3}}{48}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) - \frac{7\sqrt{3}}{3456}\pi^3\\
&+ \left(-\frac{\ln 3}{288} - \frac{\ln^2 3}{576} + \frac{S_2}{32} - \frac{1}{36}\right)\pi\sqrt{3} - \frac{\zeta(3)}{18}\Bigg]\overline{y}_{H,1}^2 + \Bigg[-\frac{1}{288} + \frac{143\pi^2}{7776} - \frac{19S_2}{64}\\
&- \left(\frac{\ln 3}{216} + \frac{59}{5184}\right)\pi\sqrt{3}\Bigg]\overline{y}_{H,1}^3 + \Bigg[\frac{55}{6912} + \frac{4145\pi^2}{373248} - \frac{257S_2}{1536} - \left(\frac{5\ln 3}{5184} + \frac{695}{62208}\right)\pi\sqrt{3}\Bigg]\overline{y}_{H,1}^4\\
&+ \Bigg[\frac{163}{34560} + \frac{3923\pi^2}{466560} - \frac{71S_2}{640} - \left(\frac{\ln 3}{2160} + \frac{289}{31104}\right)\pi\sqrt{3}\Bigg]\overline{y}_{H,1}^5 + \mathcal{O}(\overline{y}_{H,1}^6)\Bigg\}\,,\\
z_{m,\infty}^{H,\mathrm{mix}} &= \overline{y}_W^2\Bigg\{-\frac{187}{256} + \left(\frac{\ln 2}{4} - \frac{1}{64}\right)\pi^2 - \frac{3\zeta(3)}{8} - \frac{\pi}{8}\frac{1}{\overline{y}_H} + \Bigg[\frac{39}{64} - \frac{15\overline{L}_H}{64} - \frac{3\ln 2}{16}\pi^2 + \frac{9\zeta(3)}{32}\Bigg]\frac{1}{\overline{y}_H^2}\\
&. + \Bigg[\frac{173}{576} - \frac{\overline{L}_H}{24} - \frac{\ln 2}{6}\Bigg]\pi\frac{1}{\overline{y}_H^3} + \Bigg[-\frac{281}{1536} + \frac{17\overline{L}_H}{256} + \left(\frac{\ln 2}{32} + \frac{3}{256}\right)\pi^2 - \frac{3\zeta(3)}{64}\Bigg]\frac{1}{\overline{y}_H^4}\\
&+ \Bigg[-\frac{4861}{76800} + \frac{3\overline{L}_H}{320} + \frac{3\ln 2}{80}\Bigg]\pi\frac{1}{\overline{y}_H^5} + \Bigg[\frac{85}{13824} - \frac{5\pi^2}{3072}\Bigg]\frac{1}{\overline{y}_H^6} + \mathcal{O}\left(\frac{1}{\overline{y}_H^7}\right)\Bigg\}\,,
\end{aligned}
$$

$$
\begin{aligned}
z_{m,\infty}^{W,\text{mix}} = \overline{y}_W^2 \Bigg[ &-\frac{65}{256} + \frac{5\pi^2}{128} - \frac{3\zeta(3)}{32} - \left(\frac{19}{256} + \frac{\overline{L}_W}{32}\right)\frac{1}{\overline{y}_W^2} + \left(-\frac{15}{32} - \frac{15\overline{L}_W}{64} + \frac{\pi^2}{64} + \frac{9\zeta(3)}{32}\right)\frac{1}{\overline{y}_W^4} \\
&+ \left(-\frac{71}{384} + \frac{15\overline{L}_W}{64} - \frac{5\pi^2}{288} - \frac{3\zeta(3)}{16}\right)\frac{1}{\overline{y}_W^6} + \mathcal{O}\left(\frac{1}{\overline{y}_W^8}\right)\Bigg] , \\
z_{m,\infty}^{Z,\text{mix}} = \overline{y}_W^2 \Bigg[ &\frac{37}{256} - \frac{\pi^2}{64} + \left(-\frac{1}{64} - \frac{\overline{L}_Z}{64} + \left(-\frac{\ln 2}{16} + \frac{1}{32}\right)\pi^2 + \frac{3\zeta(3)}{32}\right)\frac{1}{\overline{y}_Z^2} + \frac{\pi}{32}\frac{1}{\overline{y}_Z^3} \\
&+ \left(-\frac{55}{512} + \frac{13\overline{L}_Z}{256} + \frac{\ln 2}{32}\pi^2 - \frac{3\zeta(3)}{64}\right)\frac{1}{\overline{y}_Z^4} + \left(-\frac{667}{11520} + \frac{\overline{L}_Z}{96} + \frac{\ln 2}{24}\right)\pi\frac{1}{\overline{y}_Z^5} + \left(\frac{11}{1536} - \frac{3\pi^2}{1024}\right)\frac{1}{\overline{y}_Z^6} \\
&+ \mathcal{O}\left(\frac{1}{\overline{y}_Z^7}\right)\Bigg] + a_t^2 s_W^2 \Bigg[\frac{57}{256} + \left(\frac{3\ln 2}{4} - \frac{11}{32}\right)\pi^2 - \frac{9\zeta(3)}{8} - \frac{3\pi}{8}\frac{1}{\overline{y}_Z} \\
&+ \left(\frac{7}{8} - \frac{13\overline{L}_Z}{16} + \left(-\frac{\ln 2}{2} + \frac{3}{32}\right)\pi^2 + \frac{3\zeta(3)}{4}\right)\frac{1}{\overline{y}_Z^2} + \left(\frac{167}{192} - \frac{\overline{L}_Z}{8} - \frac{\ln 2}{2}\right)\pi\frac{1}{\overline{y}_Z^3} \\
&+ \left(-\frac{71}{256} + \frac{15\overline{L}_Z}{128} + \left(\frac{\ln 2}{16} + \frac{5}{256}\right)\pi^2 - \frac{3\zeta(3)}{32}\right)\frac{1}{\overline{y}_Z^4} + \mathcal{O}\left(\frac{1}{\overline{y}_Z^5}\right)\Bigg] \\
&+ v_t^2 s_W^2 \Bigg[-\frac{71}{256} + \left(-\frac{\ln 2}{4} + \frac{5}{32}\right)\pi^2 + \frac{3\zeta(3)}{8} + \left(-\frac{3}{4} + \frac{3\pi^2}{32}\right)\frac{1}{\overline{y}_Z^2} + \left(-\frac{19}{288} + \frac{\overline{L}_Z}{24} + \frac{\ln 2}{6}\right)\pi\frac{1}{\overline{y}_Z^3} \\
&+ \left(-\frac{39}{256} + \frac{15\overline{L}_Z}{128} + \left(\frac{\ln 2}{16} - \frac{7}{256}\right)\pi^2 - \frac{3\zeta(3)}{32}\right)\frac{1}{\overline{y}_Z^4} + \mathcal{O}\left(\frac{1}{\overline{y}_Z^5}\right)\Bigg] , \\
z_m^{A,\text{mix}} = &\left[-\frac{71}{144} + \left(-\frac{4\ln 2}{9} + \frac{5}{18}\right)\pi^2 + \frac{2\zeta(3)}{3}\right] s_W^2 , \\
z_m^{tad,\text{mix}} = \overline{y}_W^2 &\left[\left(-\frac{3}{64} + \frac{3\overline{L}_H}{64}\right)\frac{1}{\overline{y}_H^2} + \left(-\frac{1}{16} + \frac{3}{16}\overline{L}_W\right)\frac{\overline{y}_H^2}{\overline{y}_W^4} + \left(-\frac{1}{32} + \frac{1}{32}\overline{L}_W\right)\frac{1}{\overline{y}_W^2}\right] \\
&+ a_t^2 s_W^2 \left[\left(-\frac{1}{8} + \frac{3\overline{L}_Z}{8}\right)\frac{\overline{y}_H^2}{\overline{y}_Z^2} + \left(-\frac{1}{16} + \frac{\overline{L}_Z}{16}\right)\right] - \frac{N_c}{2}\overline{y}_W^2\overline{y}_H^2 , \qquad (21)
\end{aligned}
$$

where

$$
\begin{aligned}
Y^\epsilon &= 8 - \frac{\zeta(3)}{3} + \frac{\pi^2}{6} + \sqrt{3}\Bigg\{-\frac{4\pi}{3} - \frac{\pi^3}{36} + \frac{2\pi\ln 3}{3} - \frac{\pi\ln^2 3}{6} \\
&\quad + 2(2-\ln 3)\left[\mathrm{Ls}_2(\pi) - \mathrm{Ls}_2\left(\frac{2\pi}{3}\right)\right] + 2\left[\mathrm{Ls}_3(\pi) - \mathrm{Ls}_3\left(\frac{2\pi}{3}\right)\right]\Bigg\} \\
&\approx 0.245815004513 , \\
S_2 &= \frac{4\sqrt{3}}{27}\mathrm{Ls}_2\left(\frac{\pi}{3}\right) \approx 0.260434137632 , \\
\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) &\approx -2.144767212569 , \qquad (22)
\end{aligned}
$$

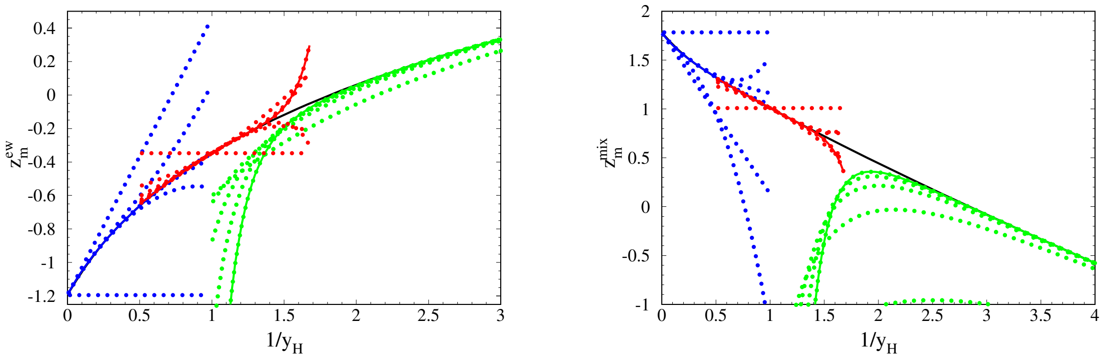
and $\zeta$ is Riemann's zeta function. The definition of $\mathrm{Ls_i(z)}$ is given in Eq. (42). The $\mu$ dependence is determined through

$$
\begin{aligned}
z_{m,0}^{(1),H,\mathrm{mix}} &= \overline{y}_W^2\left[\frac{81}{256}-\frac{27\overline{L}_H}{128}+\left(\frac{17}{128}-\frac{15\overline{L}_H}{64}\right)\overline{y}_H^2+\left(\frac{219}{512}-\frac{63\overline{L}_H}{128}\right)\overline{y}_H^4\right.\\
&\left.\quad+\left(\frac{1629}{1280}-\frac{81\overline{L}_H}{64}\right)\overline{y}_H^6+\left(\frac{5009}{1280}-\frac{231\overline{L}_H}{64}\right)\overline{y}_H^8+\mathcal{O}(\overline{y}_H^{10})\right],\\
z_{m,1}^{(1),H,\mathrm{mix}} &= \overline{y}_W^2\left[\frac{33}{64}-\frac{9\pi\sqrt{3}}{128}+\left(-\frac{3}{64}+\frac{\pi\sqrt{3}}{32}\right)\overline{y}_{H,1}+\left(-\frac{3}{128}+\frac{5\pi\sqrt{3}}{384}\right)\overline{y}_{H,1}^2\right.\\
&\left.\quad+\left(\frac{1}{128}+\frac{\pi\sqrt{3}}{288}\right)\overline{y}_{H,1}^3+\left(\frac{1}{512}+\frac{5\pi\sqrt{3}}{1728}\right)\overline{y}_{H,1}^4+\left(\frac{3}{1280}+\frac{5\pi\sqrt{3}}{2592}\right)\overline{y}_{H,1}^5+\mathcal{O}(\overline{y}_{H,1}^6)\right],\\
z_{m,\infty}^{(1),H,\mathrm{mix}} &= \overline{y}_W^2\left[\frac{63}{128}-\frac{3\pi}{16\overline{y}_H}+\left(\frac{27}{128}-\frac{9\overline{L}_H}{128}\right)\frac{1}{\overline{y}_H^2}+\left(\frac{1}{128}-\frac{3\overline{L}_H}{256}\right)\frac{1}{\overline{y}_H^4}+\frac{9\pi}{2048\overline{y}_H^5}-\frac{9}{2560\overline{y}_H^6}\right.\\
&\left.\quad+\mathcal{O}\left(\frac{1}{\overline{y}_H^7}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
z_{m,\infty}^{(1),W,\mathrm{mix}} &= \overline{y}_W^2\left[\frac{3}{16}+\left(\frac{5}{32}-\frac{3\overline{L}_W}{64}\right)\frac{1}{\overline{y}_W^2}+\left(\frac{45}{256}+\frac{9\overline{L}_W}{128}\right)\frac{1}{\overline{y}_W^4}+\left(\frac{3}{32}-\frac{9\overline{L}_W}{64}\right)\frac{1}{\overline{y}_W^6}+\mathcal{O}\left(\frac{1}{\overline{y}_W^8}\right)\right],\\
z_{m,\infty}^{(1),Z,\mathrm{mix}} &= \overline{y}_W^2\left[-\frac{15}{128}+\left(\frac{3}{128}-\frac{3\overline{L}_Z}{128}\right)\frac{1}{\overline{y}_Z^2}-\frac{3\overline{L}_Z}{256}\frac{1}{\overline{y}_Z^4}+\frac{3\pi}{512\overline{y}_Z^5}-\frac{3}{512\overline{y}_Z^6}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^7}\right)\right]\\
&\quad+a_t^2s_W^2\left[\frac{5}{64}+\left(\frac{9}{64}+\frac{3\overline{L}_Z}{16}\right)\frac{1}{\overline{y}_Z^2}-\frac{21\pi}{128\overline{y}_Z^3}+\left(\frac{21}{128}-\frac{9\overline{L}_Z}{128}\right)\frac{1}{\overline{y}_Z^4}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^5}\right)\right]\\
&\quad+v_t^2s_W^2\left[\frac{9}{64}+\frac{9}{64\overline{y}_Z^2}-\frac{9\pi}{128\overline{y}_Z^3}+\left(\frac{9}{128}-\frac{9\overline{L}_Z}{128}\right)\frac{1}{\overline{y}_Z^4}+\mathcal{O}\left(\frac{1}{\overline{y}_Z^5}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
z_m^{(1),A,\mathrm{mix}} &= \frac{s_W^2}{4},\\
z_m^{(1),tad,\mathrm{mix}} &= \overline{y}_W^2\left[\left(-\frac{3}{128}+\frac{9\overline{L}_H}{128}\right)\frac{1}{\overline{y}_H^2}+\left(\frac{3}{32}+\frac{9\overline{L}_W}{32}\right)\frac{\overline{y}_H^2}{\overline{y}_W^4}+\left(-\frac{1}{64}+\frac{3\overline{L}_W}{64}\right)\frac{1}{\overline{y}_W^2}\right]\\
&\quad+a_t^2s_W^2\left[\left(\frac{3}{16}+\frac{9\overline{L}_Z}{16}\right)\frac{\overline{y}_H^2}{\overline{y}_Z^2}-\frac{1}{32}+\frac{3\overline{L}_Z}{32}\right]+\frac{11N_c}{16}\overline{y}_W^2\overline{y}_H^2\,,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&z_m^{(2),H,\mathrm{mix}}=-\frac{9\overline{y}_W^2}{64}\,,\qquad z_m^{(2),W,\mathrm{mix}}=-\overline{y}_W^2\left[\frac{3}{64}+\frac{3}{64\overline{y}_W^2}\right],\\
&z_m^{(2),Z,\mathrm{mix}}=\frac{3\overline{y}_W^2}{64}-\frac{15a_t^2s_W^2}{64}+\frac{9v_t^2s_W^2}{64}\,,\qquad z_m^{(2),A,\mathrm{mix}}=\frac{s_W^2}{4}\,,\\
&z_m^{(2),tad,\mathrm{mix}}=\overline{y}_W^2\left[\frac{9}{128\overline{y}_H^2}+\frac{9\overline{y}_H^2}{32\overline{y}_W^4}+\frac{3}{64\overline{y}_W^2}\right]+a_t^2s_W^2\left[\frac{9\overline{y}_H^2}{16\overline{y}_Z^2}+\frac{3}{32}\right]-\frac{9N_c}{16}\overline{y}_W^2\overline{y}_H^2\,.
\end{aligned}
\tag{24}
$$




(a) (b)

Figure 3: (a) One-loop and (b) two-loop corrections to $z_m$ as a function of $1/\overline{y}_H = M_H/\overline{m}_t$. The solid (coloured) lines correspond to the highest available order for each case. The dotted curves show lower-order results and nicely demonstrate the convergence. The exact result, which is plotted (in black) over the whole $1/\overline{y}_H$ range, can be distinguished from the approximations only in small gap around $1/\overline{y}_H \approx 1.5$. For the renormalization scale $\mu^2 = \overline{m}_t^2$ has been chosen. In the plots the contributions from the tadpole diagrams are not included.

Note that the result for $z_m^A$ in the above expressions can easily be extracted from the two-loop pure QCD result [21].

Let us next discuss the numerical consequences of our result and compare the exact expressions with the compact expansions. Actually the exact result from the $W$ and $Z$ boson contribution is very well reproduced both by the large-top mass limit or the expansion around $\overline{m}_t \approx M_{W/Z}$. At the physical values for the particle masses we find a deviation from the exact result for the $W$ and $Z$ contribution below the percent level, which justifies the use of the expansion for the numerical evaluations.

The result for the one-loop coefficient of $\alpha/(\pi s_W^2)$ is shown in Fig. 3(a) for $\mu = \overline{m}_t$ as a function of $1/\overline{y}_H = M_H/\overline{m}_t$ where the tadpole contributions [9], which are numerically quite large, have been subtracted for Feynman gauge. Furthermore the following input values have been chosen

$$\overline{m}_t = 165 \text{ GeV}\,, \quad M_W = 80.425 \text{ GeV}\,, \quad M_Z = 91.19 \text{ GeV}\,, \quad c_W = M_W/M_Z\,. \tag{25}$$

Next to the (black) solid line which includes the exact result for the Higgs mass dependence and the large-$\overline{m}_t$ results for the $W$ and $Z$ contributions we also show the expansion terms in the three kinematical regions (cf. Eq. (19)) as solid lines. The lower-order results are plotted as dotted lines in order to demonstrate the convergence of the approximations. It can be seen that over almost the whole range of $\overline{y}_H$ the expansion terms provide a very good approximation to the exact result, except for a small region with $\overline{y}_H \approx 0.5\ldots0.7$ which corresponds to $M_H \approx 250\ldots300$ GeV. We want to mention that in Fig. 3(a) also the result of Ref. [7] is shown which in contrast to ours also takes into account the exact dependence on $M_W$ and $M_Z$. No visible effect is observed.

In Fig. 3(b) we show the two-loop coefficient of $\alpha\alpha_s/(\pi^2 s_W^2)$ as a function of $1/\overline{y}_H$, where again the tadpole contribution of Eq. (21) is subtracted. The result containing the exact $M_H$ dependence (cf. Eq. (43)) is plotted together with the expansion results of Eq. (21). In addition to the highest expansion terms we show as dotted lines also the lower-order ones. One can see how the approximations nicely improve while including step-by-step the higher order terms. Note that at two-loop order it is not possible to separate the result given in Ref. [7] into the contributions from the individual bosons and the tadpole contribution. Thus in order to check our results we again include the exact results from Ref. [7] in our plots (after subtracting the tadpole contributions). Again no difference is visible. In fact, adopting from Eq. (21) the large-$\overline{m}_t$ terms we obtain at the physical values of $M_W$ and $M_Z$ agreement with the exact result below the per cent level. Equivalently we can use the expansion around $\overline{m}_t \approx M_{W/Z}$ and get results with the same level of accuracy.

The excellent description of the exact result by the expansion terms together with the relative simplicity of the results in Eq. (21) provides sufficient motivation to apply in the next section the same approach to the wave function renormalization constant. In particular this means that both exact results and expansions are considered for the Higgs boson contributions and approximate formulae for large top quark mass are derived for the remaining parts.

# 4 On-shell wave function renormalization constants

In this Section we consider the wave function renormalization constants for the top quark defined through Eq. (4). Since $Z_2^{L,\mathrm{OS}}$ and $Z_2^{R,\mathrm{OS}}$ contain infra-red divergences which only cancel when considering a physical quantity, it is not possible to form a finite ratio analog to $z_m^{\mathrm{OS}}$ in Eq. (16). Thus, in the following we consider the divergent contributions and the finite parts separately. Furthermore, we switch to the vector and axial-vector contribution using the formulae

$$
\begin{aligned}
Z_2^{V,\mathrm{OS}} &= \frac{1}{2}\left(Z_2^{R,\mathrm{OS}} + Z_2^{L,\mathrm{OS}}\right) , \\
Z_2^{A,\mathrm{OS}} &= \frac{1}{2}\left(Z_2^{R,\mathrm{OS}} - Z_2^{L,\mathrm{OS}}\right) ,
\end{aligned}
\tag{26}
$$

where a non-zero contribution to $Z_2^{A,\mathrm{OS}}$ only arises from the $W$- and $Z$-boson.

In contrast to $Z_m^{\mathrm{OS}}$ the wave function renormalization needs not to be gauge parameter independent since it does not pose a physical quantity. As for the mass renormalization constant, we choose Feynman gauge for the electroweak part but allow for arbitrary $\xi$ in the QCD sector. We observe that $\xi$ drops out in the case of $Z_2^{V,\mathrm{OS}}$ up to the two-loop order, which is in analogy to QCD where only the three-loop result starts to depend on $\xi$ [30]. On the other hand $Z_2^{A,\mathrm{OS}}$ is $\xi$ dependent starting from order $\alpha\alpha_s$.

In order to present the results in a compact form we introduce the notation

$$Z_2^{X,\mathrm{OS}} = 1 + \frac{\alpha_s}{\pi} C_F \delta Z_{2,X}^{\mathrm{QCD}} + \frac{\alpha}{\pi}\left(\frac{f_{2,X}^{\mathrm{ew}}}{\epsilon} + z_{2,X}^{\mathrm{ew}}\right) + \frac{\alpha\alpha_s}{\pi^2} C_F \left(\frac{g_{2,X}^{\mathrm{mix}}}{\epsilon^2} + \frac{h_{2,X}^{\mathrm{mix}}}{\epsilon} + z_{2,X}^{\mathrm{mix}}\right), \tag{27}$$

with $X = V, A$ and the analog splitting as in Eq. (15). In particular, the results for the individual coefficients are split according to the dependence on the renormalization scale and decomposed into contributions originating from the Higgs, $W$ and $Z$ bosons and the photon[3] where for $W$ and $Z$ again the large-$m_t$ limit is adopted.

In contrast to the quotient $z_m$, where it was found convenient to use the $\overline{\mathrm{MS}}$-mass as expansion parameter, for $Z_2^{X,\mathrm{OS}}$ the natural choice is to express the results in terms of the on-shell mass. Thus, in analogy to Eq. (16) we introduce the following notation:

$$\begin{aligned}
y_H &= \frac{m_t}{M_H}, \quad y_W = \frac{m_t}{M_W}, \quad y_Z = \frac{m_t}{M_Z}, \\
L_H &= -\ln(y_H^2), \quad L_W = -\ln(y_W^2), \quad L_Z = -\ln(y_Z^2), \quad L_{\mu t} = -\ln\left(\frac{\mu^2}{m_t^2}\right).
\end{aligned} \tag{28}$$

For the numerical value of the top quark mass we use $m_t = 175$ GeV.

In the main text we again list the expansion terms and relegate the exact expression for the Higgs boson contribution to the Appendix (cf. Eq. (46)). For the QCD result and the pole part of the one-loop electro-weak corrections we obtain the following exact expressions (see, e.g., Ref. [31])

$$\begin{aligned}
\delta Z_2^{V,\mathrm{QCD}} &= \left(-\frac{3}{4\epsilon} + \frac{3}{4}L_{\mu t} - 1\right), & \delta Z_2^{A,\mathrm{QCD}} &= 0, \\
f_{2,V}^{H,\mathrm{ew}} &= -\frac{y_W^2}{32}, & f_{2,V}^{A,\mathrm{ew}} &= -\frac{s_W^2}{3}, \\
f_{2,V}^{W,\mathrm{ew}} &= -\frac{y_W^2}{32} - \frac{1}{16}, & f_{2,A}^{W,\mathrm{ew}} &= \frac{y_W^2}{32} - \frac{1}{16}, \\
f_{2,V}^{Z,\mathrm{ew}} &= -\frac{y_W^2}{32} - \frac{s_W^2\, a_t^2}{16} - \frac{s_W^2\, v_t^2}{16}, & f_{2,A}^{Z,\mathrm{ew}} &= -\frac{1}{8}s_W^2\, a_t v_t .
\end{aligned} \tag{29}$$

The finite one-loop contributions read

[3]The tadpole diagrams do not contribute to the wave function renormalization.

$$
\begin{aligned}
z_{2,V,0}^{H,\mathrm{ew}} &= y_W^2\Bigg[-\frac{1}{64}+\frac{L_H}{32}-\frac{1}{16}y_H^2-\left(\frac{19}{128}-\frac{3L_H}{32}\right)y_H^4-\left(\frac{77}{160}-\frac{3L_H}{8}\right)y_H^6 \\
&\quad -\left(\frac{519}{320}-\frac{21L_H}{16}\right)y_H^8+\mathcal{O}(y_H^{10})\Bigg]\,, \\
z_{2,V,1}^{H,\mathrm{ew}} &= y_W^2\Bigg[\frac{1}{8}-\frac{\pi\sqrt{3}}{32}+\left(\frac{1}{16}-\frac{\pi\sqrt{3}}{48}\right)y_{H,1}+\left(-\frac{1}{8}+\frac{5\pi\sqrt{3}}{288}\right)y_{H,1}^2+\left(-\frac{7}{96}+\frac{\pi\sqrt{3}}{108}\right)y_{H,1}^3 \\
&\quad +\left(-\frac{5}{128}+\frac{5\pi\sqrt{3}}{1296}\right)y_{H,1}^4+\left(-\frac{7}{240}+\frac{5\pi\sqrt{3}}{1944}\right)y_{H,1}^5+\mathcal{O}(y_{H,1}^6)\Bigg]\,, \\
z_{2,V,\infty}^{H,\mathrm{ew}} &= y_W^2\Bigg[\left(\frac{7}{32}+\frac{L_H}{8}\right)-\frac{3\pi}{16}\frac{1}{y_H}+\left(\frac{3}{32}-\frac{3L_H}{16}\right)\frac{1}{y_H^2}+\frac{15\pi}{128}\frac{1}{y_H^3} \\
&\quad +\left(-\frac{7}{64}+\frac{3L_H}{64}\right)\frac{1}{y_H^4}-\frac{21\pi}{2048}\frac{1}{y_H^5}+\frac{1}{160}\frac{1}{y_H^6}+\mathcal{O}\left(\frac{1}{y_H^7}\right)\Bigg]\,, \\
z_{2,V,\infty}^{W,\mathrm{ew}} &= y_W^2\Bigg[-\left(\frac{1}{16}+\frac{L_W}{16}\right)\frac{1}{y_W^2}-\left(\frac{21}{64}+\frac{L_W}{32}\right)\frac{1}{y_W^4}+\left(-\frac{5}{48}+\frac{3L_W}{16}\right)\frac{1}{y_W^6}+\frac{47}{384}\frac{1}{y_W^8} \\
&\quad +\mathcal{O}\left(\frac{1}{y_W^{10}}\right)\Bigg]\,, \\
z_{2,V,\infty}^{Z,\mathrm{ew}} &= y_W^2\Bigg[-\frac{1}{32}-\left(\frac{3}{32}+\frac{L_Z}{16}\right)\frac{1}{y_Z^2}+\frac{5\pi}{64}\frac{1}{y_Z^3}+\left(-\frac{5}{64}+\frac{3L_Z}{64}\right)\frac{1}{y_Z^4}-\frac{7\pi}{512}\frac{1}{y_Z^5}+\frac{1}{96}\frac{1}{y_Z^6} \\
&\quad +\mathcal{O}\left(\frac{1}{y_Z^7}\right)\Bigg]+a_t^2s_W^2\Bigg[\left(\frac{3}{4}+\frac{3L_Z}{8}\right)-\frac{9\pi}{16}\frac{1}{y_Z}+\left(\frac{3}{8}-\frac{L_Z}{2}\right)\frac{1}{y_Z^2}+\frac{35\pi}{128}\frac{1}{y_Z^3} \\
&\quad +\left(-\frac{1}{4}+\frac{3L_Z}{32}\right)\frac{1}{y_Z^4}-\frac{35\pi}{2048}\frac{1}{y_Z^5}+\frac{1}{120}\frac{1}{y_Z^6}+\mathcal{O}\left(\frac{1}{y_Z^7}\right)\Bigg]+v_t^2s_W^2\Bigg[-\left(\frac{1}{4}+\frac{L_Z}{8}\right) \\
&\quad +\frac{3\pi}{16}\frac{1}{y_Z}-\frac{3}{8}\frac{1}{y_Z^2}+\frac{15\pi}{128}\frac{1}{y_Z^3}+\left(-\frac{1}{8}+\frac{3L_Z}{32}\right)\frac{1}{y_Z^4}-\frac{63\pi}{2048}\frac{1}{y_Z^5}+\frac{1}{40}\frac{1}{y_Z^6}+\mathcal{O}\left(\frac{1}{y_Z^7}\right)\Bigg]\,, \\
z_{2,V}^{A,\mathrm{ew}} &= -\frac{4s_W^2}{9}\,, \\
z_{2,A,\infty}^{W,\mathrm{ew}} &= y_W^2\Bigg[\frac{1}{16}-\left(\frac{1}{16}+\frac{L_W}{16}\right)\frac{1}{y_W^2}+\left(-\frac{3}{64}+\frac{5L_W}{32}\right)\frac{1}{y_W^4}+\left(\frac{5}{48}-\frac{L_W}{16}\right)\frac{1}{y_W^6}-\frac{7}{384}\frac{1}{y_W^8} \\
&\quad +\mathcal{O}\left(\frac{1}{y_W^{10}}\right)\Bigg]\,, \\
z_{2,A,\infty}^{Z,\mathrm{ew}} &= a_tv_ts_W^2\Bigg[-\frac{1}{4}+\frac{\pi}{4}\frac{1}{y_Z}+\left(-\frac{1}{8}+\frac{L_Z}{4}\right)\frac{1}{y_Z^2}-\frac{5\pi}{32}\frac{1}{y_Z^3}+\left(\frac{7}{48}-\frac{L_Z}{16}\right)\frac{1}{y_Z^4}+\frac{7\pi}{512}\frac{1}{y_Z^5} \\
&\quad -\frac{1}{120}\frac{1}{y_Z^6}+\mathcal{O}\left(\frac{1}{y_Z^7}\right)\Bigg]\,,
\end{aligned}
\tag{30}
$$

and the scale-dependent terms are given by

$$
\begin{aligned}
z_{2,V}^{(1),H,\mathrm{ew}} &= \frac{y_W^2}{32}\,, & z_{2,V}^{(1),A,\mathrm{ew}} &= \frac{s_W^2}{3}\,,\\
z_{2,V}^{(1),W,\mathrm{ew}} &= \frac{y_W^2}{32} + \frac{1}{16}\,, & z_{2,A}^{(1),W,\mathrm{ew}} &= -\frac{y_W^2}{32} + \frac{1}{16}\,,\\
z_{2,V}^{(1),Z,\mathrm{ew}} &= \frac{y_W^2}{32} + \frac{a_t^2 s_W^2}{16} + \frac{v_t^2 s_W^2}{16}\,, & z_{2,A}^{(1),Z,\mathrm{ew}} &= \frac{a_t v_t s_W^2}{8}\,.
\end{aligned}
\tag{31}
$$

The double-pole parts of the two-loop result are still quite compact and can be cast into the form

$$
\begin{aligned}
g_{2,V}^{H,\mathrm{mix}} &= \frac{3y_W^2}{64}\,, & g_{2,V}^{A,\mathrm{mix}} &= \frac{s_W^2}{4}\,,\\
g_{2,V}^{W,\mathrm{mix}} &= \frac{3y_W^2}{64} + \frac{3}{64}\,, & g_{2,A}^{W,\mathrm{mix}} &= -\frac{(2+\xi)y_W^2}{128} - \frac{1-\xi}{64}\,,\\
g_{2,V}^{Z,\mathrm{mix}} &= \frac{3y_W^2}{64} + \frac{3}{64}s_W^2 a_t^2 + \frac{3}{64}s_W^2 v_t^2\,, & g_{2,A}^{Z,\mathrm{mix}} &= -\frac{a_t v_t s_W^2(1-\xi)}{32}\,,
\end{aligned}
\tag{32}
$$

whereas for the single poles we obtain the expansions[4]

$$
\begin{aligned}
h_{2,V,0}^{H,\mathrm{mix}} &= y_W^2\Bigg[\frac{31}{256} - \frac{3L_H}{128} + \frac{3}{64}y_H^2 + \left(\frac{57}{512} - \frac{9L_H}{128}\right)y_H^4 + \left(\frac{231}{640} - \frac{9L_H}{32}\right)y_H^6\\
&\quad + \left(\frac{1557}{1280} - \frac{63L_H}{64}\right)y_H^8 + \mathcal{O}(y_H^{10})\Bigg]\,,\\
h_{2,V,1}^{H,\mathrm{mix}} &= y_W^2\Bigg[\frac{1}{64} + \frac{3\pi\sqrt{3}}{128} + \left(-\frac{3}{64} + \frac{\pi\sqrt{3}}{64}\right)y_{H,1} + \left(\frac{3}{32} - \frac{5\pi\sqrt{3}}{384}\right)y_{H,1}^2 + \left(\frac{7}{128} - \frac{\pi\sqrt{3}}{144}\right)y_{H,1}^3\\
&\quad + \left(\frac{15}{512} - \frac{5\pi\sqrt{3}}{1728}\right)y_{H,1}^4 + \left(\frac{7}{320} - \frac{5\pi\sqrt{3}}{2592}\right)y_{H,1}^5 + \mathcal{O}(y_{H,1}^6)\Bigg]\,,\\
h_{2,V,\infty}^{H,\mathrm{mix}} &= y_W^2\Bigg[-\left(\frac{7}{128} + \frac{3L_H}{32}\right) + \frac{9\pi}{64}\frac{1}{y_H} + \left(-\frac{9}{128} + \frac{9L_H}{64}\right)\frac{1}{y_H^2} - \frac{45\pi}{512}\frac{1}{y_H^3}\\
&\quad + \left(\frac{21}{256} - \frac{9L_H}{256}\right)\frac{1}{y_H^4} + \frac{63\pi}{8192}\frac{1}{y_H^5} + \mathcal{O}\left(\frac{1}{y_H^6}\right)\Bigg]\,,
\end{aligned}
$$

[4]Note, that the exact expressions for the Higgs boson results are given in Eq. (51).

$$
\begin{aligned}
h_{2,V,\infty}^{W,\text{mix}} &= y_W^2\left[\frac{7}{64}+\left(\frac{17}{128}+\frac{3L_W}{64}\right)\frac{1}{y_W^2}+\left(\frac{63}{256}+\frac{3L_W}{128}\right)\frac{1}{y_W^4}+\left(\frac{5}{64}-\frac{9L_W}{64}\right)\frac{1}{y_W^6}+\mathcal{O}\left(\frac{1}{y_W^8}\right)\right],\\
h_{2,V,\infty}^{Z,\text{mix}} &= y_W^2\left[\frac{17}{128}+\left(\frac{9}{128}+\frac{3L_Z}{64}\right)\frac{1}{y_Z^2}-\frac{15\pi}{256}\frac{1}{y_Z^3}+\left(\frac{15}{256}-\frac{9L_Z}{256}\right)\frac{1}{y_Z^4}+\frac{21\pi}{2048}\frac{1}{y_Z^5}-\frac{1}{128}\frac{1}{y_Z^6}\right.\\
&\left.+\mathcal{O}\left(\frac{1}{y_Z^7}\right)\right]+a_t^2s_W^2\left[-\left(\frac{61}{128}+\frac{9L_Z}{32}\right)+\frac{27\pi}{64}\frac{1}{y_Z}+\left(-\frac{9}{32}+\frac{3L_Z}{8}\right)\frac{1}{y_Z^2}-\frac{105\pi}{512}\frac{1}{y_Z^3}\right.\\
&\left.+\left(\frac{3}{16}-\frac{9L_Z}{128}\right)\frac{1}{y_Z^4}+\mathcal{O}\left(\frac{1}{y_Z^5}\right)\right]\\
&+v_t^2s_W^2\left[\left(\frac{35}{128}+\frac{3L_Z}{32}\right)-\frac{9\pi}{64}\frac{1}{y_Z}+\frac{9}{32}\frac{1}{y_Z^2}-\frac{45\pi}{512}\frac{1}{y_Z^3}+\left(\frac{3}{32}-\frac{9L_Z}{128}\right)\frac{1}{y_Z^4}+\mathcal{O}\left(\frac{1}{y_Z^5}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
h_{2,V}^{A,\text{mix}} &= \frac{17s_W^2}{24},\\
h_{2,A,\infty}^{W,\text{mix}} &= y_W^2\left[-\frac{(3+2\xi)}{64}+\left(\frac{3(-1+2\xi)}{128}-\frac{(1-\xi)L_W}{64}\right)\frac{1}{y_W^2}+\left(-\frac{3(1-\xi)}{256}+\frac{5(1-\xi)L_W}{128}\right)\frac{1}{y_W^4}\right],\\
h_{2,A,\infty}^{Z,\text{mix}} &= a_tv_ts_W^2\left[\frac{(-5+8\xi)}{64}+\frac{(1-\xi)\pi}{16y_Z}+\left(-\frac{1}{32}+\frac{L_Z}{16}\right)\frac{1-\xi}{y_Z^2}-\frac{5(1-\xi)\pi}{128y_Z^3}+\left(\frac{7}{192}-\frac{L_Z}{64}\right)\frac{1-\xi}{y_Z^4}\right].
\end{aligned}
\tag{33}
$$

The scale dependence is ruled by the following coefficients:

$$
\begin{aligned}
h_{2,V}^{(1),H,\text{mix}} &= -\frac{3y_W^2}{32}, & h_{2,V}^{(1),A,\text{mix}} &= -\frac{s_W^2}{2},\\
h_{2,V}^{(1),W,\text{mix}} &= -\frac{3y_W^2}{32}-\frac{3}{32}, & h_{2,A}^{(1),W,\text{mix}} &= \frac{(2+\xi)y_W^2}{64}+\frac{(1-\xi)}{32},\\
h_{2,V}^{(1),Z,\text{mix}} &= -\frac{3y_W^2}{32}-\frac{3a_t^2s_W^2}{32}-\frac{3v_t^2s_W^2}{32}, & h_{2,A}^{(1),Z,\text{mix}} &= \frac{(1-\xi)a_tv_ts_W^2}{16}.
\end{aligned}
\tag{34}
$$

Let us in the following discuss the results for the finite contribution to $Z_2^{\text{OS}}$. The result from the Higgs boson exchange expressed in terms of master integrals is given in Eq. (50) of Appendix B. The expansion terms read

$$
\begin{aligned}
z_{2,V,0}^{H,\text{mix}} = y_W^2\Bigg[&\frac{119}{512}-\frac{29L_H}{256}-\frac{3L_H^2}{256}+\left(\frac{509}{2304}+\frac{L_H}{192}-\frac{3L_H^2}{128}+\frac{13\pi^2}{384}\right)y_H^2\\
&+\left(\frac{7627}{9216}-\frac{13L_H}{512}-\frac{31L_H^2}{128}-\frac{71\pi^2}{768}\right)y_H^4+\left(\frac{739079}{256000}+\frac{39533L_H}{57600}-\frac{2861L_H^2}{3840}-\frac{1219\pi^2}{3840}\right)y_H^6\\
&+\left(\frac{1962407}{230400}+\frac{12253L_H}{3840}-\frac{305L_H^2}{128}-\frac{23\pi^2}{24}\right)y_H^8+\mathcal{O}(y_H^{10})\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
z_{2,V,1}^{H,\mathrm{mix}} = y_W^2 \Bigg[ & -\frac{79}{768} - \frac{Y^\epsilon}{24} + \left(-\frac{\ln 3}{36} + \frac{245}{2304}\right)\pi^2 - \left(\frac{3\ln 3}{8} + \frac{519}{256}\right) S_2 + \frac{\sqrt{3}}{12}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{7\sqrt{3}}{864}\pi^3 \\
& + \left(\frac{73\ln 3}{1152} + \frac{\ln^2 3}{144} - \frac{S_2}{8} - \frac{41}{1152}\right)\pi\sqrt{3} + \frac{2\zeta(3)}{9} \\
& + \left(-\frac{29}{32} - \frac{3Y^\epsilon}{32} + \left(-\frac{\ln 3}{16} + \frac{79}{3456}\right)\pi^2 + \left(-\frac{27\ln 3}{32} + \frac{531}{128}\right) S_2 + \frac{3\sqrt{3}}{16}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{7\sqrt{3}}{384}\pi^3\right. \\
& \left. + \left(-\frac{7\ln 3}{192} + \frac{\ln^2 3}{64} - \frac{9S_2}{32} + \frac{47}{576}\right)\pi\sqrt{3} + \frac{\zeta(3)}{2}\right) y_{H,1} \\
& + \left(-\frac{7}{192} - \frac{Y^\epsilon}{32} - \left(\frac{\ln 3}{48} + \frac{1417}{20736}\right)\pi^2 + \left(-\frac{9\ln 3}{32} + \frac{669}{256}\right) S_2 + \frac{\sqrt{3}}{16}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{7\sqrt{3}}{1152}\pi^3\right. \\
& \left. + \left(\frac{19\ln 3}{1152} + \frac{\ln^2 3}{192} - \frac{3S_2}{32} + \frac{139}{3456}\right)\pi\sqrt{3} + \frac{\zeta(3)}{6}\right) y_{H,1}^2 \\
& + \left(\frac{295}{3456} - \frac{7867\pi^2}{186624} + \frac{865 S_2}{768} + \left(\frac{55\ln 3}{2592} + \frac{113}{15552}\right)\pi\sqrt{3}\right) y_{H,1}^3 \\
& + \left(\frac{277}{6912} - \frac{4367\pi^2}{186624} + \frac{631 S_2}{768} + \left(\frac{29\ln 3}{5184} + \frac{71}{7776}\right)\pi\sqrt{3}\right) y_{H,1}^4 \\
& + \left(\frac{4009}{207360} - \frac{98903\pi^2}{5598720} + \frac{15463 S_2}{23040} + \left(\frac{253\ln 3}{77760} + \frac{4447}{466560}\right)\pi\sqrt{3}\right) y_{H,1}^5 + \mathcal{O}(y_{H,1}^6)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
z_{2,V,\infty}^{H,\mathrm{mix}} = y_W^2 \Bigg[ & \frac{123}{256} - \frac{L_H}{2} + \frac{3L_H^2}{64} + \left(\frac{15}{128} - \frac{\ln 2}{2}\right)\pi^2 + \frac{3\zeta(3)}{4} + \left(\frac{15}{16} - \frac{9L_H}{64} - \frac{9\ln 2}{32}\right)\pi\frac{1}{y_H} \\
& + \left(-\frac{1585}{1152} + \frac{329L_H}{384} - \frac{11L_H^2}{128} + \left(-\frac{7}{192} + \frac{15\ln 2}{32}\right)\pi^2 - \frac{45\zeta(3)}{64}\right)\frac{1}{y_H^2} + \left(-\frac{2053}{2304} + \frac{311L_H}{1536}\right. \\
& \left. + \frac{487\ln 2}{768}\right)\pi\frac{1}{y_H^3} + \left(\frac{29029}{115200} - \frac{271L_H}{1920} + \frac{11L_H^2}{512} - \left(\frac{13}{384} + \frac{3\ln 2}{32}\right)\pi^2 + \frac{9\zeta(3)}{64}\right)\frac{1}{y_H^4} \\
& + \left(\frac{28439}{153600} - \frac{1563L_H}{40960} - \frac{2811\ln 2}{20480}\right)\pi\frac{1}{y_H^5} + \mathcal{O}\left(\frac{1}{y_H^6}\right)\Bigg],
\end{aligned}
$$

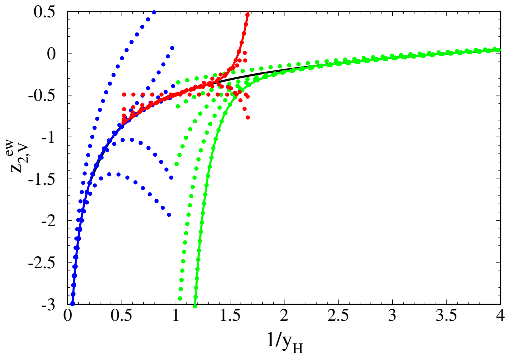
$$
\begin{aligned}
z_{2,V,\infty}^{W,\mathrm{mix}} = y_W^2 \Bigg[ & \frac{79}{256} - \frac{\pi^2}{64} + \frac{3\zeta(3)}{32} + \left(-\frac{3}{256} + \frac{7L_W}{64} - \frac{3L_W^2}{128} - \frac{\pi^2}{128} - \frac{3\zeta(3)}{16}\right)\frac{1}{y_W^2} \\
& + \left(\frac{661}{512} - \frac{L_W}{256} - \frac{3L_W^2}{256} + \frac{5\pi^2}{96} - \frac{15\zeta(3)}{32}\right)\frac{1}{y_W^4} + \left(-\frac{317}{768} + \frac{L_W}{32} + \frac{9L_W^2}{128} + \frac{79\pi^2}{576} + \frac{9\zeta(3)}{16}\right)\frac{1}{y_W^6} \\
& + \mathcal{O}\left(\frac{1}{y_W^8}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
z^{Z,\mathrm{mix}}_{2,V,\infty} = y_W^2 \Bigg[ & \frac{107}{256} - \frac{\pi^2}{128} + \left( \frac{19}{128} + \frac{17L_Z}{128} - \frac{3L_Z^2}{128} + \left( -\frac{5}{64} + \frac{5\ln 2}{32} \right)\pi^2 - \frac{15\zeta(3)}{64} \right) \frac{1}{y_Z^2} \\
& + \left( -\frac{3}{16} + \frac{15L_Z}{256} + \frac{15\ln 2}{128} \right) \pi \frac{1}{y_Z^3} + \left( \frac{757}{4608} - \frac{11L_Z}{96} + \frac{11L_Z^2}{512} + \left( \frac{1}{768} - \frac{3\ln 2}{32} \right)\pi^2 + \frac{9\zeta(3)}{64} \right) \frac{1}{y_Z^4} \\
& + \left( \frac{7837}{46080} - \frac{271L_Z}{6144} - \frac{479\ln 2}{3072} \right) \pi \frac{1}{y_Z^5} + \left( \frac{1091}{76800} + \frac{L_Z}{1280} + \frac{21\pi^2}{2048} \right) \frac{1}{y_Z^6} + \mathcal{O}\left( \frac{1}{y_Z^7} \right) \Bigg] \\
& + a_t^2 s_W^2 \Bigg[ -\frac{815}{256} - \frac{9L_Z}{16} + \frac{9L_Z^2}{64} + \left( \frac{125}{128} - \frac{3\ln 2}{2} \right)\pi^2 + \frac{9\zeta(3)}{4} + \left( \frac{27}{32} - \frac{27L_Z}{64} - \frac{27\ln 2}{32} \right) \pi \frac{1}{y_Z} \\
& + \left( \frac{259}{288} + \frac{67L_Z}{96} - \frac{L_Z^2}{8} + \left( -\frac{59}{192} + \frac{5\ln 2}{4} \right)\pi^2 - \frac{15\zeta(3)}{8} \right) \frac{1}{y_Z^2} + \left( -\frac{337}{192} + \frac{281L_Z}{512} + \frac{457\ln 2}{256} \right) \pi \frac{1}{y_Z^3} \\
& + \left( -\frac{34577}{57600} + \frac{19L_Z}{480} + \frac{11L_Z^2}{256} - \left( \frac{43}{768} + \frac{3\ln 2}{16} \right)\pi^2 + \frac{9\zeta(3)}{32} \right) \frac{1}{y_Z^4} + \mathcal{O}\left( \frac{1}{y_Z^5} \right) \Bigg] \\
& + v_t^2 s_W^2 \Bigg[ \frac{369}{256} + \frac{L_Z}{8} - \frac{3L_Z^2}{64} + \left( -\frac{51}{128} + \frac{\ln 2}{2} \right)\pi^2 - \frac{3\zeta(3)}{4} + \left( -\frac{3}{16} + \frac{9L_Z}{64} + \frac{9\ln 2}{32} \right) \pi \frac{1}{y_Z} \\
& + \left( \frac{217}{144} - \frac{19L_Z}{96} + \frac{L_Z^2}{32} - \frac{49\pi^2}{192} \right) \frac{1}{y_Z^2} + \left( \frac{991}{2304} - \frac{41L_Z}{1536} - \frac{217\ln 2}{768} \right) \pi \frac{1}{y_Z^3} \\
& + \left( -\frac{17873}{57600} + \frac{L_Z}{480} + \frac{11L_Z^2}{256} + \left( \frac{65}{768} - \frac{3\ln 2}{16} \right)\pi^2 + \frac{9\zeta(3)}{32} \right) \frac{1}{y_Z^4} + \mathcal{O}\left( \frac{1}{y_Z^5} \right) \Bigg] , \\
z^{A,\mathrm{mix}}_{2,V} = \Bigg[ & \frac{433}{144} + \left( \frac{8\ln 2}{9} - \frac{49}{72} \right)\pi^2 - \frac{4\zeta(3)}{3} \Bigg] s_W^2 ,
\end{aligned}
$$

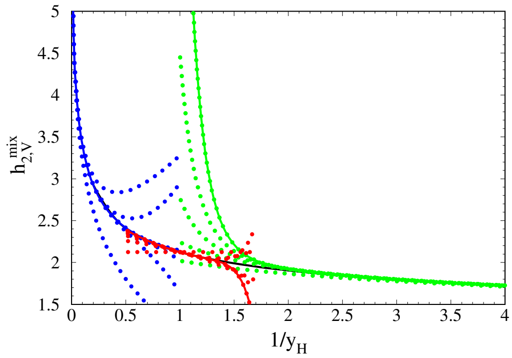
$$
\begin{aligned}
z^{W,\mathrm{mix}}_{2,A,\infty} = y_W^2 \Bigg[ & -\frac{31}{256} - \frac{3\xi}{32} - \frac{\pi^2}{32} + \frac{\xi\pi^2}{256} + \frac{3\zeta(3)}{32} \\
& + \left( \frac{13}{256} + \frac{9\xi}{64} - \frac{3L_W}{64} + \frac{3\xi L_W}{64} + \frac{L_W^2}{128} - \frac{\xi L_W^2}{128} + \frac{13\pi^2}{384} - \frac{7\xi\pi^2}{384} - \frac{3\zeta(3)}{8} \right) \frac{1}{y_W^2} \\
& + \left( -\frac{129}{512} - \frac{3\xi}{512} + \frac{21L_W}{256} - \frac{21\xi L_W}{256} - \frac{5L_W^2}{256} + \frac{5\xi L_W^2}{256} + \frac{11\pi^2}{192} + \frac{5\xi\pi^2}{192} + \frac{15\zeta(3)}{32} \right) \frac{1}{y_W^4} \\
& + \left( \frac{193}{2304} - \frac{29\xi}{1152} - \frac{5L_W}{96} + \frac{\xi L_W}{48} + \frac{L_W^2}{128} - \frac{\xi L_W^2}{128} - \frac{11\pi^2}{144} - \frac{\pi^2\xi}{96} - \frac{3\zeta(3)}{16} \right) \frac{1}{y_W^6} + \mathcal{O}\left( \frac{1}{y_W^8} \right) \Bigg] , \\
z^{Z,\mathrm{mix}}_{2,A,\infty} = a_t v_t s_W^2 \Bigg[ & \frac{9}{128} + \frac{3\xi}{8} - \left( \frac{37}{192} - \frac{\ln 2}{4} \right)\pi^2 + \frac{\pi^2\xi}{192} - \frac{3\zeta(3)}{8} + \left( \frac{3}{16} - \frac{L_Z}{16} - \frac{\ln 2}{8} \right) \pi \frac{1-\xi}{y_Z} \\
& + \left( \frac{1}{8} - \frac{3\xi}{16} - \frac{3L_Z}{32} - \frac{3\xi L_Z}{32} - \frac{(1-\xi)L_Z^2}{32} + \left( \frac{5}{32} - \frac{\ln 2}{2} \right)\pi^2 + \frac{3\zeta(3)}{4} \right) \frac{1}{y_Z^2} \\
& + \left( \frac{595}{1152} + \frac{13\xi}{128} - \frac{49L_Z}{384} - \frac{5\xi L_Z}{128} - \frac{113\ln 2}{192} - \frac{5\xi\ln 2}{64} \right) \pi \frac{1}{y_Z^3} \\
& + \left( -\frac{1}{32} + \frac{31\xi}{576} + \frac{5L_Z}{192} + \frac{\xi L_Z}{96} - \frac{(1+\xi)L_Z^2}{128} + \left( \frac{17}{768} + \frac{\ln 2}{8} \right)\pi^2 - \frac{3\zeta(3)}{16} \right) \frac{1}{y_Z^4} + \mathcal{O}\left( \frac{1}{y_Z^5} \right) \Bigg] . \quad (35)
\end{aligned}
$$

The results covering the scale dependence are given by

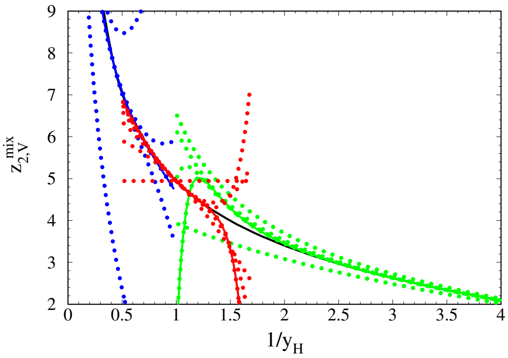
$$
\begin{aligned}
z_{2,V,0}^{(1),H,\text{mix}} &= y_W^2\Bigg[-\frac{31}{128}+\frac{3L_H}{64}-\frac{3y_H^2}{32}+\left(-\frac{57}{256}+\frac{9L_H}{64}\right)y_H^4+\left(-\frac{231}{320}+\frac{9L_H}{16}\right)y_H^6 \\
&\quad +\left(-\frac{1557}{640}+\frac{63L_H}{32}\right)y_H^8+\mathcal{O}(y_H^{10})\Bigg], \\
z_{2,V,1}^{(1),H,\text{mix}} &= y_W^2\Bigg[-\frac{1}{32}-\frac{3\pi\sqrt{3}}{64}+\left(\frac{3}{32}-\frac{\pi\sqrt{3}}{32}\right)y_{H,1}+\left(-\frac{3}{16}+\frac{5\pi\sqrt{3}}{192}\right)y_{H,1}^2 \\
&\quad +\left(-\frac{7}{64}+\frac{\pi\sqrt{3}}{72}\right)y_{H,1}^3+\left(-\frac{15}{256}+\frac{5\pi\sqrt{3}}{864}\right)y_{H,1}^4+\left(-\frac{7}{160}+\frac{5\pi\sqrt{3}}{1296}\right)y_{H,1}^5+\mathcal{O}(y_{H,1}^6)\Bigg], \\
z_{2,V,\infty}^{(1),H,\text{mix}} &= y_W^2\Bigg[\frac{7}{64}+\frac{3L_H}{16}-\frac{9\pi}{32y_H}+\left(\frac{9}{64}-\frac{9L_H}{32}\right)\frac{1}{y_H^2}+\frac{45\pi}{256y_H^3}+\left(-\frac{21}{128}+\frac{9L_H}{128}\right)\frac{1}{y_H^4} \\
&\quad -\frac{63\pi}{4096y_H^5}+\mathcal{O}\left(\frac{1}{y_H^6}\right)\Bigg], \\
z_{2,V,\infty}^{(1),W,\text{mix}} &= y_W^2\Bigg[-\frac{7}{32}-\left(\frac{17}{64}+\frac{3L_W}{32}\right)\frac{1}{y_W^2}-\left(\frac{63}{128}+\frac{3L_W}{64}\right)\frac{1}{y_W^4}+\left(-\frac{5}{32}+\frac{9L_W}{32}\right)\frac{1}{y_W^6} \\
&\quad +\mathcal{O}\left(\frac{1}{y_W^8}\right)\Bigg], \\
z_{2,V,\infty}^{(1),Z,\text{mix}} &= y_W^2\Bigg[-\frac{17}{64}-\left(\frac{9}{64}+\frac{3L_Z}{32}\right)\frac{1}{y_Z^2}+\frac{15\pi}{128y_Z^3}+\left(-\frac{15}{128}+\frac{9L_Z}{128}\right)\frac{1}{y_Z^4}-\frac{21\pi}{1024y_Z^5}+\frac{1}{64y_Z^6} \\
&\quad +\mathcal{O}\left(\frac{1}{y_Z^7}\right)\Bigg]+a_t^2s_W^2\Bigg[\frac{61}{64}+\frac{9L_Z}{16}-\frac{27\pi}{32y_Z}+\left(\frac{9}{16}-\frac{3L_Z}{4}\right)\frac{1}{y_Z^2}+\frac{105\pi}{256y_Z^3} \\
&\quad +\left(-\frac{3}{8}+\frac{9L_Z}{64}\right)\frac{1}{y_Z^4}+\mathcal{O}\left(\frac{1}{y_Z^5}\right)\Bigg]+v_t^2s_W^2\Bigg[-\frac{35}{64}-\frac{3L_Z}{16}+\frac{9\pi}{32y_Z}-\frac{9}{16y_Z^2}+\frac{45\pi}{256y_Z^3} \\
&\quad +\left(-\frac{3}{16}+\frac{9L_Z}{64}\right)\frac{1}{y_Z^4}+\mathcal{O}\left(\frac{1}{y_Z^5}\right)\Bigg], \\
z_{2,V}^{(1),A,\text{mix}} &= -\frac{17s_W^2}{12}, \\
z_{2,A}^{(1),W,\text{mix}} &= y_W^2\Bigg[\frac{3+2\xi}{32}+\left(\frac{3-6\xi}{64}+\frac{1-\xi}{32}L_W\right)\frac{1}{y_W^2}+\left(\frac{3}{128}-\frac{5L_W}{64}\right)\frac{1-\xi}{y_W^4} \\
&\quad +\left(-\frac{5}{96}+\frac{L_W}{32}\right)\frac{1-\xi}{y_W^6}+\mathcal{O}\left(\frac{1}{y_W^8}\right)\Bigg], \\
z_{2,A}^{(1),Z,\text{mix}} &= a_tv_ts_W^2\Bigg[\frac{5}{32}-\frac{\xi}{4}-\frac{(1-\xi)\pi}{8y_Z}+\left(\frac{1}{16}-\frac{L_Z}{8}\right)\frac{1-\xi}{y_Z^2}+\frac{5(1-\xi)\pi}{64} \\
&\quad +\left(-\frac{7}{96}+\frac{L_Z}{32}\right)\frac{1-\xi}{y_Z^4}+\mathcal{O}\left(\frac{1}{y_Z^5}\right)\Bigg],
\end{aligned}
$$




(a)




(b)




(c)

Figure 4: (a) One-loop, (b) $1/\epsilon$ pole and (c) constant part of the two-loop corrections to $Z_2^{V,\mathrm{OS}}$ as a function of $1/y_H = M_H/m_t$. The solid (coloured) lines correspond to the highest available order for each case. The dotted curves show lower-order results and nicely demonstrate the convergence. The exact result, which is plotted (in black) over the whole $1/y_H$ range is only visible in a small gap around $1/y_H \approx 2$.

$$
\begin{aligned}
z_{2,V}^{(2),H,\mathrm{mix}} &= \frac{3y_W^2}{32}\,, & z_{2,V}^{(2),W,\mathrm{mix}} &= \frac{3y_W^2}{32} + \frac{3}{32}\,, \\
z_{2,V}^{(2),Z,\mathrm{mix}} &= \frac{3y_W^2}{32} + \frac{3a_t^2 s_W^2}{32} + \frac{3v_t^2 s_W^2}{32}\,, & z_{2,V}^{(2),A,\mathrm{mix}} &= \frac{s_W^2}{2}\,, \\
z_{2,A}^{(2),W,\mathrm{mix}} &= -\frac{(2+\xi)y_W^2}{64} - \frac{1-\xi}{32}\,, & z_{2,A}^{(2),Z,\mathrm{mix}} &= -\frac{(1-\xi)a_t v_t s_W^2}{16}\,.
\end{aligned}
\tag{36}
$$

In Fig. 4 we discuss our analytic results in numerical form. Fig. 4(a) shows the finite part of the one-loop contribution where the exact result is represented by the solid line and the expansions are plotted as dotted curves. Similarly to $z_m$ (cf. Fig. 3) it can be seen that the dotted lines nicely converge to the exact curve after including successively higher order expansion terms. As one can see, after taking into account the result from the regions $y_H \to 0, 1$ and $\infty$ there remains only a quite small range for $y_H$ ($1/y_H \approx 1.5 \ldots 2.0$) where the (black) solid curve is still visible and the simple expansions fail to provide good approximations. The situation is very similar for the divergent $1/\epsilon$ and finite two-loop contribution which are shown in Figs. 4(b) and 4(c), respectively. Thus we can conclude that it is possible to avoid the use of the quite complicated exact expressions for $Z_2^V$ but to adopt an adequate simple expansion. For Higgs boson masses outside the range $M_H \approx 250 \ldots 300$ GeV one can simply use the corresponding formula from Eq. (35) while for 250 GeV$\leq M_H \leq 300$ GeV a straightforward interpolation provides a sufficiently good approximation.

It is interesting to mention that the result obtained in the gaugeless limit approximates the full result within approximately 20% accuracy for Higgs boson masses between 100 GeV and 800 GeV. For $z_m$ the situation is similar once the tadpole contributions are discarded. In case the latter are included the relative deviation between the full result and the one in the gaugeless limit becomes smaller [9].

# 5 Conclusions

The renormalization constants constitute building blocks for the evaluation of quantum corrections to various processes. In this paper the on-shell top quark mass and wave function renormalization constants have been considered in the SM up to order $\alpha\alpha_s$. The inclusion of electroweak effects introduces a further scale into the problem, as compared to the QCD or QED corrections, which makes the calculation of the integrals significantly more complicated.

We expressed $Z_m^{\rm OS}$ and $Z_2^{V/A,{\rm OS}}$ as a linear combination of a handful master integrals which are known analytically, however, contain quite involved functions. For the complicated two-scale master integrals we applied the powerful method of asymptotic expansion in three different kinematical regions, which leads to power expansions multiplied by simple logarithms. We could reproduce the result for $Z_m^{\rm OS}$ available in the literature. The expression for $Z_2^{V/A,{\rm OS}}$ is new and constitutes a building block, e.g., in the mixed electroweak/QCD corrections for top quark pair production at threshold. We checked that both for $Z_m^{\rm OS}$ and $Z_2^{V/A,{\rm OS}}$ the simple expansions agree very well with the exact result. In particular, it has been shown that the expansion for large top quark mass leads to compact formulae which approximate the exact results quite precisely — almost up to the point $m_t = M$ with $M = M_W, M_Z$ of $M_H$. As far as the Higgs boson mass dependence is concerned, also the expansions around $m_t = M_H$ and $m_t \ll M_H$ have been considered.

## Acknowledgements

We would like the thank M. Faisst for useful discussions and for providing expressions which we could use for cross checks. We thank C. Anastasiou for discussions about `AIR`, A. Onishchenko, V. Smirnov and O. Veretin for carefully reading the paper and helpful suggestions and J. Piclum for discussions on the tadpole contributions. This work was supported by the "Impuls- und Vernetzungsfonds" of the Helmholtz Association, contract

number VH-NG-008 and the SFB/TR 9.

# A Analytic results for $z_m^{H,\mathrm{ew}}$ and $z_m^{H,\mathrm{mix}}$

The exact dependence on $y_H$ of the one-loop electroweak contribution reads

$$z_m^{H,\mathrm{ew}} = \frac{y_W^2}{16}\left[\frac{3}{2\epsilon} + \frac{B_p(0,1)}{2m_t^2} - \frac{B_p(1,0)}{2m_t^2} - 2\left(1-\frac{1}{4y_H^2}\right)B_p(1,1)\right], \tag{37}$$

where the function $B_p(n_1,n_2)$ corresponds to the one-loop on-shell integral with a internal top quark and Higgs boson line

$$B_p(n_1,n_2) = \int \frac{\mathrm{d}^d k}{i\pi^{d/2}} \frac{e^{\epsilon\gamma_E}}{(k^2-M_H^2)^{n_1}(k^2+2kq)^{n_2}}. \tag{38}$$

The special cases needed in this paper are given by

$$\begin{aligned}
B_p(0,1) &= m^2\left(\frac{\mu^2}{m^2}\right)^\epsilon\left(\frac{1}{\epsilon}+1+\epsilon\left(1+\frac{\pi^2}{12}\right)+\epsilon^2\left(1+\frac{\pi^2}{12}-\frac{\zeta(3)}{3}\right)\right), \\
B_p(1,0) &= B_p(0,1)\Big|_{m\to M}, \\
B_p(1,1) &= \frac{\left(\frac{\mu^2}{M^2}\right)^\epsilon-(1-2y_H^2)\left(\frac{\mu^2}{m^2}\right)^\epsilon}{2y_H^2\epsilon}\left[1+2\epsilon+\left(4+\frac{\pi^2}{12}\right)\epsilon^2+\left(8+\frac{\pi^2}{6}-\frac{\zeta(3)}{3}\right)\epsilon^3\right] \\
&\quad+\left(\frac{y_H^4}{(4y_H^2-1)}\right)^\epsilon\left(\frac{\mu^2}{m^2}\right)^\epsilon\left\{\left[1+2\epsilon+\left(4+\frac{\pi^2}{12}\right)\epsilon^2\right]\Psi_1+2\epsilon\,(1+2\epsilon)\,\Psi_2+2\epsilon^2\Psi_3\right\},
\end{aligned} \tag{39}$$

where we have included the order $\epsilon$ terms which are needed for the two-loop expressions, and $\Psi_i$ is a shorthand notation for ($y=m/M$):

$$\Psi_i = \frac{1}{y}\sqrt{4-\frac{1}{y^2}}\left(\mathrm{Ls}_i(\pi)-\frac{1}{2}\mathrm{Ls}_i\left(t_1\right)-\frac{1}{2}\mathrm{Ls}_i\left(t_2\right)\right), \tag{40}$$

with

$$\begin{aligned}
t_1 &= 2\arccos\left(\frac{1}{2y}\right), \\
t_2 &= 2\arccos\left(1-\frac{1}{2y^2}\right).
\end{aligned} \tag{41}$$

The function $\mathrm{Ls}_i(z)$ is defined through

$$\mathrm{Ls}_i(z) = -\int_0^z \mathrm{d}x \ln^{i-1}\left[2\sin\left(\frac{x}{2}\right)\right]. \tag{42}$$

At two-loop order it is convenient to write the finite quantity $z_m^{H,\text{mix}}$ in the form

$$z_m^{H,\text{mix}} = a_m^{H,\text{mix}} + b_m^{H,\text{mix}}, \tag{43}$$

where $a_m^{H,\text{mix}}$ corresponds to the generic two-loop result and $b_m^{H,\text{mix}}$ origins from counterterm contributions and products of one-loop results. We obtain

$$\begin{aligned}
a_m^{H,\text{mix}} &= -\frac{\overline{y}_W^2}{64}\Bigg[-\frac{(2-\epsilon+3\epsilon^2+4\epsilon^3)(-1+4\overline{y}_H^2)}{2\epsilon\overline{y}_H^2}\frac{H_2}{\overline{m}_t^2}+(-1+2\epsilon)\frac{H_4}{\overline{m}_t^2}\\
&\quad-\frac{(1+\epsilon)(3-3\epsilon+8\epsilon^2)}{\epsilon}\frac{H_1}{\overline{m}_t^4}-\frac{(2+\epsilon+4\epsilon^2+8\epsilon^3)(-1+4\overline{y}_H^2)}{\epsilon\overline{y}_H^2}H_5\\
&\quad+\frac{(-2+\epsilon-3\epsilon^2-4\epsilon^3)+\overline{y}_H^2\left(4+10\epsilon-2\epsilon^2+20\epsilon^3\right)}{2\epsilon\overline{y}_H^2}\frac{H_3}{\overline{m}_t^2}\\
&\quad-\frac{(-4-\epsilon-7\epsilon^2-12\epsilon^3)+\overline{y}_H^2\left(12\epsilon-8\epsilon^2+12\epsilon^3\right)}{2\epsilon}\frac{Y_2}{\overline{m}_t^4}\\
&\quad+\frac{(-4-\epsilon-7\epsilon^2-12\epsilon^3)+\overline{y}_H^2\left(8+14\epsilon+6\epsilon^2+36\epsilon^3\right)}{2\epsilon\overline{y}_H^2}\frac{Y_3}{\overline{m}_t^2}\\
&\quad+\frac{2(-1+\epsilon)^2}{\epsilon}\frac{Y_1}{\overline{m}_t^2}-\frac{-2+4\epsilon-2\epsilon^2+\overline{y}_H^2\left(4-16\epsilon+8\epsilon^2\right)}{\epsilon\overline{y}_H^2}Y_4\Bigg],\\
b_m^{H,\text{mix}} &= \frac{y_W^2}{4}\Bigg\{\frac{-18-3\epsilon+15\epsilon^2}{32\epsilon^2}-\frac{3}{32}\overline{L}_{\mu t}+\Bigg[-\frac{3(-1+2\overline{y}_H^2)}{32}\epsilon\overline{L}_{\mu t}^2+\frac{(3+\epsilon)(-1+2\overline{y}_H^2)}{16}\overline{L}_{\mu t}\\
&\quad-\frac{-18-16\epsilon-16\epsilon^2-\epsilon^2\pi^2+\overline{y}_H^2\left(24+8\epsilon+32\epsilon^2+2\epsilon^2\pi^2\right)}{64\epsilon}\Bigg]\frac{B_p(1,0)}{\overline{m}_t^2}\\
&\quad+\Bigg[\frac{3(-1+3\overline{y}_H^2)}{32\overline{y}_H^2}\epsilon\overline{L}_{\mu t}^2+\frac{3+\epsilon+\overline{y}_H^2\left(-9+6\epsilon\right)}{16\overline{y}_H^2}\overline{L}_{\mu t}\\
&\quad+\frac{-18-16\epsilon-16\epsilon^2-\epsilon^2\pi^2+\overline{y}_H^2\left(108+24\epsilon+3\epsilon^2\pi^2\right)}{64\epsilon\overline{y}_H^2}\Bigg]B_p(1,1)\\
&\quad+(-1+4\overline{y}_H^2)\Bigg[-\frac{3}{64\overline{y}_H^2}\epsilon\overline{L}_{\mu t}^2+\frac{(3+4\epsilon)\overline{L}_{\mu t}}{32\overline{y}_H^2}-\frac{(16+32\epsilon+\epsilon\pi^2)}{128\overline{y}_H^2}\Bigg]\frac{y_H\mathrm{d}}{\mathrm{d}y_H}B_p(1,1)\Bigg\}.
\end{aligned}\tag{44}$$

with $\overline{y}_W$, $\overline{y}_H$ and $\overline{L}_{\mu t}$ are defined in Eq. (16). The poles which are still present in $a_m^{H,\text{mix}}$ and $b_m^{H,\text{mix}}$ cancel in the proper sum.

Finally, the one- and two-loop coefficients determining the $\mu$ dependence are given by

$$\begin{aligned}
z_m^{(1),H,\text{ew}} &= \frac{3y_W^2}{32},\\
z_m^{(1),H,\text{mix}} &= y_W^2\Bigg[-\frac{3\sqrt{1-4\overline{y}_H^2}\ln\left(\overline{K}_H\right)}{256\overline{y}_H^4}(8\overline{y}_H^2+1)-\frac{3\overline{L}_H}{256\overline{y}_H^4}(6\overline{y}_H^2+1)\\
&\quad+\frac{3}{128\overline{y}_H^2}(21\overline{y}_H^2+1)\Bigg],\\
z_m^{(2),H,\text{mix}} &= -\frac{9y_W^2}{64},
\end{aligned}\tag{45}$$

where $\overline{K}_H = \frac{1-\sqrt{1-4\overline{y}_H^2}}{1+\sqrt{1-4\overline{y}_H^2}}$.

# B Analytic results for $Z_{2,V}^{H,\mathrm{ew}}$ and $Z_{2,V}^{H,\mathrm{mix}}$

The one-loop contribution from the Higgs boson exchange diagram to $Z_{2,V}^{H,\mathrm{ew}}$ reads

$$Z_{2,V}^{H,\mathrm{ew}} = -\frac{y_W^2}{32}\left[4(1-\epsilon)y_H^2\frac{B_p(1,0)}{m_t^2}+\frac{3-2\epsilon}{y_H^2}B_p(1,1)\right.$$
$$\left.+\frac{3-2\epsilon}{m_t^2}\left(B_p(0,1)-B_p(1,0)-2m_t^2B_p(1,1)\right)\right], \tag{46}$$

which — after inserting the results for $B_p(n_1,n_2)$ from Eq. (39), with $m = m_t$ and $M = M_H$, and expanding in $\epsilon$ — can be cast in the form

$$f_{2,V}^{H,\mathrm{ew}} = -\frac{y_W^2}{32},$$
$$z_{2,V}^{H,\mathrm{ew}} = -y_W^2\left[\frac{3\sqrt{1-4y_H^2}\ln(K_H)}{64y_H^4}(2y_H^2-1)+\frac{L_H}{64y_H^4}(-8y_H^4+12y_H^2-3)\right.$$
$$\left.+\frac{(-7y_H^2+3)}{32y_H^2}\right], \tag{47}$$

where $y_H$ and $L_H$ are defined in Eq. (28) and $K_H = \frac{1-\sqrt{1-4y_H^2}}{1+\sqrt{1-4y_H^2}}$.

At two-loop order we again split $Z_{2,V}^{H,\mathrm{OS}}$ into two parts,

$$Z_{2,V}^{H,\mathrm{mix}} = A_2^{H,\mathrm{mix}} + B_2^{H,\mathrm{mix}}, \tag{48}$$

where $A_2^{H,\mathrm{mix}}$ corresponds to the generic two-loop result and $B_2^{H,\mathrm{mix}}$ origins from counterterm contributions and squared one-loop results. We obtain

$$
\begin{aligned}
A_2^{H,\mathrm{mix}} \;=\; & -\frac{y_W^2}{64}\Bigg\{-\frac{(9-\epsilon+12\epsilon^2+26\epsilon^3)-(30+28\epsilon+22\epsilon^2+100\epsilon^3)y_H^2+(56\epsilon+16\epsilon^2-72\epsilon^3)y_H^4}{(4y_H^2-1)\epsilon}\frac{H_1}{m_t^4} \\
& +\frac{(-6+5\epsilon-13\epsilon^2+2\epsilon^3)+(14-22\epsilon+96\epsilon^2-160\epsilon^3)y_H^2}{2\epsilon y_H^2}\frac{H_2}{m_t^2} \\
& +\Bigg[\frac{(-6+5\epsilon-13\epsilon^2+2\epsilon^3)+(30+58\epsilon^2-16\epsilon^3)y_H^2}{2(4y_H^2-1)\epsilon y_H^2} \\
& +\frac{(-16-44\epsilon-12\epsilon^2+56\epsilon^3)y_H^2+(32\epsilon-160\epsilon^3)y_H^4}{(4y_H^2-1)\epsilon}\Bigg]\frac{H_3}{m_t^2} \\
& -\frac{-2+20\epsilon^2-54\epsilon^3}{\epsilon}\frac{H_4}{m_t^2}+\frac{-(6+12\epsilon^2+14\epsilon^3)+(14-12\epsilon+72\epsilon^2-72\epsilon^3)y_H^2}{\epsilon y_H^2}H_5 \\
& +\frac{(6-16\epsilon+14\epsilon^2-4\epsilon^3)+(-20+58\epsilon-56\epsilon^2+16\epsilon^3)y_H^2}{(4y_H^2-1)\epsilon}\frac{Y_1}{m_t^2} \\
& -\frac{(-12-5\epsilon-13\epsilon^2-46\epsilon^3)+(40+88\epsilon-68\epsilon^2+380\epsilon^3)y_H^2+(-200\epsilon+352\epsilon^2-744\epsilon^3)y_H^4}{2(4y_H^2-1)\epsilon}\frac{Y_2}{m_t^4} \\
& +\Bigg[\frac{-(12+3\epsilon+11\epsilon^2+54\epsilon^3)+(60+54\epsilon-6\epsilon^2+348\epsilon^3)y_H^2}{2(4y_H^2-1)\epsilon y_H^2} \\
& +\frac{(-32-64\epsilon+68\epsilon^2-236\epsilon^3)y_H^2+(24\epsilon-40\epsilon^2+16\epsilon^3)y_H^4}{(4y_H^2-1)\epsilon}\Bigg]\frac{Y_3}{m_t^2} \\
& -\frac{(-6+16\epsilon-14\epsilon^2+4\epsilon^3)+(30-96\epsilon+88\epsilon^2-24\epsilon^3)y_H^2+(-32+144\epsilon-128\epsilon^2+32\epsilon^3)y_H^4}{(4y_H^2-1)\epsilon y_H^2}Y_4\Bigg\},
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
B_2^{H,\mathrm{mix}} \;=\; & \frac{y_W^2}{4}\Bigg\{\Bigg[\frac{21+10\epsilon-(108+36\epsilon)y_H^2+(24+8\epsilon)y_H^4}{16(4y_H^2-1)}L_{\mu t}-\frac{3\epsilon(7-36y_H^2+8y_H^4)}{32(4y_H^2-1)}L_{\mu t}^2 \\
& +\frac{-(21+10\epsilon+38\epsilon^2)+(108+36\epsilon+180\epsilon^2)y_H^2-(24+8\epsilon+32\epsilon^2)y_H^4}{16(4y_H^2-1)\epsilon}+\frac{(-7+36y_H^2-8y_H^4)\epsilon\pi^2}{64(4y_H^2-1)}\Bigg] \\
& \times\frac{B_p(0,1)}{m_t^2}+\Bigg[\frac{-(21+10\epsilon)+(132+44\epsilon)y_H^2-(156+28\epsilon)y_H^4}{16(4y_H^2-1)}L_{\mu t}-\frac{3\epsilon\,(-7+44y_H^2-52y_H^4)}{32(4y_H^2-1)}L_{\mu t}^2 \\
& +\frac{21+10\epsilon+38\epsilon^2-(132+44\epsilon+212\epsilon^2)y_H^2+(156+28\epsilon+200\epsilon^2)y_H^4}{16\epsilon(4y_H^2-1)} \\
& -\frac{(-7+44y_H^2-52y_H^4)\epsilon\pi^2}{64(4y_H^2-1)}\Bigg]\frac{B_p(1,0)}{m_t^2}+\Bigg[\frac{3\epsilon\,(-7+52y_H^2-92y_H^4+8y_H^6)}{32(4y_H^2-1)y_H^2}L_{\mu t}^2 \\
& +\frac{21+10\epsilon-(156+52\epsilon)y_H^2+(276+68\epsilon)y_H^4+(-24+16\epsilon)y_H^6}{16(4y_H^2-1)y_H^2}L_{\mu t} \\
& +\frac{-(21+10\epsilon+38\epsilon^2)+(156+52\epsilon+268\epsilon^2)y_H^2-(276+68\epsilon+456\epsilon^2)y_H^4+(24-16\epsilon)y_H^6}{16(4y_H^2-1)\epsilon y_H^2} \\
& +\frac{(-7+52y_H^2-92y_H^4+8y_H^6)\epsilon\pi^2}{64(4y_H^2-1)y_H^2}\Bigg]B_p(1,1)\Bigg\}.
\end{aligned}
\tag{50}
$$

Note that the individual contributions $A_2^{H,\mathrm{mix}}$ and $B_2^{H,\mathrm{mix}}$ still depend on the QCD gauge parameter $\xi$ which cancels in the proper sum. For simplicity these terms have already been omitted in Eqs. (49) and (50).

From the formulae (49) and (50) it is straightforward to extract exact expressions for the pole parts which are given by

$$
\begin{aligned}
g_{2,V}^{\rm mix} &= \frac{3y_W^2}{64}\,, \\
h_{2,V}^{H,\rm mix} &= y_W^2\Bigg[-\frac{\ln(K_H)}{y_H^4}\sqrt{1-4y_H^2}\left(\frac{7}{256}+\frac{1}{128(1-4y_H^2)}\right)-\frac{9L_H}{256y_H^4} \\
&\quad +\frac{\ln(K_H)}{y_H^2}\sqrt{1-4y_H^2}\left(\frac{11}{128}+\frac{1}{64(1-4y_H^2)}\right)+\frac{9L_H}{64y_H^2}+\frac{9}{128y_H^2} \\
&\quad -\frac{3}{32}L_H+\frac{\ln(K_H)}{16\sqrt{1-4y_H^2}}-\frac{7}{128}\Bigg]\,,
\end{aligned}
\tag{51}
$$

where $K_H$ is defined after Eq. (47).

For completeness we want to provide the exact result covering the renormalization scale dependence for $Z_2^{V,H,\rm OS}$. The corresponding coefficients are defined in analogy to the ones for $z_m$ in Eq. (16) and read

$$
\begin{aligned}
z_{2,V}^{(1),H,\rm ew} &= \frac{y_W^2}{32}\,, \\
h_{2,V}^{(1),H,\rm mix} &= -\frac{3y_W^2}{32}\,, \\
z_{2,V}^{(1),H,\rm mix} &= y_W^2\Bigg[\frac{\sqrt{1-4y_H^2}\ln(K_H)}{128y_H^4}(-22y_H^2+7)-\frac{\ln(K_H)}{64y_H^4\sqrt{1-4y_H^2}}(8y_H^4+2y_H^2-1) \\
&\quad +\frac{3L_H}{128y_H^4}(8y_H^4-12y_H^2+3)+\frac{(7y_H^2-9)}{64y_H^2}\Bigg]\,, \\
z_{2,V}^{(2),\rm mix} &= \frac{3y_W^2}{32}\,.
\end{aligned}
\tag{52}
$$

# C Master integrals

In this Section we discuss the master integrals occuring in our calculation. The master integrals needed for the neutral boson exchange which reduce to products of one-loop integrals, to two-loop one-scale integrals or to two-loop vacuum integrals are given by (see, e.g., Refs. [32, 33])

$$
\begin{aligned}
H_1 &= m^4\left(\frac{\mu^2}{m^2}\right)^{2\epsilon}\left[\frac{1}{\epsilon^2}+\frac{2}{\epsilon}+3+\frac{\pi^2}{6}+\epsilon\left(4-\frac{2}{3}\zeta(3)+\frac{\pi^2}{3}\right)+\mathcal{O}\left(\epsilon^2\right)\right]\,,\\
H_2 &= M^2\left(\frac{\mu^2}{M^2}\right)^{2\epsilon}\left[\frac{1}{2\epsilon^2}+\frac{3}{2\epsilon}+\frac{7}{2}+\frac{\pi^2}{4}+\epsilon\left(\frac{15}{2}-\frac{4}{3}\zeta(3)+\frac{3\pi^2}{4}\right)+\mathcal{O}\left(\epsilon^2\right)\right]\,,\\
H_3 &= m^2\left(\frac{\mu^2}{m^2}\right)^{2\epsilon}\left\{\frac{1}{\epsilon^2}\left(1+\frac{1}{2y^2}\right)+\frac{1}{\epsilon}\left(3+\frac{3}{2y^2}+\frac{1}{y^2}\ln\left(y^2\right)\right)\right.\\
&\quad+\left(7+\frac{\pi^2}{6}\right)\left(1+\frac{1}{2y^2}\right)+\frac{3}{y^2}\ln\left(y^2\right)+\frac{1}{2y^2}\ln^2\left(y^2\right)+2\Omega_2\\
&\quad+\epsilon\left[\left(15+\frac{\pi^2}{2}-\frac{2\zeta(3)}{3}\right)\left(1+\frac{1}{2y^2}\right)+\left(7+\frac{\pi^2}{6}\right)\frac{1}{y^2}\ln\left(y^2\right)+\frac{3}{2y^2}\ln^2\left(y^2\right)\right.\\
&\quad\left.\left.+\frac{1}{6y^2}\ln^3\left(y^2\right)+\left(6-2\ln\left(\frac{4}{y^2}-\frac{1}{y^4}\right)\right)\Omega_2+2\Omega_3\right]+\mathcal{O}\left(\epsilon^2\right)\right\}\,,
\end{aligned}
$$

$$
\begin{aligned}
Y_1 &= m^2\left(\frac{\mu^2}{m^2}\right)^{2\epsilon}\left[\frac{3}{2\epsilon^2}+\frac{17}{4\epsilon}+\frac{59}{8}+\frac{\pi^2}{4}+\epsilon\left(\frac{65}{16}-\zeta(3)+\frac{49\pi^2}{24}\right)+\mathcal{O}\left(\epsilon^2\right)\right]\,,\\
Y_2 &= y^2M^4\left(\frac{\mu^2}{m^2}\right)^{-\epsilon}\left(\frac{\mu^2}{M^2}\right)^{\epsilon}\frac{H_1}{m^4}\,,\\
Y_3 &= B_p(0,1)B_p(1,1)\,, \qquad (53)
\end{aligned}
$$

with $y = m/M$ and

$$
\Omega_i \;=\; \Psi_i - \frac{1}{2y}\sqrt{4-\frac{1}{y^2}}\mathrm{Ls}_i\left(t_1\right)\,, \qquad (54)
$$

where $\Psi_i$, $t_1$ and $t_2$ are defined in Eqs. (40) and (41).

As already mentioned in the main text, analytical results for the complicated two-scale master integrals can be found in Ref. [7]. We refrain from repeating them here. Instead we perform expansions in the limits $y \to 0$, $y \to 1$ and $y \to \infty$ which we derived independently with the help of asymptotic expansions [34]. In many phenomenological applications it is advantageous to use the handy approximation formulae in favour of the complicated exact expressions.

The expansions in the three different limits require a strategy of its own. In the case of a large boson mass $M$ one obtains, next to on-shell, also vacuum integrals up to two loops (both for charged and neutral boson exchange) which are, e.g., implemented in MATAD [35]. The subdiagrams contributing in this limit can be obtained in a completely automated way with the help of `exp` [11, 12].

In the limit $y \to 1$, the case of a neutral boson exchange reduces to a simple Taylor expansion. The resulting integrals are well studied within QCD and documented in a program code [32]. The expansion for $y \to 1$ of the diagrams involving a $W$ boson is more complicated, which is due to the appearance of additional massless particles, the bottom quarks. We did not perform the calculation of the diagrams in this limit since for the practical applications only the limit $m_t \gg M_W$ is needed.

Finally for the case $y \to \infty$, which is the limit of main interest, a careful inspection of the regions [36, 37] contributing to the integral under consideration is in order.

In the following we present a pedagogical example which illustrates the procedure in more detail. Let us consider the master integral $Y_4$ defined through

$$Y_4 = \frac{e^{2\epsilon\gamma_E}}{(i\pi^{d/2})^2} \int \frac{\mathrm{d}^d k \mathrm{d}^d l}{(k^2+2kq)\,(l^2-2lq)\,((k-l)^2+2q(k-l))\,(l^2-M^2)}\,. \tag{55}$$

In the limit $y \to 0$, i.e. $m \ll M$ one has to consider the cases (i) $|\vec{k}| \sim |\vec{l}| \sim M$, (ii) $|\vec{k}| \sim m$, $|\vec{l}| \sim M$, (iii) $|\vec{k}| \sim |\vec{l}| \sim m$, and (iv) $|\vec{k}| \sim |\vec{l}| \sim M$ but with $|\vec{k} - \vec{l}| \sim m$, which we denote as hard-hard (HH), soft-hard (SH), soft-soft (SS) and hard-soft (HS) region. In each region one has to perform a simple Taylor expansion of each propagator in the small quantities. E.g., if $\vec{k}$ is hard the propagator $1/(k^2+2kq)$ is expanded in $2kq/k^2$ since we have $q^2 = m^2$. After adding the contributions from all regions one finally obtains the result for $Y_4$

$$Y_{4,0} = Y_4(\mathrm{HH}) + Y_4(\mathrm{SH}) + Y_4(\mathrm{SS}) + Y_4(\mathrm{HS})\,, \tag{56}$$

where the integrals in the individual region have the form

$$\begin{aligned}
Y_4(\mathrm{HH}) &= \sum_{n_1,\ldots,n_5} \mathcal{C}^{\mathrm{HH}}_{n_1\ldots n_5} \int \frac{\mathrm{d}^d k \mathrm{d}^d l\ (-2kq)^{n_1}\,(-2lq)^{n_2}}{(k^2)^{n_3}\,((k-l)^2-M^2)^{n_4}\,(l^2)^{n_5}}\,,\\
Y_4(\mathrm{SH/HS}) &= \sum_{n_1,\ldots,n_5} \mathcal{C}^{\mathrm{SH/HS}}_{n_1\ldots n_5} \int \frac{\mathrm{d}^d k \mathrm{d}^d l\ (-2kq)^{n_1}\,(-2kl)^{n_2}\,(-2lq)^{n_3}}{(k^2-M^2)^{n_4}\,(l^2+2lq)^{n_5}}\,,\\
Y_4(\mathrm{SS}) &= \sum_{n_1,\ldots,n_5} \mathcal{C}^{\mathrm{SS}}_{n_1\ldots n_5} \int \frac{\mathrm{d}^d k \mathrm{d}^d l\ (k^2)^{n_1}\,(l^2)^{n_2}}{(k^2+2kq)^{n_3}\,(l^2-2lq)^{n_4}((k-l)^2+2q(k-l))^{n_5}}\,.
\end{aligned} \tag{57}$$

We have used partial fractioning and rearranged the different propagators in a convenient way. The indexes $n_1$ to $n_5$ run over appropriate integer values, and the $\mathcal{C}^{ab}_{n_1\ldots n_5}$ are shorthand notations for a set of functions depending on both scales $M$ and $m$ and the dimension $d$. The first four terms of $Y_4$ expanded for $y \to 0$ read

$$\begin{aligned}
Y_{4,0} = \left(\frac{\mu^2}{m^2}\right)^{2\epsilon} \Bigg\{ & \frac{1}{2\epsilon^2} + \left[\frac{3}{2} + \ln(y^2) + \left(\frac{1}{2} + \ln(y^2)\right) y^2 + \left(\frac{5}{3} + 2\ln(y^2)\right) y^4 \right.\\
& \left. + \left(\frac{59}{12} + 5\ln(y^2)\right) y^6 \right] \frac{1}{\epsilon} + \frac{7}{2} + \frac{\pi^2}{4} + 3\ln(y^2) + \ln^2(y^2) + \\
& \left(2 - \frac{\pi^2}{6} + \frac{11\ln(y^2)}{2}\right) y^2 + \left(\frac{85}{9} - \frac{\pi^2}{3} + \frac{26\ln(y^2)}{3}\right) y^4\\
& + \left(\frac{244}{9} - \frac{5\pi^2}{6} + \frac{241\log(y^2)}{12}\right) y^6 \Bigg\} + \mathcal{O}(y^8)\,,
\end{aligned} \tag{58}$$

where we refrain from listing the contributions of order $\epsilon$.

In the region $y \to \infty$, where $M \ll m$ a straightforward inspection reveals only two different ranges for the integration momenta $\vec{k}$ and $\vec{l}$. In the first one (hh) both momenta are of order $m$, which now is hard; for the second one (hs) $\vec{k}$ is hard and $\vec{l}$ is soft, i.e. of order $M$. Thus $Y_4$ can be written as

$$Y_4 = Y_4(\mathrm{hh}) + Y_4(\mathrm{hs})\,, \tag{59}$$

where the following integrals occur

$$\begin{aligned} Y_4(\mathrm{hh}) &= \sum_{n_1,\ldots,n_5} \mathbb{C}^{\mathrm{hh}}_{n_1\ldots n_5} \int \frac{\mathrm{d}^dk\mathrm{d}^dl\,(M^2)^{n_5}}{(k^2+2kq)^{n_1}\,(l^2-2lq)^{n_2}\,((k-l)^2+2q(k-l))^{n_3}\,(l^2)^{n_4}}\,, \\ Y_4(\mathrm{hs}) &= \sum_{n_1,\ldots,n_5} \mathbb{C}^{\mathrm{hs}}_{n_1\ldots n_5} \int \frac{\mathrm{d}^dk\mathrm{d}^dl\;(2kl)^{n_1}\,(-2lq)^{n_2}}{(k^2+2kq)^{n_3}\,(k^2)^{n_4}\,(l^2-M^2)^{n_5}}\,. \end{aligned} \tag{60}$$

Note that the collection of integrals in $Y_4$(hh) and the ones in $Y_4$(SS) are equivalent, and, at the same time, equal to the ones one would need to solve in order to get an arbitrary two loop $\mathcal{O}(\alpha_s^2)$ strong QCD correction. The first seven terms of $Y_4$ expanded for $y \to \infty$ read

$$\begin{aligned} Y_{4,\infty} &= \left(\frac{\mu^2}{m^2}\right)^{2\epsilon} \left\{ \frac{1}{2\epsilon^2} + \left[\frac{5}{2} - \frac{\pi}{y} + \left(1 + \frac{\ln(y^2)}{2}\right)\frac{1}{y^2} + \frac{\pi}{8y^3} - \frac{1}{12y^4} + \frac{\pi}{128y^5} \right.\right. \\ &\left. -\frac{1}{120y^6}\right]\frac{1}{\epsilon} + \frac{19}{2} - \frac{7\pi^2}{12} + \left(-2 + 2\ln 2 - \ln(y^2)\right)\frac{\pi}{y} \\ &+ \left(-3 + \frac{\pi^2}{6} + \frac{\ln^2(y^2)}{4}\right)\frac{1}{y^2} - \left(\frac{1}{6} + \frac{\ln 2}{4} - \frac{\ln(y^2)}{8}\right)\frac{\pi}{y^3} + \left(\frac{2}{3} + \frac{\ln(y^2)}{6}\right)\frac{1}{y^4} \\ &\left. + \left(\frac{19}{960} - \frac{\ln 2}{64} + \frac{\ln(y^2)}{128}\right)\frac{\pi}{y^5} + \left(\frac{131}{3600} + \frac{\ln(y^2)}{60}\right)\frac{1}{y^6} + \mathcal{O}\left(\frac{1}{y^7}\right)\right\}\,. \end{aligned} \tag{61}$$

Finally, let us consider limit the $y \to 1$. Here the asymptotic expansion leads to only one region which corresponds to a naive Taylor expansion in the quantity $\Delta = (m^2 - M^2)$. This leads to the integrals of the type

$$Y_4 = \sum_{n_1} \mathbf{C}_{n_1} \int \frac{\mathrm{d}^dk\mathrm{d}^dl\,(\Delta)^{n_1-1}}{(k^2+2kq)\,(l^2-2lq)\,((k-l)^2+2q(k-l))\,(l^2-m^2)^{n_1}}\,. \tag{62}$$

The first six terms in the expansion are given by

$$
\begin{aligned}
Y_{4,1} &= \left(\frac{\mu^2}{m^2}\right)^{2\epsilon}\left\{\frac{1}{2\epsilon^2}+\left[\frac{5}{2}-\frac{\pi\sqrt{3}}{3}+\frac{\pi\sqrt{3}}{9}y_1+\left(-\frac{1}{6}+\frac{2\pi\sqrt{3}}{27}\right)y_1^2+\frac{2\pi\sqrt{3}}{81}y_1^3+\frac{4\pi\sqrt{3}}{243}y_1^4\right.\right.\\
&\quad\left.+\left(\frac{1}{180}+\frac{8\pi\sqrt{3}}{729}\right)y_1^5\right]\frac{1}{\epsilon}+\frac{19}{2}+\left(\frac{\ln 3}{3}-\frac{4}{3}\right)\pi\sqrt{3}-\frac{63}{4}S_2+\left[-\frac{\pi^2}{36}+\frac{21}{4}S_2\right.\\
&\quad\left.-\left(\frac{\ln 3}{9}+\frac{1}{9}\right)\pi\sqrt{3}\right]y_1+\left[-\frac{1}{3}+\frac{7}{2}S_2+\left(-\frac{2\ln 3}{27}+\frac{1}{54}\right)\pi\sqrt{3}\right]y_1^2+\left[-\frac{11}{54}+\frac{7}{6}S_2\right.\\
&\quad\left.+\left(-\frac{2\ln 3}{81}+\frac{23}{486}\right)\pi\sqrt{3}\right]y_1^3+\left[-\frac{25}{162}+\frac{7}{9}S_2+\left(-\frac{4\ln 3}{243}+\frac{119}{2916}\right)\pi\sqrt{3}\right]y_1^4\\
&\quad\left.+\left[-\frac{473}{4860}+\frac{14}{27}S_2+\left(-\frac{8\ln 3}{729}+\frac{1363}{43740}\right)\pi\sqrt{3}\right]y_1^5+\mathcal{O}(y_1^6)\right\}, \qquad (63)
\end{aligned}
$$

with $y_1 = 1 - 1/y^2 = \Delta/m^2$ and $S_2$ as defined in Eq. (22).

We proceeded in an analog way for the remaining two integrals in Eq. (9). Furthermore, for the seven integrals in Eqs. (11) and (12) we evaluated the phenomenological limit $y \to \infty$. However, we refrain from listing the results explicitly.

# References


[1] see, e.g., `http://linearcollider.org/`

[2] A. H. Hoang *et al.*, Eur. Phys. J. directC **2** (2000) 1 [arXiv:hep-ph/0001286].

[3] B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Nucl. Phys. **B635** (2002) 357.

[4] A.A. Penin and M. Steinhauser, Phys. Lett. B 538 (2002) 335.

[5] B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Phys. Rev. Lett. 90 (2003) 212001; Erratum *ibid.* 91 (2003) 139903.

[6] A.H. Hoang, Phys. Rev. D 69 (2004) 034009.

[7] F. Jegerlehner and M. Y. Kalmykov, Nucl. Phys. B **676** (2004) 365 [arXiv:hep-ph/0308216].

[8] M. Faisst, J. H. Kühn, T. Seidensticker and O. Veretin, Nucl. Phys. B **665** (2003) 649 [arXiv:hep-ph/0302275].

[9] M. Faisst, J. H. Kühn and O. Veretin, Phys. Lett. B **589** (2004) 35 [arXiv:hep-ph/0403026].

[10] P. Nogueira, J. Comput. Phys. **105** (1993) 279.

[11] R. Harlander, T. Seidensticker and M. Steinhauser, Phys. Lett. B **426** (1998) 125 [hep-ph/9712228].

[12] T. Seidensticker, hep-ph/9905298.

[13] J. A. M. Vermaseren, arXiv:math-ph/0010025.

[14] K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

[15] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [arXiv:hep-ph/0102033].

[16] C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046 [arXiv:hep-ph/0404258].

[17] Y. Schröder, Nucl. Phys. Proc. Suppl. **116** (2003) 402 [hep-ph/0211288].

[18] C. Sturm, PhD thesis (June 2005), Karlsruhe Univ., Germany (unpublished).

[19] K. G. Chetyrkin, J. H. Kühn, P. Mastrolia and C. Sturm, Eur. Phys. J. C **40** (2005) 361 [hep-ph/0412055].

[20] Y. Schröder and M. Steinhauser, Phys. Lett. B **622** (2005) 124 [hep-ph/0504055].

[21] N. Gray, D. J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C **48** (1990) 673.

[22] J. Fleischer, F. Jegerlehner, O. V. Tarasov and O. L. Veretin, Nucl. Phys. B **539** (1999) 671 [Erratum-ibid. B **571** (2000) 511] [arXiv:hep-ph/9803493].

[23] K. G. Chetyrkin and M. Steinhauser, Phys. Rev. Lett. **83** (1999) 4001 [arXiv:hep-ph/9907509].

[24] K. G. Chetyrkin and M. Steinhauser, Nucl. Phys. B **573** (2000) 617 [arXiv:hep-ph/9911434].

[25] K. Melnikov and T. v. Ritbergen, Phys. Lett. B **482** (2000) 99 [arXiv:hep-ph/9912391].

[26] R. Hempfling and B. A. Kniehl, Phys. Rev. D **51** (1995) 1386 [arXiv:hep-ph/9408313].

[27] B. A. Kniehl, J. H. Piclum and M. Steinhauser, Nucl. Phys. B **695** (2004) 199 [arXiv:hep-ph/0406254].

[28] S. P. Martin, Phys. Rev. D **72** (2005) 096008 [arXiv:hep-ph/0509115].

[29] J.H. Piclum, Diplomarbeit, University of Hamburg, 2004, (unpublished).

[30] K. Melnikov and T. van Ritbergen, Nucl. Phys. B **591** (2000) 515 [arXiv:hep-ph/0005131].

[31] A. Denner, Fortsch. Phys. **41** (1993) 307.

[32] J. Fleischer and M. Y. Kalmykov, Comput. Phys. Commun. **128** (2000) 531 [arXiv:hep-ph/9907431].

[33] A. I. Davydychev and M. Y. Kalmykov, Nucl. Phys. B **605** (2001) 266 [arXiv:hep-th/0012189].

[34] V.A. Smirnov, Applied Asymptotic Expansions in Momenta and Masses, Springer-Verlag, Berlin-Heidelberg, 2001.

[35] M. Steinhauser, Comput. Phys. Commun. **134** (2001) 335 [arXiv:hep-ph/0009029].

[36] M. Beneke and V. A. Smirnov, Nucl. Phys. B **522** (1998) 321 [arXiv:hep-ph/9711391].

[37] V. A. Smirnov and E. R. Rakhmetov, Theor. Math. Phys. **120** (1999) 870 [Teor. Mat. Fiz. **120** (1999) 64] [arXiv:hep-ph/9812529].