Summing next-to-next-to-leading logarithms in $b \to c$ transitions at zero recoil

A.G. Grozin

Institut für Theoretische Teilchenphysik, Universität Karlsruhe

Abstract

The HQET-1 $\to$ HQET-2 matching coefficient for the $b \to c$ currents at zero recoil has been found with two-loop accuracy. The next-to-next-to-leading logarithms has been summed in the leading order in m_c/m_b. Higher-order corrections are estimated in the large-β_0 limit.

Key words:
PACS: 12.38.Bx, 12.39.Hg

1 Introduction

Weak $b \to c$ transitions are described by QCD currents $\bar{c}\Gamma b$ with $\Gamma = \gamma^\mu$ and $\gamma_5\gamma^\mu$. If we are only interested in their matrix elements for $p_b = m_b v + k$, $p_c = m_c v' + k'$, where the residual momenta k, k' are small as compared to the heavy-quark masses $m_{c,b}$, then we may expand these QCD currents in HQET operators of the form $\bar{c}_{v'} \cdots b_v$ with the appropriate quantum numbers divided by powers of $m_{c,b}$. The coefficients are calculated by matching. This is descussed, e.g., in the review [1] or in the textbook [2]. In this paper we consider the zero-recoil case $v' = v$. Then, in the v rest frame,

$$\begin{aligned}
\bar{c}\gamma_0 b &= \eta_V\, \bar{c}_v b_v + \mathcal{O}(1/m_{c,b}^2)\,,\\
\bar{c}\gamma_5\vec{\gamma} b &= \eta_A\, \bar{c}_v \gamma_5\vec{\gamma} b_v + \mathcal{O}(1/m_{c,b}^2)\,.
\end{aligned} \tag{1}$$

We use $\overline{\mathrm{MS}}$ renormalization with anticommuting γ_5; then the vector and axial currents in QCD do not depend on the normalization scale μ. The HQET currents with $v' = v$ are also μ-independent, and hence the matching coefficients

Email address: grozin@particle.uni-karlsruhe.de (A.G. Grozin).

$\eta_{V,A}$ do not depend on any normalization scale. We shall not discuss $1/m_{c,b}^2$ power corrections here.

The coefficients can be obtained by matching QCD matrix elements with those in HQET:

$$\eta_\Gamma = 1 + \sum_n \eta_\Gamma^{(n)} \left(\frac{\alpha_s^{(5)}(\mu_0)}{4\pi} \right)^n , \quad \mu_0 = \sqrt{m_b m_c} \,. \tag{2}$$

Here $m_{b,c}$ are the on-shell heavy-quark masses. In this way, we know the exact dependence of $\eta_\Gamma^{(n)}$ on $r = m_c/m_b$; however, we cannot sum terms with leading powers of $L = \log(m_b/m_c)$ in all orders of perturbation theory. The two-loop correction $\eta_\Gamma^{(2)}$ has been calculated in [3] and confirmed by an independent re-calculation [4].

Alternatively, at $r \ll 1$ we can perform the matching in two steps. First, we match QCD with HQET-1, where b is static and c is dynamic, at $\mu \sim m_b$. Second, we match HQET-1 with HQET-2, where both b and c are static, at $\mu \sim m_c$. In this way, we obtain $\eta_{V,A}$ as series in r:

$$\eta_\Gamma = C_\Gamma(m_b) G(m_b, m_c) E(m_c) + \mathcal{O}(r) \,. \tag{3}$$

In practice, few terms of this expansion can be found, because the number of relevant operators in HQET-1 grows fast with the dimensionality. However, in each term of the expansion in r, we can sum leading, next-to-leading, etc., powers of L. The leading term in the r-expansion is known at the next-to-leading logarithmic order [5]. The $\mathcal{O}(r)$ term is known at the leading logarithmic order [6]. The $\mathcal{O}(r^2)$ term is discussed in [7]. The two approaches can be combined: we subtract several first terms of expansion of (2) in r and add the RG-improved results instead.

The QCD $\to$ HQET-1 matching coefficients

$$C_\Gamma(m_b) = 1 + \sum_n c_\Gamma^{(n)} \left(\frac{\alpha_s^{(5)}(m_b)}{4\pi} \right)^n \tag{4}$$

are known at the next-to-next-to-leading (NNL) order [8,9]. The running factor $G(m_b, m_c)$ is determined by the anomalous dimension γ of the heavy-light current in HQET-1. Until recently, it was only known at two loops [10,11]. Now it is has been calculated at three loops [12], i.e., also at the NNL order, using the methods of [13]. Finally, the HQET-1 $\to$ HQET-2 matching coefficient

$$E(m_c) = 1 + \sum_n e_n \left(\frac{\alpha_s^{(4)}(m_c)}{4\pi} \right)^n \tag{5}$$

is currently known at one loop [5] only. In the present paper, we shall find the two-loop coefficient e_2, thus completing the NNL order calculation of $\eta_{V,A}$ at $\mathcal{O}(r^0)$.

Quite a different kind of summation was performed in [14]: terms with the highest powers of β_0 in the perturbative series for $\eta_{V,A}$ were summed to all orders.

2 Summing NNL logarithms

By requiring that the two-step matching result (3) reproduces the $\mathcal{O}(r^0)$ term in $\eta_{V,A}^{(2)}$ [3], we have two independent derivations of the matching coefficient $E(m_c)$:

$$\begin{aligned} E(m_c) = {} & 1 - 4C_F \frac{\alpha_s(m_c)}{4\pi} \\ & + C_F \Biggl[C_F \left(-20\pi^2 \log 2 + 14\zeta_3 + 73\zeta_2 + \frac{81}{16} \right) \\ & \quad + C_A \left(10\pi^2 \log 2 - 41\zeta_3 - 6\zeta_2 + \frac{503}{48} \right) \\ & \quad + T_F \left(-17\zeta_2 + \frac{233}{12} \right) + T_F n_l \left(-4\zeta_2 - \frac{31}{12} \right) \Biggr] \left(\frac{\alpha_s(m_c)}{4\pi} \right)^2 \\ \approx {} & 1 - \frac{4}{3} \frac{\alpha_s(m_c)}{\pi} + 4.00 \left(\frac{\alpha_s(m_c)}{\pi} \right)^2 . \end{aligned} \tag{6}$$

Here $n_l = 3$ is the number of flavours lighter than c; $\alpha_s^{(4)}(\mu)$ is written simply as $\alpha_s(\mu)$. This result can also be obtained by a direct calculation; this will be discussed in a separate publication.

Now we can sum the logarithms (up to NNL ones) to all orders:

$$\eta_\Gamma = x^{-\gamma_0/(2\beta_0)} \left[1 + h_\Gamma^{(1)} \frac{\alpha_s(m_b)}{4\pi} + h_\Gamma^{(2)} \left(\frac{\alpha_s(m_b)}{4\pi} \right)^2 + \cdots \right] . \tag{7}$$

Here

$$x = \frac{\alpha_s(m_c)}{\alpha_s(m_b)} , \tag{8}$$

$\alpha_s(\mu)$ and β_0 are for 4 flavours. The NL correction is

$$
\frac{h^{(1)}_{V,A}}{C_F} = -\frac{1}{4}\left(11x + \left\{\begin{matrix} 13 \\ 21 \end{matrix}\right\}\right) + \left[\left(8\zeta_2 - \frac{23}{4}\right) C_F - 2(\zeta_2 + 4) C_A\right] \frac{x-1}{\beta_0}
+ \frac{3C_A(11C_F + 7C_A)(x-1)}{2\beta_0^2}\,, \tag{9}
$$

where the upper number in braces is for V and the lower one for A. The NNL correction is

$$
\begin{aligned}
\frac{h^{(2)}_{V,A}}{C_F} &= \left[\left(3\zeta_2 + \frac{29}{24}\right) x^2 - 3\zeta_2 - \frac{1}{24}\left\{\begin{matrix} 53 \\ 205 \end{matrix}\right\}\right] \beta_0 \\
&+ \Bigg[\left(-20\pi^2 \log 2 + 3\zeta_3 + \frac{247}{3}\zeta_2 + \frac{643}{96}\right) x^2 + \left\{\begin{matrix} 143 \\ 231 \end{matrix}\right\} \frac{x}{16} \\
&\quad + \left\{\begin{matrix} 4 \\ 28/3 \end{matrix}\right\} \pi^2 \log 2 - \left\{\begin{matrix} 3 \\ 11 \end{matrix}\right\} \zeta_3 - \left\{\begin{matrix} 73/3 \\ 67 \end{matrix}\right\} \zeta_2 + \frac{1}{96}\left\{\begin{matrix} 1283 \\ 2899 \end{matrix}\right\}\Bigg] C_F \\
&+ \Bigg[\left(10\pi^2 \log 2 - \frac{63}{2}\zeta_3 - \frac{58}{3}\zeta_2 - \frac{41}{96}\right) x^2 \\
&\quad - \left\{\begin{matrix} 2 \\ 14/3 \end{matrix}\right\} \pi^2 \log 2 - \left\{\begin{matrix} 9 \\ 1 \end{matrix}\right\} \frac{\zeta_3}{2} + \left\{\begin{matrix} 55 \\ 103 \end{matrix}\right\} \frac{\zeta_2}{3} - \frac{1}{96}\left\{\begin{matrix} 343 \\ 919 \end{matrix}\right\}\Bigg] C_A \\
&+ \left[-(13\zeta_2 - 22)x^2 - \left\{\begin{matrix} 24 \\ 40/3 \end{matrix}\right\} \zeta_2 + \left\{\begin{matrix} 727/18 \\ 133/6 \end{matrix}\right\}\right] T_F \\
&+ \Bigg\{\Bigg[\left(20\zeta_4 + 9\zeta_3 - 50\zeta_2 + \frac{483}{16}\right) x \\
&\qquad + 20\zeta_4 + 9\zeta_3 - \left\{\begin{matrix} 54 \\ 70 \end{matrix}\right\} \zeta_2 + \frac{1}{16}\left\{\begin{matrix} 529 \\ 713 \end{matrix}\right\}\Bigg] C_F^2 \\
&\quad + \Bigg[\left(4\zeta_4 + \frac{57}{2}\zeta_3 - \frac{77}{6}\zeta_2 + \frac{629}{24}\right) x \\
&\qquad + 4\zeta_4 + \frac{57}{2}\zeta_3 - \left\{\begin{matrix} 71 \\ 47 \end{matrix}\right\} \frac{\zeta_2}{6} + \frac{1}{24}\left\{\begin{matrix} 725 \\ 1109 \end{matrix}\right\}\Bigg] C_F C_A \\
&\quad + \left(6\zeta_4 - 33\zeta_3 + \frac{19}{3}\zeta_2 + \frac{103}{8}\right)(x+1) C_A^2 \Bigg\} \frac{x-1}{\beta_0} \\
&+ \Bigg\{\frac{1}{32}(32\zeta_2 - 23)^2 (x-1) C_F^3 - \left(7\zeta_2 + \frac{609}{32}\right)(x+1) C_A^3
\end{aligned}
$$

$$
\begin{aligned}
&+\left[-\left(40\zeta_4+\frac{17}{2}\zeta_2+\frac{479}{8}\right)x+40\zeta_4+\frac{193}{2}\zeta_2-\frac{1}{8}\left\{\begin{matrix}1281\\1545\end{matrix}\right\}\right]C_F^2C_A\\
&+\left[\left(5\zeta_4+33\zeta_2-\frac{395}{4}\right)x-5\zeta_4+\zeta_2-\left\{\begin{matrix}168\\189\end{matrix}\right\}\right]C_FC_A^2\Bigg\}\frac{x-1}{\beta_0^2}\\
&+3C_A(11C_F+7C_A)\left\{\left(4\zeta_2-\frac{23}{8}\right)(x-1)C_F^2+\frac{7}{4}(x+1)C_A^2\right.\\
&\left.+\left[-\left(\zeta_2+\frac{5}{4}\right)x+\zeta_2+\frac{27}{4}\right]C_FC_A\right\}\frac{x-1}{\beta_0^3}\\
&+\frac{9C_FC_A^2(11C_F+7C_A)^2(x-1)^2}{8\beta_0^4}\,.
\end{aligned}
\tag{10}
$$

Numerically,

$$
\begin{aligned}
\eta_{V,A}=x^{6/25}\Bigg[&1-\left(1.10588x+\left\{\begin{matrix}0.89412\\1.56079\end{matrix}\right\}\right)\frac{\alpha_s(m_b)}{\pi}\\
&+\left(3.94156x^2+\left\{\begin{matrix}0.98879\\1.72604\end{matrix}\right\}x-\left\{\begin{matrix}2.44780\\8.22155\end{matrix}\right\}\right)\left(\frac{\alpha_s(m_b)}{\pi}\right)^2+\cdots\Bigg]\,.
\end{aligned}
\tag{11}
$$

3 Higher orders in the large-β_0 limit

It would be extremely difficult to sum the N^{3}L logarithms. However, it is easy to find the β_0^2 term in $h_\Gamma^{(3)}$. The highest powers of β_0 in all orders of perturbation theory for γ were found in [8]:

$$
\begin{aligned}
\gamma&=-\frac{C_F}{3}\frac{\beta}{\beta_0}\frac{3+2\beta}{B(2+\beta,2+\beta)\Gamma(1-\beta)\Gamma(3+\beta)}\\
&=-3C_F\frac{\beta}{\beta_0}\left[1+\frac{5}{6}\beta-\frac{35}{36}\beta^2-\left(2\zeta_3-\frac{83}{72}\right)\beta^3+\cdots\right]\,.
\end{aligned}
\tag{12}
$$

where $\beta=\beta_0\alpha_s/(4\pi)$. The matching coefficients $C_\Gamma(m_b)$ were also considered in [8]:

$$
\begin{aligned}
C_{V,A}(m_b)=1-C_F\frac{\beta}{\beta_0}\Bigg[&\left\{\begin{matrix}2\\4\end{matrix}\right\}+\left(3\zeta_2+\left\{\begin{matrix}47/16\\445/48\end{matrix}\right\}\right)\beta\\
&+\left(7\zeta_3+\left\{\begin{matrix}13\\21\end{matrix}\right\}\zeta_2+\frac{1}{432}\left\{\begin{matrix}-1751\\7993\end{matrix}\right\}\right)\beta^2+\cdots\Bigg]\,.
\end{aligned}
\tag{13}
$$

Here we use the same method for E. The solution of the RG equation can be written as

$$
\begin{aligned}
E(\mu) &= \hat{E} \left(\frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right)^{\gamma_0/(2\beta_0)} K(\alpha_s(\mu)) \,, \\
K(\alpha_s) &= \exp \int\limits_0^{\alpha_s} \left(\frac{\gamma(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0}{2\beta_0} \right) \frac{d\alpha_s}{\alpha_s} \,.
\end{aligned}
\tag{14}
$$

The RG invariant $\hat{E}$ is, at the first order in $1/\beta_0$,

$$
\hat{E} = 1 + \frac{1}{\beta_0} \int\limits_0^{\infty} S(u) e^{-u/\beta} du + \mathcal{O}\left(\frac{1}{\beta_0^2} \right) \,,
\tag{15}
$$

where $\beta = \beta_0 \alpha_s(m_c)/(4\pi)$. We obtain

$$
S(u) = 3C_F \left[e^{(5/3)u} \frac{\Gamma(u)\Gamma(2-2u)}{\Gamma(3-u)} \frac{1-u-u^2}{1+2u} - \frac{1}{2u} \right] .
\tag{16}
$$

Note that the leading infrared renormalon is at $u = 1$. Collecting all this together, we obtain

$$
\begin{aligned}
E(m_c) = 1 + C_F \frac{\beta}{\beta_0} \Bigg[& -4 + \left(3\zeta_2 + \frac{31}{16} \right) \beta \\
& + \left(7\zeta_3 - 11\zeta_2 - \frac{7655}{432} \right) \beta^2 + \cdots \Bigg] .
\end{aligned}
\tag{17}
$$

This reproduces the β_0 term at two loops (6), and gives us the β_0^2 term at three loops.

Using this result, we obtain

$$
\begin{aligned}
\frac{h_{V,A}^{(3)}}{C_F} = & \left[\left(6\zeta_3 - 11\zeta_2 - \frac{3703}{216} \right) x^3 - 6\zeta_3 - \left\{ \begin{array}{c} 13 \\ 21 \end{array} \right\} \zeta_2 + \frac{1}{216} \left\{ \begin{array}{c} 751 \\ -4121 \end{array} \right\} \right] \beta_0^2 \\
& + \mathcal{O}(\beta_0) \,.
\end{aligned}
\tag{18}
$$

Numerically, this means adding the term

$$
- \left(40.5461 x^3 + \left\{ \begin{array}{c} 36.3421 \\ 88.0131 \end{array} \right\} \right) \left(\frac{\alpha_s(m_b)}{\pi} \right)^3
$$

inside the bracket in (11). Of course, this is an estimate only, because the terms $\mathcal{O}(\beta_0)$ and $\mathcal{O}(1)$ are not known.

4 What is known about the three-loop coefficients?

Re-expressing $\alpha_s(m_b)$, $\alpha_s(m_c)$ via $\alpha_s^{(5)}(\mu_0)$, we can write η_Γ in the form (2). From our NNL result, we obtain the L^3, L^2, and L terms in the three-loop coefficients $\eta_\Gamma^{(3)}$. The β_0^2 part of the non-logarithmic term can also be found. We can even make one more improvement, and obtain the β_0^2 part of $\eta_\Gamma^{(3)}$ exactly as a function of r (including power corrections). The highest powers of β_0 in all orders of perturbation theory for η_Γ were found in [14]. They have the form similar to (15), with $\beta = \beta_0\alpha_s(\mu_0)/(4\pi)$ and

$$\begin{aligned}
S_V(u) &= 6C_F e^{(5/3)u}\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}\frac{1-u-u^2}{1+2u}\left[(1-2u)R_0 - R_1\right] , \\
S_A(u) &= 2C_F e^{(5/3)u}\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}\frac{1}{1+2u} \\
&\quad\times\Big[3(1-2u)(1-u-u^2)R_0 - (3-u+u^2)R_1\Big] , \qquad (19)
\end{aligned}$$

where

$$R_0 = \cosh(Lu)\,, \quad R_1 = \frac{\sinh\frac{(1-2u)L}{2}}{\sinh\frac{L}{2}}\,. \qquad (20)$$

This gives us the β_0^2 parts of $\eta_{V,A}^{(3)}$ as exact functions of r.

Collecting all this together, we obtain

$$\begin{aligned}
\frac{\eta_{V,A}^{(3)}}{C_F} &= \Bigg[L\coth\frac{L}{2}\left(L^2 + 12\zeta_2 + \begin{Bmatrix}41/6\\39/2\end{Bmatrix}\right) \\
&\quad - \begin{Bmatrix}6\\8\end{Bmatrix}L^2 - \begin{Bmatrix}24\\32\end{Bmatrix}\zeta_2 - \begin{Bmatrix}41/3\\326/9\end{Bmatrix}\Bigg]\beta_0^2 \\
&\quad + \left(\frac{9}{2}C_F^2 + 12C_FT_F + \frac{16}{3}T_F^2\right)L^3 \\
&\quad + \Bigg[\left(\begin{Bmatrix}3\\15\end{Bmatrix}\frac{C_F}{2} + \begin{Bmatrix}4\\20\end{Bmatrix}\frac{T_F}{3}\right)\beta_0 + 3\left(16\zeta_2 - \frac{1}{2}\begin{Bmatrix}23\\29\end{Bmatrix}\right)C_F^2
\end{aligned}$$

$$
\begin{aligned}
&- 3(4\zeta_2+1)C_F C_A + \frac{16}{3}\left(8\zeta_2 - \begin{Bmatrix} 8 \\ 11 \end{Bmatrix}\right) C_F T_F \\
&- \frac{4}{3}(8\zeta_2-13)C_A T_F - \begin{Bmatrix} 32/3 \\ 128/9 \end{Bmatrix} T_F^2 \Bigg] L^2 \\
+ &\Bigg[\left(-\begin{Bmatrix} 48 \\ 176/3 \end{Bmatrix}\pi^2\log 2 + \begin{Bmatrix} 12 \\ 28 \end{Bmatrix}\zeta_3 + \begin{Bmatrix} 640 \\ 896 \end{Bmatrix}\frac{\zeta_2}{3} - \begin{Bmatrix} 79/12 \\ 213/4 \end{Bmatrix}\right) C_F\beta_0 \\
&+ \left(\begin{Bmatrix} 24 \\ 88/3 \end{Bmatrix}\pi^2\log 2 - \begin{Bmatrix} 54 \\ 62 \end{Bmatrix}\zeta_3 - \begin{Bmatrix} 226 \\ 322 \end{Bmatrix}\frac{\zeta_2}{3} + \frac{1}{3}\begin{Bmatrix} 56 \\ 122 \end{Bmatrix}\right) C_A\beta_0 \\
&+ \left(\begin{Bmatrix} 22 \\ 2/3 \end{Bmatrix}\zeta_2 - \frac{1}{9}\begin{Bmatrix} 355 \\ 179 \end{Bmatrix}\right) T_F\beta_0 \\
&+ \left(-\begin{Bmatrix} 48 \\ 32 \end{Bmatrix}\pi^2\log 2 + 80\zeta_4 + \begin{Bmatrix} 36 \\ 12 \end{Bmatrix}\zeta_3 + \begin{Bmatrix} 14 \\ -146 \end{Bmatrix}\zeta_2 + \frac{1}{2}\begin{Bmatrix} 241 \\ 385 \end{Bmatrix}\right) C_F^2 \\
&+ \left(\begin{Bmatrix} 24 \\ 16 \end{Bmatrix}\pi^2\log 2 + 16\zeta_4 + \begin{Bmatrix} 6 \\ 18 \end{Bmatrix}\zeta_3 + \begin{Bmatrix} 47 \\ 215 \end{Bmatrix}\frac{\zeta_2}{3} - \frac{1}{12}\begin{Bmatrix} 1313 \\ 1769 \end{Bmatrix}\right) C_F C_A \\
&+ \left(24\zeta_4 - 132\zeta_3 + \frac{16}{3}\zeta_2 + \frac{1}{2}\begin{Bmatrix} 27 \\ -1 \end{Bmatrix}\right) C_A^2 \\
&+ \left(-\begin{Bmatrix} 128/3 \\ 256/9 \end{Bmatrix}\pi^2\log 2 - \begin{Bmatrix} 0 \\ 64 \end{Bmatrix}\frac{\zeta_3}{3} + \begin{Bmatrix} 393 \\ -343 \end{Bmatrix}\frac{\zeta_2}{9} + \begin{Bmatrix} 481/2 \\ 4273/18 \end{Bmatrix}\right) C_F T_F \\
&+ \left(\begin{Bmatrix} 64/3 \\ 128/9 \end{Bmatrix}\pi^2\log 2 - \begin{Bmatrix} 96 \\ 256/3 \end{Bmatrix}\zeta_3 + \begin{Bmatrix} -8/3 \\ 40 \end{Bmatrix}\zeta_2 - \begin{Bmatrix} 152/3 \\ 80 \end{Bmatrix}\right) C_A T_F \\
&- \frac{4}{9}\left(\begin{Bmatrix} 222 \\ 158 \end{Bmatrix}\zeta_2 - \begin{Bmatrix} 1123/3 \\ 265 \end{Bmatrix}\right) T_F^2 \Bigg] L \\
+ &\,\mathcal{O}(\beta_0) \qquad (21)
\end{aligned}
$$

Non-logarithmic terms $\mathcal{O}(\beta_0)$ and $\mathcal{O}(1)$ are not known. Numerically, the best estimate of the three-loop contribution to (2) is

$$
\begin{aligned}
&\Bigg[\left(1.80787 + 2.89352\frac{r}{1-r}\right) L^3 - \begin{Bmatrix} 8.14599 \\ 9.76174 \end{Bmatrix} L^2 \\
+ &\left(\begin{Bmatrix} 27.63733 \\ 45.56374 \end{Bmatrix} + \begin{Bmatrix} 76.88814 \\ 113.53938 \end{Bmatrix}\frac{r}{1-r}\right) L
\end{aligned}
$$

$$
-\left\{\begin{array}{c} 76.88814 \\ 128.55919 \end{array}\right\} + \mathcal{O}(\beta_0)\Bigg] \left(\frac{\alpha_s^{(5)}(\mu_0)}{\pi}\right)^3 \tag{22}
$$

(we used $\coth(L/2) = (1+r)/(1-r)$).

5 Results and conclusion

The program RunDec [15] was used for numerical estimates:

$$
r \approx 0.34\,, \quad L \approx 1.08\,, \quad \alpha_s(m_b) \approx 0.216\,, \quad x \approx 1.56\,, \quad \alpha_s^{(5)}(\mu_0) \approx 0.258\,.
$$

To combine the results of the two-loop calculation [3] with our resummation, we subtract from [3] its $\mathcal{O}(r^0)$ term and add the resummed result (11) instead. This results in the corrections

$$
\Delta\eta_V \approx 0.021\,, \quad \Delta\eta_A \approx 0.029\,, \tag{23}
$$

to be added to the results of [3]. If we also add the β_0^2 term (18) of the $\mathrm{N}^3\mathrm{L}$ logarithmic correction, these numbers change significantly:

$$
\Delta\eta_V \approx -0.049\,, \quad \Delta\eta_A \approx -0.059\,. \tag{24}
$$

We don't know $\mathcal{O}(\beta_0)$ and $\mathcal{O}(1)$ terms in this correction; if we suppose that they (taken together) don't exceed the $\mathcal{O}(\beta_0^2)$ terms, then the theoretical error in $\eta_{V,A}$ is about 0.08 — not a very encouraging estimate.

We can also discuss the situation from another side. The α_s^3 terms (22) in (2) are

$$
\begin{aligned}
&\Delta\eta_V \approx 0.012 - 0.018 \approx -0.005 \pm 0.04\,,\\
&\Delta\eta_A \approx 0.022 - 0.035 \approx -0.013 \pm 0.07\,,
\end{aligned} \tag{25}
$$

where the first contribution is all terms with powers of L and without $\mathcal{O}(r)$ correction; the second contribution is the β_0^2 part of the non-logarithmic term and of the power corrections; the β_0^2 part of the non-logarithmic term is taken as a measure of the theoretical error. We see that summation of powers of the "large" logarithm $L \approx 1$ does not make much sense: the logarithmic terms at $\mathcal{O}(\alpha_s^3)$ are largely canceled by the non-logarithmic $\mathcal{O}(\beta_0^2)$ terms, leaving very small corrections.

A substantial improvement of the theoretical accuracy of $\eta_{V,A}$ can only be achieved by the full three-loop calculation (which seems very difficult), or at least by the calculation of the $\mathcal{O}(\beta_0)$ part of the three-loop term (this reduced to light-quark loop insertions into the two-loop diagrams of [3], and is also quite difficult, because this loop can be inserted not only into a gluon propagator, but also into a three-gluon vertex).

References

[1] M. Neubert, Phys. Reports **245** (1994) 259.

[2] A.G. Grozin, *Heavy Quark Effective Theory*, Springer (2004).

[3] A. Czarnecki and K. Melnikov, Nucl. Phys. B **505** (1997) 65.

[4] J. Franzkowski and J.B. Tausk, Eur. Phys. J. C **5** (1998) 517.

[5] M. Neubert, Phys. Rev. D **46** (1992) 2212.

[6] A.F. Falk and B. Grinstein, Phys. Lett. B **247** (1990) 406.

[7] C. Balzereit and T. Ohl, Phys. Lett. B **398** (1997) 365.

[8] D.J. Broadhurst and A.G. Grozin, Phys. Rev. D **52** (1995) 4082.

[9] A.G. Grozin, Phys. Lett. B **445** (1997) 165.

[10] X. Ji and M.J. Musolf, Phys. Lett. B **257** (1991) 409.

[11] D.J. Broadhurst and A.G. Grozin, Phys. Lett. B **267** (1991) 105.

[12] K.G. Chetyrkin and A.G. Grozin, Nucl. Phys. B **666** (2003) 289.

[13] A.G. Grozin, J. High Energy Phys. **03** (2000) 013.

[14] M. Neubert, Phys. Lett. B **341** (1995) 367.

[15] K.G. Chetyrkin, J.H. Kühn, M. Steinhauser, Comput. Phys. Commun. **133** (2000) 43