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# Two-Loop Electroweak Logarithms in Four-Fermion Processes at High Energy

B. Jantzen<sup>\*a,b</sup>, J.H. Kühn<sup>b</sup>, A.A. Penin<sup>b,c</sup>, and V.A. Smirnov<sup>d,e</sup>

<sup>a</sup>*Paul Scherrer Institut  
5232 Villigen PSI, Switzerland*

<sup>b</sup>*Institut für Theoretische Teilchenphysik, Universität Karlsruhe  
76128 Karlsruhe, Germany*

<sup>c</sup>*Institute for Nuclear Research of Russian Academy of Sciences  
117312 Moscow, Russia*

<sup>d</sup>*Nuclear Physics Institute of Moscow State University  
119992 Moscow, Russia*

<sup>e</sup>*II. Institut für Theoretische Physik, Universität Hamburg  
22761 Hamburg, Germany*

## Abstract

We present the complete analytical result for the two-loop logarithmically enhanced contributions to the high energy asymptotic behavior of the vector form factor and the four-fermion cross section in a spontaneously broken  $SU(2)$  gauge model. On the basis of this result we derive the dominant two-loop electroweak corrections to the neutral current four-fermion processes at high energies. Previously neglected effects of the gauge boson mass difference are included through the next-to-next-to-leading logarithmic approximation.

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\*B. Feucht in previous publications.

# 1 Introduction

Recently a new wave of interest in the Sudakov asymptotic regime [1, 2] has risen in connection with higher-order corrections to electroweak processes at high energies [3–21]. Experimental and theoretical studies of electroweak interactions have traditionally explored the range from very low energies, *e.g.* through parity violation in atoms, up to energies comparable to the masses of the  $W$ - and  $Z$ -bosons at LEP or the Tevatron. The advent of multi-TeV colliders like the LHC during the present decade or a future linear electron-positron collider will give access to a completely new energy domain. Once the characteristic energies  $\sqrt{s}$  are far larger than the masses of the  $W$ - and  $Z$ -bosons,  $M_{W,Z}$ , exclusive reactions like electron-positron (or quark-antiquark) annihilation into a pair of fermions or gauge bosons will receive virtual corrections enhanced by powers of the large *electroweak* logarithm  $\ln(s/M_{W,Z}^2)$ .

In Refs. [9, 10] we have extended the leading logarithmic (LL) analysis of [8]. The next-to-leading logarithmic (NLL) and next-to-next-to-leading logarithmic (NNLL) corrections to the high energy asymptotic behavior of the neutral current four-fermion processes have been resummed to all orders using the *evolution equation* approach. Only the light quark case was considered and the mass difference between the neutral and charged gauge bosons was neglected. On the basis of this result the logarithmically enhanced part of the phenomenologically important two-loop corrections to the total cross section and to various asymmetries was obtained including the  $\ln^n(s/M_{W,Z}^2)$  terms with  $n = 2, 3, 4$ . The results up to NLL have been confirmed by the explicit one-loop [5, 12, 13] and two-loop [14–17, 20] calculations. The subleading logarithms in the TeV region are comparable to the leading terms due to their large numerical coefficients. Thus, the calculation of the remaining two-loop linear logarithms is necessary to control the convergence of the logarithmic expansion. These corrections represent the next-to-next-to-next-to-leading logarithmic (N<sup>3</sup>LL) contribution. They are of special interest both from the phenomenological and conceptual point of view because, in contrast to the higher powers of the logarithm, they are sensitive to the details of the gauge boson mass generation. The first results beyond the NNLL approximation have been obtained in [16, 21, 29].

In this paper we extend our previous analysis and complete the calculation of the two-loop logarithmic corrections to the neutral current four-fermion processes. We incorporate previously neglected effects of the gauge boson mass difference. The two-loop logarithmic terms are derived within the *expansion by regions* approach [22–25] by inspecting the structure of singularities of the contributions of different regions. The calculation is significantly simplified by taking the exponentiation of the logarithmic corrections in the Sudakov limit into account. This property naturally appears and can be fully elaborated within the evolution equation approach [26–28]. To identify the pure QED infrared logarithms which are compensated by soft real photon radiation we combine the *hard* evolution equation which governs the dependence of the amplitudes on  $s$  with the *infrared* evolution equation [8] which describes the dependence of the amplitude on an infrared regulator. The main results of the present paper have been reported earlier in a short letter [29].

In Section 2 we present the complete result for the two-loop logarithmic corrections to

the form factor which describes fermion scattering in an external Abelian field. Two models are considered: a non-Abelian  $SU(2)$  gauge theory with the Higgs mechanism of gauge boson mass generation and an Abelian  $U(1) \times U(1)$  theory with two gauge bosons of essentially different masses [21]. In Section 3 we generalize the result to the four-fermion process. Finally, in Section 4, we apply this result to electroweak processes. A brief summary and conclusions are given in Section 5.

## 2 Abelian form factor in the Sudakov limit

The vector form factor  $\mathcal{F}$  determines the fermion scattering amplitude in an external Abelian field. It plays a special role since it is the simplest quantity which includes the complete information about the universal *collinear* logarithms. This information is directly applicable to a process with an arbitrary number of fermions. The form factor is a function of the Euclidean momentum transfer  $Q^2 = -(p_1 - p_2)^2$  where  $p_{1,2}$  is the incoming/outgoing fermion momentum. In the next two sections we consider two characteristic examples: (i) the  $SU(2)$  gauge model with gauge bosons of a nonzero mass  $M$  which emulates the massive gauge boson sector of the standard model and (ii) the  $U(1) \times U(1)$  gauge model with two gauge bosons of essentially different masses which emulates the effect of the  $Z - \gamma$  mass gap. We focus on the asymptotic behavior of the form factor in the Sudakov limit  $M/Q \ll 1$  with on-shell massless fermions,  $p_1^2 = p_2^2 = 0$ .

In the Sudakov limit the coefficients of the perturbative series in the coupling constant can be expanded in  $M^2/Q^2$ . To compute the leading term of the series in  $M^2/Q^2$  we use the expansion by regions approach [22–25]. It is based on separating the contributions of dynamical modes or *regions* characteristic for different asymptotic regimes and consists of the following steps:

- (i) consider various regions of a loop momentum  $k$  and expand, in every region, the integrand in a Taylor series with respect to the parameters considered small in this region;
- (ii) integrate the expanded integrand over the whole integration domain of the loop momenta;
- (iii) put to zero any scaleless integral.

In step (ii) dimensional regularization with  $d = 4 - 2\epsilon$  space-time dimensions is used to handle the divergences. In the Sudakov limit under consideration the following regions are relevant [30–32]:

$$\begin{aligned}
 \text{hard (h):} & \quad k \sim Q, \\
 \text{1-collinear (1c):} & \quad k_+ \sim Q, \quad k_- \sim M^2/Q, \quad \underline{k} \sim M, \\
 \text{2-collinear (2c):} & \quad k_- \sim Q, \quad k_+ \sim M^2/Q, \quad \underline{k} \sim M, \\
 \text{soft (s):} & \quad k \sim M,
 \end{aligned} \tag{1}$$

where  $k_{\pm} = k_0 \pm k_3$ ,  $\underline{k} = (k_1, k_2)$  and we choose  $p_{1,2} = (Q/2, 0, 0, \mp Q/2)$  so that  $2p_1 p_2 = Q^2 = -s$ . By  $k \sim Q$ , *etc.* we mean that every component of  $k$  is of order  $Q$ . The expansion procedure can be facilitated by a technique based on Mellin-Barnes representation which can be used not only for evaluating Feynman integrals but also to pick up contributions of regions (see Sect. 4.8 of [33] for a general discussion and [34] where the MB representation was used for this purpose.)

## 2.1 $SU(2)$ model with massive gauge boson

Let us apply the method to compute the corrections to the form factor in the  $SU(2)$  model. In one loop the expansion by regions leads to the following decomposition

$$\mathcal{F}^{(1)} = \mathcal{F}_h^{(1)} + \mathcal{F}_c^{(1)} + \mathcal{F}_s^{(1)}, \quad (2)$$

where the subscript  $c$  denotes the contribution of both collinear regions, for a perturbative function  $f(\alpha)$  we define

$$f(\alpha) = \sum_n \left( \frac{\alpha}{4\pi} \right)^n f^{(n)}, \quad (3)$$

and the form factor in the Born approximation is normalized to 1. The hard region contribution, which we will later need, reads

$$\mathcal{F}_h^{(1)} = \frac{C_F}{Q^{2\epsilon}} \left[ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{\pi^2}{6} - 8 + \left( -16 + \frac{\pi^2}{4} + \frac{14}{3}\zeta(3) \right) \epsilon \right] + \mathcal{O}(\epsilon^2), \quad (4)$$

where  $C_F = (N^2 - 1)/(2N)$  for a  $SU(N)$  gauge group,  $\zeta(3) = 1.202057\dots$  is the value of the Riemann's zeta-function and all the power-suppressed terms are neglected. For convenience we do not include the standard factor  $(4\pi e^{-\gamma_E \epsilon} (\mu^2)^\epsilon)$  per loop, where  $\gamma_E = 0.577216\dots$  is Euler's constant. The contributions of all the regions [10] add up to the well known finite result

$$\mathcal{F}^{(1)} = -C_F \left( \mathcal{L}^2 - 3\mathcal{L} + \frac{7}{2} + \frac{2\pi^2}{3} \right), \quad (5)$$

where  $\mathcal{L} = \ln(Q^2/M^2)$ . A similar decomposition can be performed in two loops

$$\mathcal{F}^{(2)} = \mathcal{F}_{hh}^{(2)} + \mathcal{F}_{hc}^{(2)} + \mathcal{F}_{cc}^{(2)} + \dots \quad (6)$$

The hard-hard part reads

$$\begin{aligned} \mathcal{F}_{hh}^{(2)} = & \left( \frac{1}{2}\mathcal{F}_h^{(1)} - \frac{\beta_0}{\epsilon} \right) \mathcal{F}_h^{(1)} + \frac{C_F}{Q^{4\epsilon}} \left\{ \left[ -\frac{11}{6\epsilon^3} + \left( -\frac{83}{9} + \frac{\pi^2}{6} \right) \frac{1}{\epsilon^2} + \left( -\frac{4129}{108} - \frac{11}{36}\pi^2 \right. \right. \right. \\ & \left. \left. + 13\zeta(3) \right) \frac{1}{\epsilon} \right] C_A + \left[ \frac{2}{3\epsilon^3} + \frac{28}{9\epsilon^2} + \left( \frac{353}{27} + \frac{\pi^2}{9} \right) \frac{1}{\epsilon} \right] T_F n_f + \left[ \frac{1}{6\epsilon^3} + \frac{17}{18\epsilon^2} \right. \\ & \left. \left. + \left( \frac{455}{108} + \frac{\pi^2}{36} \right) \frac{1}{\epsilon} \right] T_F n_s + \left[ -\frac{3}{4} + \pi^2 - 12\zeta(3) \right] \frac{C_F}{\epsilon} \right\} + \mathcal{O}(\epsilon^0), \quad (7) \end{aligned}$$

for  $\alpha$  defined in the  $\overline{MS}$  scheme. Here  $C_A = N$ ,  $T_F = 1/2$ ,  $\beta_0 = 11C_A/3 - 4T_F n_f/3 - T_F n_s/3$  is the one-loop beta-function, and  $n_f$  ( $n_s$ ) is the number of Dirac fermions (scalars) in the fundamental representation. Eq. (7) coincides with the known massless two-loop QCD result [35, 36] except for the scalar loop contribution which is new. With  $\alpha$  renormalized at the scale  $M$  in the one-loop result, the total two-loop contribution takes the form

$$\begin{aligned} \mathcal{F}^{(2)} = & \frac{1}{2}\mathcal{F}^{(1)2} + C_F \left\{ \left[ \frac{11}{9}\mathcal{L} + \left( -\frac{233}{18} + \frac{\pi^2}{3} \right) \right] C_A + \left[ -\frac{4}{9}\mathcal{L} + \frac{38}{9} \right] T_F n_f + \left[ -\frac{1}{9}\mathcal{L} + \frac{25}{18} \right] \right. \\ & \left. \times T_F n_s \right\} \mathcal{L}^2 + \left[ \Delta_{NA}^{(2)} + \Delta_f^{(2)} + \Delta_s^{(2)} + \Delta_A^{(2)} \right] \mathcal{L} + \mathcal{O}(\mathcal{L}^0). \end{aligned} \quad (8)$$

The coefficients of the second and higher powers of the logarithm are insensitive to the infrared structure of the model as explained below. In contrast, the coefficient of the linear logarithm does in general depend on the whole mass spectrum of the model. The purely Abelian term reads [21]

$$\Delta_A^{(2)} = \left( \frac{3}{2} - 2\pi^2 + 24\zeta(3) \right) C_F^2. \quad (9)$$

Massless Dirac fermions give [16]

$$\Delta_f^{(2)} = -\frac{34}{3} C_F T_F n_f. \quad (10)$$

The non-Abelian contribution depends on the details of the gauge boson mass generation. For the spontaneously broken gauge group  $SU(2)$  with a single Higgs boson of mass  $M_H = M$  in the fundamental representation explicit calculation leads to

$$\Delta_{NA}^{(2)} + \Delta_s^{(2)} = \frac{749}{16} + \frac{43}{24}\pi^2 - 44\zeta(3) + \frac{15}{4}\sqrt{3}\pi + \frac{13}{2}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right). \quad (11)$$

Here  $\text{Cl}_2\left(\frac{\pi}{3}\right) = 1.014942\dots$  is the value of the Clausen function. For comparison, in the (hypothetical) case of a light Higgs boson  $M_H \ll M$  this contribution becomes

$$\Delta_{NA}^{(2)} + \Delta_s^{(2)} = \frac{747}{16} + \frac{97}{48}\pi^2 - 44\zeta(3) + \frac{33}{8}\sqrt{3}\pi + \frac{21}{4}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right). \quad (12)$$

In the electroweak theory inspired model with the  $SU(2)_L$  gauge group, six left-handed fermion doublets, and  $M_H = M$ , the result for the two-loop logarithmic corrections reads

$$\begin{aligned} \mathcal{F}^{(2)} = & \frac{9}{32}\mathcal{L}^4 - \frac{19}{48}\mathcal{L}^3 - \left( \frac{463}{48} - \frac{7}{8}\pi^2 \right) \mathcal{L}^2 + \left( 29 - \frac{11}{24}\pi^2 - \frac{61}{2}\zeta(3) + \frac{15}{4}\sqrt{3}\pi \right. \\ & \left. + \frac{13}{2}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) \right) \mathcal{L} + \mathcal{O}(\mathcal{L}^0). \end{aligned} \quad (13)$$

The asymptotic dependence of the form factor on  $Q$  in the Sudakov limit is governed by the hard evolution equation [26–28]

$$\frac{\partial}{\partial \ln Q^2} \mathcal{F} = \left[ \int_{M^2}^{Q^2} \frac{dx}{x} \gamma(\alpha(x)) + \zeta(\alpha(Q^2)) + \xi(\alpha(M^2)) \right] \mathcal{F}. \quad (14)$$

Its solution is

$$\mathcal{F} = F_0(\alpha(M^2)) \exp \left\{ \int_{M^2}^{Q^2} \frac{dx}{x} \left[ \int_{M^2}^x \frac{dx'}{x'} \gamma(\alpha(x')) + \zeta(\alpha(x)) + \xi(\alpha(M^2)) \right] \right\}. \quad (15)$$

By calculating the functions entering the evolution equation order by order in  $\alpha$  one gets the logarithmic approximations for the form factor. For example, the LL approximation includes all the terms of the form  $\alpha^n \mathcal{L}^{2n}$  and is determined by the one-loop value of  $\gamma(\alpha)$ ; the NLL approximation includes all the terms of the form  $\alpha^n \mathcal{L}^{2n-m}$  with  $m = 0, 1$  and requires the one-loop values of  $\gamma(\alpha)$ ,  $\zeta(\alpha)$  and  $\xi(\alpha)$  as well as the one-loop running of  $\alpha$  in  $\gamma(\alpha)$ ; and so on. The functions entering the evolution equation can in principle be determined by comparing Eq. (15) expanded in the coupling constant to the fixed order result for the form factor. Within the expansion by regions approach the logarithmic contributions show up as singularities of the different regions. One can identify the regions relevant for determining a given parameter of the evolution equation and compute them separately up to the required accuracy which facilitates the analysis by far. For example, the anomalous dimensions  $\gamma(\alpha)$  and  $\zeta(\alpha)$  are known to be mass-independent and determined by the singularities of the contribution with all the loop momenta being hard [26–28]. The dimensionally regularized hard contribution exponentiates as well [28, 37] with the functions  $\gamma(\alpha)$  and  $\zeta(\alpha)$  parameterizing the double and single pole contribution to the exponent. The one- and two-loop hard contribution can be written as

$$\begin{aligned} \mathcal{F}_h^{(1)} &= \frac{1}{Q^{2\epsilon}} \left( \frac{\gamma^{(1)}}{\epsilon^2} - \frac{\zeta^{(1)}}{\epsilon} + F_{0h}^{(1)} \right) + \mathcal{O}(\epsilon), \\ \mathcal{F}_{hh}^{(2)} &= \left( \frac{1}{2} \mathcal{F}_h^{(1)} - \frac{\beta_0}{\epsilon} \right) \mathcal{F}_h^{(1)} + \frac{1}{Q^{4\epsilon}} \left\{ \left[ \frac{1}{\epsilon^3} \frac{\gamma^{(1)} \beta_0}{4} + \frac{1}{\epsilon^2} \left( \frac{\gamma^{(2)}}{4} - \frac{\zeta^{(1)} \beta_0}{2} \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{2\epsilon} \left( -\zeta^{(2)} + F_{0h}^{(1)} \beta_0 \right) \right] \right\} + \mathcal{O}(\epsilon^0), \end{aligned} \quad (16)$$

From Eqs. (4, 7, 16) we find

$$\begin{aligned} \gamma^{(1)} &= -2C_F, \\ \zeta^{(1)} &= 3C_F, \\ \gamma^{(2)} &= C_F \left[ \left( -\frac{134}{9} + \frac{2}{3}\pi^2 \right) C_A + \frac{40}{9} T_F n_f + \frac{16}{9} T_F n_s \right], \\ \zeta^{(2)} &= C_F \left[ \left( \frac{2545}{54} + \frac{11}{9}\pi^2 - 26\zeta(3) \right) C_A - \left( \frac{418}{27} + \frac{4}{9}\pi^2 \right) T_F n_f \right. \\ &\quad \left. - \left( \frac{311}{54} + \frac{\pi^2}{9} \right) T_F n_s + \left( \frac{3}{2} - 2\pi^2 + 24\zeta(3) \right) C_F \right]. \end{aligned} \quad (17)$$

At the same time the functions  $\xi(\alpha)$  and  $F_0(\alpha)$  fix the initial conditions for the evolution equation at  $Q = M$  and do depend on the infrared structure of the model. To determine the function  $\xi(\alpha)$  one has to know the singularities of the collinear region contribution while

$F_0(\alpha)$  requires the complete information on the contributions of all the regions. The total one- and two-loop form factor can be expressed through the parameters of the evolution equation as follows

$$\begin{aligned}\mathcal{F}^{(1)} &= \frac{1}{2}\gamma^{(1)}\mathcal{L}^2 + (\xi^{(1)} + \zeta^{(1)})\mathcal{L} + F_0^{(1)}. \\ \mathcal{F}^{(2)} &= \frac{1}{8}(\gamma^{(1)})^2\mathcal{L}^4 + \frac{1}{2}\left(\xi^{(1)} + \zeta^{(1)} - \frac{1}{3}\beta_0\right)\gamma^{(1)}\mathcal{L}^3 + \frac{1}{2}\left(\gamma^{(2)} + (\xi^{(1)} + \zeta^{(1)})^2\right. \\ &\quad \left. - \beta_0\zeta^{(1)} + F_0^{(1)}\gamma^{(1)}\right)\mathcal{L}^2 + (\zeta^{(2)} + \xi^{(2)} + F_0^{(1)}(\zeta^{(1)} + \xi^{(1)}))\mathcal{L} + \mathcal{O}(\mathcal{L}^0).\end{aligned}\quad (18)$$

With the known values of  $\gamma^{(1,2)}$  and  $\zeta^{(1,2)}$  it is straightforward to obtain the result for the remaining functions

$$\begin{aligned}\xi^{(1)} &= 0, \\ F_0^{(1)} &= -C_F\left(\frac{7}{2} + \frac{2\pi^2}{3}\right) \\ \xi^{(2)} &= \xi_{NA}^{(2)} + \xi_f^{(2)} + \xi_s^{(2)} + \xi_A^{(2)},\end{aligned}\quad (19)$$

where the Abelian contribution vanishes  $\xi_A^{(2)} = 0$ , the massless Dirac fermions give

$$\xi_f^{(2)} = \left(\frac{112}{27} + \frac{4}{9}\pi^2\right)C_FT_Fn_f,\quad (20)$$

and for the spontaneously broken  $SU(2)$  model with  $M_H = M$  we get

$$\xi_{NA}^{(2)} + \xi_s^{(2)} = -\frac{391}{18} - 5\zeta(3) + \frac{15}{4}\sqrt{3}\pi + \frac{13}{2}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right).\quad (21)$$

Note that the functions  $\gamma(\alpha)$  and  $\xi(\alpha)$  are protected against the Abelian multiloop corrections by the properties of the light-cone Wilson loop [27, 28, 38].

The analysis of the evolution equation gives a lot of insight into the structure of the logarithmic corrections. For example, Eq. (18) tells us that, up to the NNLL approximation, the information on the infrared structure of the model enters through the one-loop coefficients  $\xi^{(1)}$  and  $F_0^{(1)}$  which are insensitive to the details of the mass generation. Thus one can compute the coefficient of the two-loop quadratic logarithm with the gauge boson mass introduced by hand [10].

Note that to complete the N<sup>3</sup>LL approximation one needs the three-loop value of  $\gamma(\alpha)$ . For  $n_s = 0$  this has been recently obtained in the context of QCD splitting functions [39]

$$\begin{aligned}\gamma^{(3)} &= C_F\left[\left(-\frac{245}{3} + \frac{268}{27}\pi^2 - \frac{44}{3}\zeta(3) - \frac{22}{45}\pi^4\right)C_A^2 + \left(\frac{836}{27} - \frac{80}{27}\pi^2 + \frac{112}{3}\zeta(3)\right)C_AT_Fn_f\right. \\ &\quad \left.+ \left(\frac{110}{3} - 32\zeta(3)\right)C_FT_Fn_f + \frac{32}{27}(T_Fn_f)^2\right].\end{aligned}\quad (22)$$

## 2.2 $U(1) \times U(1)$ model with mass gap

Let us now discuss the second example, a  $U(1) \times U(1)$  model with  $\lambda$ ,  $\alpha'$  and  $M$ ,  $\alpha$  for masses and coupling constants, respectively. We consider the limit  $\lambda \ll M$  and make use of the *infrared* evolution equation which governs the dependence of the form factor  $\mathcal{F}(\lambda, M, Q)$  on  $\lambda$  [8]. The virtual corrections become divergent in the limit  $\lambda \rightarrow 0$ . According to the Kinoshita-Lee-Nauenberg theorem [40, 41], these divergences are cancelled against the ones of the corrections due to the emission of real light gauge bosons of vanishing energy and/or collinear to one of the on-shell fermion lines. The singular behavior of the form factor must be the same in the full  $U_{\alpha'}(1) \times U_{\alpha}(1)$  theory and the effective  $U_{\alpha'}(1)$  model with only the light gauge boson. For  $\lambda \ll M \ll Q$  the solution of the infrared evolution equation is given by the Abelian part of the exponent (15) with  $M$ ,  $\alpha$  replaced by  $\lambda$ ,  $\alpha'$ . Thus the form factor can be written in a factorized form

$$\mathcal{F}(\lambda, M, Q) = \tilde{F}(M, Q)\mathcal{F}_{\alpha'}(\lambda, Q) + \mathcal{O}(\lambda/M), \quad (23)$$

where  $\mathcal{F}_{\alpha'}(\lambda, Q)$  stands for the  $U_{\alpha'}(1)$  form factor and  $\tilde{F}(M, Q)$  depends both on  $\alpha$  and  $\alpha'$ , and incorporates all the logarithms of the form  $\ln(Q^2/M^2)$ . It can be obtained directly by calculating the ratio

$$\tilde{F}(M, Q) = \left[ \frac{\mathcal{F}(\lambda, M, Q)}{\mathcal{F}_{\alpha'}(\lambda, Q)} \right]_{\lambda \rightarrow 0}. \quad (24)$$

Since the function  $\tilde{F}(M, Q)$  does not depend on the infrared regularization, the ratio in Eq. (24) can be evaluated with  $\lambda = 0$  using dimensional regularization for the infrared divergences. The resulting two-parameter perturbative expansion is

$$\tilde{F}(M, Q) = \sum_{n,m} \frac{\alpha'^n \alpha^m}{(4\pi)^{n+m}} \tilde{F}^{(n,m)} \quad (25)$$

where

$$\tilde{F}^{(0,0)} = 1, \quad \tilde{F}^{(n,0)} = 0, \quad \tilde{F}^{(0,m)} = \mathcal{F}^{(m)}, \quad (26)$$

and the two-loop interference term reads [21]

$$\tilde{F}^{(1,1)} = (3 - 4\pi^2 + 48\zeta(3)) \mathcal{L} + \mathcal{O}(\mathcal{L}^0). \quad (27)$$

In the equal mass case,  $\lambda = M$ , we have an additional reparameterization symmetry, and the form factor is determined by Eq. (15) with the effective coupling  $\bar{\alpha} = \alpha' + \alpha$  so that  $\mathcal{F}(M, M, Q) = \mathcal{F}_{\bar{\alpha}}(M, Q)$ . We can now write down the matching relation

$$\mathcal{F}(M, M, Q) = C(M, Q)\tilde{F}(M, Q)\mathcal{F}_{\alpha'}(M, Q), \quad (28)$$

where the matching coefficient  $C(M, Q)$  represents the effect of the power-suppressed terms neglected in Eq. (23). By combining the explicit results for  $\mathcal{F}_{\alpha'}(M, Q)$  and  $\tilde{F}(M, Q)$  the matching coefficient has found to be  $C(M, Q) = 1 + \mathcal{O}(\alpha'\alpha\mathcal{L}^0)$  [21]. In two-loops it does



not contain logarithmic terms, and up to the N<sup>3</sup>LL accuracy, the product  $\tilde{F}(M, Q)\mathcal{F}_{\alpha'}(\lambda, Q)$  continuously approaches  $\mathcal{F}(M, M, Q)$  as  $\lambda$  goes to  $M$ . Therefore, to get *all* the logarithms of the heavy gauge boson mass in two-loop approximation for the theory with mass gap, it is sufficient to divide the form factor  $\mathcal{F}_{\bar{\alpha}}(M, Q)$  of the symmetric phase by the form factor  $\mathcal{F}_{\alpha'}(\lambda, Q)$  of the effective  $U_{\alpha'}(1)$  theory taken at the symmetric point  $\lambda = M$ . Thus we have reduced the calculation in the theory with mass gap to the one in the symmetric theory with a single mass parameter. The logarithmic terms in the expansion of  $\tilde{F}(M, Q)$  exponentiate by construction and one can describe the exponent with a set of functions  $\tilde{\gamma}(\alpha, \alpha')$ ,  $\tilde{\zeta}(\alpha, \alpha')$ ,  $\tilde{\xi}(\alpha, \alpha')$  and  $\tilde{F}_0(\alpha, \alpha')$  in analogy with Eq. (15). The matching procedure can naturally be formulated in terms of these functions. For the mass-independent functions we have the all order relation

$$\begin{aligned}\tilde{\gamma}(\alpha', \alpha) &= \gamma(\bar{\alpha}) - \gamma(\alpha'), \\ \tilde{\zeta}(\alpha', \alpha) &= \zeta(\bar{\alpha}) - \zeta(\alpha').\end{aligned}\tag{29}$$

In two loops we obtain by explicit calculation

$$\tilde{\xi}(\alpha', \alpha) = \xi(\bar{\alpha}) = 0,\tag{30}$$

which holds in higher orders for the Abelian model due to the nonrenormalization properties discussed in the previous section. Thus, the only nontrivial two-loop matching is for the coefficient  $\tilde{F}_0^{(1,1)}$  due to the non-logarithmic contribution to  $C(M, Q)$  which is beyond the accuracy of our analysis.

Note that the absence of the two-loop linear-logarithmic term in  $C(M, Q)$  is an exceptional feature of the Abelian corrections. The general analysis of the evolution equation [10] shows that the terms neglected in Eq. (23) contribute starting from the N<sup>3</sup>LL approximation. Indeed, the solution of the hard evolution equation for  $F(\lambda, M, Q)$  which determines its dependence on  $Q$  is of the form (15) with the infrared sensitive quantities  $F_0$  and  $\xi$  being functions of the ratio  $\lambda/M$ . A nontrivial dependence on the mass ratio in general emerges first through the two-loop coefficient  $\xi^{(2)}$  due to the interference diagrams with both massive and massless gauge bosons. The matching is necessary to take care of the difference  $\xi^{(2)}|_{\lambda/M=1} \neq \xi^{(2)}|_{\lambda/M=0}$ . Thus, for a non-Abelian theory with the mass gap of the standard model type, *i.e.* with interaction between the heavy and light gauge bosons, the matching becomes nontrivial already in N<sup>3</sup>LL approximation.

### 3 Four-fermion amplitude

We consider the four-fermion scattering at fixed angles in the limit where all the kinematical invariants are of the same order and are far larger than the gauge boson mass,  $|s| \sim |t| \sim |u| \gg M^2$ . The analysis of the four-fermion amplitude is complicated by the additional kinematical variable and the presence of different isospin and Lorentz structures. We adopt the following notation

$$\mathcal{A}^\lambda = \bar{\psi}_2 t^a \gamma_\mu \psi_1 \bar{\psi}_4 t^a \gamma_\mu \psi_3,$$

$$\begin{aligned}
\mathcal{A}_{LL}^\lambda &= \bar{\psi}_{2L} t^a \gamma_\mu \psi_{1L} \bar{\psi}_{4L} t^a \gamma_\mu \psi_{3L}, \\
\mathcal{A}_{LR}^d &= \bar{\psi}_{2L} \gamma_\mu \psi_{1L} \bar{\psi}_{4R} \gamma_\mu \psi_{3R},
\end{aligned}
\tag{31}$$

*etc.* Here  $t^a$  denotes the  $SU(N)$  “isospin” generator,  $p_i$  the momentum of the  $i$ th fermion and  $p_1, p_3$  are incoming, and  $p_2, p_4$  outgoing momenta respectively. Hence  $t = (p_1 - p_4)^2 = -sx_-$  and  $u = (p_1 + p_3)^2 = -sx_+$  where  $x_\pm = (1 \pm \cos \theta)/2$  and  $\theta$  is the angle between  $\mathbf{p}_1$  and  $\mathbf{p}_4$ . The complete basis consists of four independent chiral amplitudes, each of them of two possible isospin structures. For the moment we consider a parity conserving theory, hence only two chiral amplitudes are not degenerate. The Born amplitude is given by

$$\mathcal{A}_B = \frac{ig^2}{s} \mathcal{A}^\lambda.
\tag{32}$$

The collinear divergences in the hard part of the virtual corrections and the corresponding *collinear* logarithms are known to factorize. They are responsible, in particular, for the double logarithmic contribution and depend only on the properties of the external on-shell particles but not on the specific process [26–28, 42–45]. This fact is especially clear if a physical (Coulomb or axial) gauge is used for the calculation. In this gauge the collinear divergences are present only in the self energy insertions to the external particles [28, 43, 45]. Thus, for each fermion-antifermion pair of the four-fermion amplitude the collinear logarithms are the same as for the form factor  $\mathcal{F}$  discussed in the previous section. Let us denote by  $\tilde{\mathcal{A}}$  the amplitude with the collinear logarithms factored out. For convenience we separate from  $\tilde{\mathcal{A}}$  all the corrections entering Eq. (15) so that

$$\mathcal{A} = \frac{ig^2}{s} \mathcal{F}^2 \tilde{\mathcal{A}}.
\tag{33}$$

The resulting amplitude  $\tilde{\mathcal{A}}$  contains the logarithms of the *soft* nature corresponding to the soft divergences of the hard region contribution<sup>1</sup>, and the renormalization group logarithms. It can be represented as a vector in the color/chiral basis and satisfies the following evolution equation [45–47]:

$$\frac{\partial}{\partial \ln Q^2} \tilde{\mathcal{A}} = \chi(\alpha(Q^2)) \tilde{\mathcal{A}},
\tag{34}$$

where  $\chi(\alpha)$  is the matrix of the soft anomalous dimensions. Note that we do not include in Eq. (34) the pure renormalization group logarithms which can be absorbed by fixing the renormalization scale of  $g$  in the Born amplitude (32) to be  $Q$ . The solution of Eq. (34) is given by the path-ordered exponent

$$\tilde{\mathcal{A}} = \text{Pexp} \left[ \int_{M^2}^{Q^2} \frac{dx}{x} \chi(\alpha(x)) \right] \mathcal{A}_0(\alpha(M^2)),
\tag{35}$$

---

<sup>1</sup>Note that within the expansion by regions the soft  $\times$  collinear double logarithmic divergence of the hard region contribution is canceled against the ultraviolet divergence of the collinear region *i.e.* there is no one-to-one correspondence between the *soft logarithms* and the ultraviolet divergences of the *soft region* contributions.

where  $\tilde{\mathcal{A}}_0(\alpha)$  determines the initial conditions for the evolution equation at  $Q = M$ . The matrix of the soft anomalous dimensions is determined by the coefficients of the single pole of the hard region contribution to the exponent (35).

In one loop the elements of the matrix  $\chi(\alpha)$  do not depend on chirality and read [9]

$$\begin{aligned}\chi_{\lambda\lambda}^{(1)} &= -2C_A (\ln(x_+) + i\pi) + 4 \left( C_F - \frac{T_F}{N} \right) \ln \left( \frac{x_+}{x_-} \right), \\ \chi_{\lambda d}^{(1)} &= 4 \frac{C_F T_F}{N} \ln \left( \frac{x_+}{x_-} \right), \\ \chi_{d\lambda}^{(1)} &= 4 \ln \left( \frac{x_+}{x_-} \right), \\ \chi_{dd}^{(1)} &= 0.\end{aligned}\tag{36}$$

In terms of the functions introduced above the one-loop correction reads

$$\begin{aligned}\mathcal{A}^{(1)} &= \frac{ig^2}{s} \left[ (\gamma^{(1)} \mathcal{L}^2(Q^2) + (2\zeta^{(1)} + 2\zeta^{(1)} + \chi_{\lambda\lambda}^{(1)}) \mathcal{L}(Q^2) + 2F_0^{(1)}) \mathcal{A}^\lambda \right. \\ &\quad \left. + \chi_{\lambda d}^{(1)} \mathcal{L}(Q^2) \mathcal{A}^d + \tilde{\mathcal{A}}_0^{(1)} \right],\end{aligned}\tag{37}$$

where

$$\tilde{\mathcal{A}}_0^{(1)} = \tilde{A}_0^{(1)\lambda}{}_{LL} \mathcal{A}_{LL}^\lambda + \tilde{A}_0^{(1)\lambda}{}_{LR} \mathcal{A}_{LR}^\lambda + \dots\tag{38}$$

For the present two-loop analysis of the annihilation cross section only the real part of the coefficients  $\tilde{A}_0^{(1)}$  is needed (see *e.g.* [10]),

$$\begin{aligned}\text{Re} \left[ \tilde{A}_0^{(1)\lambda}{}_{LL} \right] &= \left( C_F - \frac{T_F}{N} \right) f(x_+, x_-) + C_A \left( \frac{85}{9} + \pi^2 \right) - \frac{20}{9} T_F n_f - \frac{8}{9} T_F n_s, \\ \text{Re} \left[ \tilde{A}_0^{(1)\lambda}{}_{LR} \right] &= - \left( C_F - \frac{T_F}{N} - \frac{C_A}{2} \right) f(x_-, x_+) + C_A \left( \frac{85}{9} + \pi^2 \right) - \frac{20}{9} T_F n_f - \frac{8}{9} T_F n_s, \\ \text{Re} \left[ \tilde{A}_0^{(1)d}{}_{LL} \right] &= \frac{C_F T_F}{N} f(x_+, x_-), \\ \text{Re} \left[ \tilde{A}_0^{(1)d}{}_{LR} \right] &= - \frac{C_F T_F}{N} f(x_-, x_+),\end{aligned}\tag{39}$$

where

$$f(x_+, x_-) = \frac{2}{x_+} \ln x_- + \frac{x_- - x_+}{x_+^2} \ln^2 x_-.\tag{40}$$

The two-loop matrix  $\chi^{(2)}$  can be extracted from the result for the single pole part of the hard contribution to the four-quark amplitude [48–50]. For vanishing beta-function it reads

$$\chi^{(2)}|_{\beta_0=0} = \frac{\gamma^{(2)}}{\gamma^{(1)}} \chi^{(1)},\tag{41}$$

After averaging over the isospin Eq. (41) agrees with the result of Ref. [51]. An alternative approach to compute the soft anomalous dimension matrix is based on studying the renormalization properties of the product of light-like Wilson lines (see, *e.g.* [52]). This approach has been used in Ref. [53] for the calculation of the soft anomalous dimension matrix in two loops. The result of Ref. [53] is in agreement with Eq. (41).

The part of the matrix  $\chi^{(2)}$  proportional to the beta-function originates from the running of the coupling constant in the one-loop nonlogarithmic term. We find that Eq. (39) gets contributions only from the hard region so that the effective renormalization scale of  $\alpha$  there is  $Q$ . Hence

$$\chi_{\lambda\lambda}^{(2)} \mathcal{A}^\lambda \Big|_{\beta_0} = -\beta_0 \left( \tilde{A}_0^{(1)\lambda}{}_{LL} \mathcal{A}_{LL}^\lambda + \tilde{A}_0^{(1)\lambda}{}_{LR} \mathcal{A}_{LR}^\lambda + \tilde{A}_0^{(1)d}{}_{LL} \mathcal{A}_{LL}^d + \tilde{A}_0^{(1)d}{}_{LR} \mathcal{A}_{LR}^d \right), \quad (42)$$

*i.e.* in this order the soft anomalous dimension matrix does depend on the chirality. The matrix  $\chi^{(1,2)}$  as well as the coefficients  $\tilde{A}_0^{(1)i}{}_{IJ}$  are determined by the hard region contribution and are insensitive to the details of the gauge boson mass generation.

Note that all the nontrivial soft region contributions in the expansion of the multiscale two-loop integrals in the Sudakov limit are power-suppressed [23–25]. The unsuppressed soft region contributions originate from one-scale self-energy insertions into an external on-shell fermion or into a collinear gluon propagator. On the other hand one power of the logarithm in the double logarithmic contribution is of soft nature. Thus the two-loop soft logarithms in  $\tilde{\mathcal{A}}$  can be related to the collinear divergences though the latter factorize and do not contribute to the soft anomalous dimension matrix themselves. This, in particular, explains the presence of the  $\gamma^{(2)}$  contribution to  $\chi^{(2)}$ .

The two-loop correction to the four-fermion amplitude is obtained by the direct generalization of the form factor analysis with the only complication related to the matrix structure of Eq. (35)

$$\begin{aligned} \mathcal{A}^{(2)} = & \frac{ig^2(Q^2)}{s} \left\{ \frac{1}{2} (\gamma^{(1)})^2 \mathcal{L}^4 \mathcal{A}^\lambda + \left[ \left( 2\zeta^{(1)} + \chi_{\lambda\lambda}^{(1)} - \frac{1}{3}\beta_0 \right) \mathcal{A}^\lambda + \chi_{\lambda d}^{(1)} \mathcal{A}^d \right] \gamma^{(1)} \mathcal{L}^3 \right. \\ & + \left[ \left( \gamma^{(2)} + (2\zeta^{(1)} - \beta_0) \zeta^{(1)} + 2F_0^{(1)} \gamma^{(1)} + \frac{1}{2} \left( (4\zeta^{(1)} - \beta_0) \chi_{\lambda\lambda}^{(1)} + \chi_{\lambda\lambda}^{(1)2} + \chi_{\lambda d}^{(1)} \chi_{d\lambda}^{(1)} \right) \right) \mathcal{A}^\lambda \right. \\ & + \frac{1}{2} \left( (4\zeta^{(1)} - \beta_0) \chi_{\lambda d}^{(1)} + \chi_{\lambda\lambda}^{(1)} \chi_{\lambda d}^{(1)} \right) \mathcal{A}^d + \gamma^{(1)} \tilde{\mathcal{A}}_0^{(1)} \left. \right] \mathcal{L}^2 + \left[ (2\zeta^{(2)} + 2\xi^{(2)} + 2F_0^{(1)} (2\zeta^{(1)} \right. \\ & \left. + \chi_{\lambda\lambda}^{(1)}) + \chi_{\lambda\lambda}^{(2)} \right) \mathcal{A}^\lambda + (2F_0^{(1)} \chi_{\lambda d}^{(1)} + \chi_{\lambda d}^{(2)}) \mathcal{A}^d + (2\zeta^{(1)} + \chi^{(1)}) \tilde{\mathcal{A}}_0^{(1)} \left. \right] \mathcal{L} + \mathcal{O}(\mathcal{L}^0) \left. \right\}, \quad (43) \end{aligned}$$

where we used the fact that  $\xi^{(1)} = \chi_{dd}^{(1)} = 0$  and the matrix structure of the product  $\chi^{(1)} \tilde{\mathcal{A}}_0^{(1)}$  is implied.

Let us again discuss the standard model inspired example considered in the previous section. With the result for the amplitudes it is straightforward to compute the one- and two-loop corrections to the total cross section of the four-fermion annihilation process. For the annihilation process  $f' \bar{f}' \rightarrow f \bar{f}$  one has to make the analytical continuation of the above result to the Minkowskian region of negative  $Q^2 = -s$  according to the  $s + i0$  prescription.

The above approximation is formally not valid for the small angle region  $\theta < M/\sqrt{s}$ , which, however, gives only a power-suppressed contribution to the total cross section. For the  $SU(2)_L$  model we obtain

$$\begin{aligned}\sigma^{(2)} &= \left[ \frac{9}{2} \mathcal{L}^4(s) - \frac{449}{6} \mathcal{L}^3(s) + \left( \frac{4855}{18} + \frac{37}{3} \pi^2 \right) \mathcal{L}^2(s) + \left( \frac{48049}{216} - \frac{1679}{18} \pi^2 - 122\zeta(3) \right. \right. \\ &\quad \left. \left. + 15\sqrt{3}\pi + 26\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) \right) \mathcal{L}(s) \right] \sigma_B \\ &\approx \left( 4.50 \mathcal{L}^4(s) - 74.83 \mathcal{L}^3(s) + 391.45 \mathcal{L}^2(s) - 717.49 \mathcal{L}(s) \right) \sigma_B,\end{aligned}\quad (44)$$

and

$$\begin{aligned}\sigma^{(2)} &= \left[ \frac{9}{2} \mathcal{L}^4(s) - \frac{125}{6} \mathcal{L}^3(s) - \left( \frac{799}{9} - \frac{37}{3} \pi^2 \right) \mathcal{L}^2(s) + \left( \frac{38005}{216} - \frac{383}{18} \pi^2 - 122\zeta(3) \right. \right. \\ &\quad \left. \left. + 15\sqrt{3}\pi + 26\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) \right) \mathcal{L}(s) \right] \sigma_B \\ &\approx \left( 4.50 \mathcal{L}^4(s) - 20.83 \mathcal{L}^3(s) + 32.95 \mathcal{L}^2(s) - 53.38 \mathcal{L}(s) \right) \sigma_B,\end{aligned}\quad (45)$$

for the initial and final state fermions of the same or opposite eigenvalue of the isospin operator  $t^3$ , respectively. Here  $\sigma_B$  is the Born cross section with the  $\overline{MS}$  coupling constants renormalized at the scale  $\sqrt{s}$  and  $\mathcal{L}(s) = \ln(s/M^2)$ .

## 4 Two-loop electroweak logarithms

In Born approximation, the amplitude for the neutral current process  $f' \bar{f}' \rightarrow f \bar{f}$  reads

$$\mathcal{A}_B = \frac{ig^2}{s} \sum_{I,J=L,R} \left( T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \mathcal{A}_{IJ}^{f'f}, \quad (46)$$

where

$$\mathcal{A}_{IJ}^{f'f} = \bar{f}'_I \gamma_\mu f'_I \bar{f}_J \gamma_\mu f_J, \quad (47)$$

$t_W \equiv \tan \theta_W$  with  $\theta_W$  being the Weinberg angle and  $T_f$  ( $Y_f$ ) is the isospin (hypercharge) of the fermion which depends on the fermion chirality. The one-loop electroweak correction to the amplitude is well known (see [54] for the most general result). The calculation of the two-loop electroweak corrections even in the high energy limit is a challenging theoretical problem at the limit of available computational techniques. It is complicated in particular by the presence of the mass gap and mixing in the gauge sector. In Sect. 4.1 we develop the approach of Ref. [21] and reduce the analysis of the dominant two-loop logarithmic electroweak corrections to a problem with a single mass parameter which has been solved in the previous section. In Sect. 4.2 we present the explicit expression for the two-loop virtual linear-logarithmic electroweak correction to the four-fermion amplitudes. In Sect. 4.3 we complete the analysis of Refs. [9, 10] by computing the leading effects of the  $W - Z$  mass splitting through NNLL approximation. In Sect. 4.4 we introduce infrared safe semi-inclusive cross sections and in Sect. 4.5 we give numerical results for the two-loop corrections to the cross sections and various asymmetries.

## 4.1 Separating QED infrared logarithms

The main difference between the analysis of the electroweak standard model with the spontaneously broken  $SU_L(2) \times U(1)$  gauge group and the treatment of the pure  $SU_L(2)$  case considered above is the presence of the massless photon which results in infrared divergences of fully exclusive cross sections. We regularize these divergences by giving the photon a small mass  $\lambda$ . The dependence of the virtual corrections on  $\lambda$  in the limit  $\lambda^2 \ll M^2 \ll Q^2$  is governed by the QED infrared evolution equation and is given by the factor

$$\begin{aligned} \mathcal{U} = & U_0(\alpha_e(Q^2)) \\ & \times \exp \left\{ \frac{\alpha_e(\lambda^2)}{4\pi} \left[ - \left( Q_f^2 + Q_{f'}^2 - \frac{\alpha_e}{\pi} \left( \frac{76}{27} N_g (Q_f^2 + Q_{f'}^2) + \frac{16}{9} N_g \ln \left( \frac{x_+}{x_-} \right) Q_{f'} Q_f \right) \right) \right. \right. \\ & \times \ln^2 \left( \frac{Q^2}{\lambda^2} \right) + \left[ \left( 3 + \frac{\alpha_e}{\pi} \left( \left( -\frac{40}{9} + \frac{16}{27} \pi^2 \right) N_g + \left( \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta(3) \right) Q_f^2 \right) \right) Q_f^2 + (f \leftrightarrow f') \right. \\ & \left. \left. + \left( 4 - \frac{\alpha_e}{\pi} \frac{160}{27} N_g \right) \ln \left( \frac{x_+}{x_-} \right) Q_{f'} Q_f \right] \ln \left( \frac{Q^2}{\lambda^2} \right) - \frac{\alpha_e}{\pi} \frac{8}{27} N_g (Q_f^2 + Q_{f'}^2) \ln^3 \left( \frac{Q^2}{\lambda^2} \right) \right] \right\}, \quad (48) \end{aligned}$$

where  $\alpha_e$  is the  $\overline{MS}$  QED coupling constant and  $Q_f$  is the electric charge of the fermion. The NNLL approximation for  $\mathcal{U}$  can be found in [10] and to derive the two-loop linear logarithm in the exponent (48) we use the following expressions for the QED parameters:

$$\begin{aligned} \zeta_e^{(2)} + \xi_e^{(2)} &= \left[ -\frac{272}{9} N_g + \left( \frac{3}{2} - 2\pi^2 + 24\zeta(3) \right) Q_f^2 \right] Q_f^2, \\ \chi_e^{(2)}|_{\beta_0=0} &= -\frac{640}{27} \ln \left( \frac{x_+}{x_-} \right) N_g Q_{f'} Q_f. \end{aligned} \quad (49)$$

The expressions for  $\zeta_e^{(2)}$ ,  $\xi_e^{(2)}$  and  $\chi_e^{(2)}$  can be obtained by substituting  $n_f \rightarrow 8N_g/3$  into the general formulae. The coefficient  $U_0(\alpha_e(Q^2))$  in Eq. (48) is a two-component vector in the chiral basis. We have a freedom in definition of this quantity because it does not depend on  $\lambda$  and can be absorbed into the electroweak part of the corrections. It is convenient to normalize the QED factor to  $\mathcal{U}|_{Q=\lambda} = 1$  so that  $U_0^{(1)} = 0$ . Note that the structure of Eq. (48) differs from the one of the solution of the hard evolution equations (15, 35) because  $\alpha_e$  in the preexponential factor  $U_0(\alpha_e)$  is renormalized at  $Q$  so that all the dependence on  $\lambda$  is contained in the exponent. By contrast, in  $F_0(\alpha)$  and  $\tilde{A}_0(\alpha)$  the coupling constant is renormalized at  $M$  so that all the dependence on  $Q$  is contained in the exponent. This is why there is no  $\chi_e^{(2)}|_{\beta_0}$  contribution to the exponent (48) and the  $N_g Q_f^2$  part of the linear logarithm coefficient there differs from the one of  $\zeta_e^{(2)} + \xi_e^{(2)}$ .

Our goal now is to separate the above infrared divergent QED contribution from the total two-loop corrections to get the pure electroweak logarithms  $\ln(Q^2/M_{W,Z}^2)$ . Within the evolution equation approach [10] it has been found that the electroweak and QED logarithms up to the NNLL approximation can be disentangled by means of the following two-step procedure:

- (i) the corrections are evaluated using the fields of unbroken symmetry phase with all the gauge bosons of the same mass  $M \approx M_{Z,W}$ , *i.e.* without mass gap;
- (ii) the QED contribution (48) with  $\lambda = M$  is factorized leaving the pure electroweak logarithms.

This reduces the calculation of the two-loop electroweak logarithms up to the quadratic term to a problem with a single mass parameter. Then the effect of the  $Z - W$  boson mass splitting can systematically be taken into account within an expansion around the equal mass approximation [21]. In general the above two-step procedure is not valid in the N<sup>3</sup>LL approximation which is sensitive to fine details of the gauge boson mass generation. For the exact calculation of the coefficient of the two-loop linear-logarithmic term one has to use the true mass eigenstates of the standard model. The evaluation of the corrections in this case becomes a very complicated multiscale problem. The analysis, however, is drastically simplified in a model with a Higgs boson of zero hypercharge. In this model the mixing is absent and the above two-step procedure can be applied to disentangle all the two-loop logarithms of the  $SU_L(2)$  gauge boson mass from the infrared logarithms associated with the massless hypercharge gauge boson (see the discussion below). In the standard model the mixing of the gauge bosons results in a linear-logarithmic contribution which is not accounted for in this approximation. It is, however, suppressed by the small factor  $\sin^2 \theta_W \equiv s_W^2 \approx 0.2$ . Therefore, the approximation gives an estimate of the coefficient in front of the linear electroweak logarithm with 20% accuracy. As we will see, a 20% error in the coefficient in front of the two-loop linear electroweak logarithm leads to an uncertainty comparable to the nonlogarithmic contribution and is practically negligible. If we also neglect the difference between  $M_H$  and  $M_{Z,W}$ , the calculation involves a single mass parameter at every step and the results of the previous sections can directly be applied to the isospin  $SU(2)_L$  gauge group with the coupling  $g$  and the hypercharge  $U(1)$  gauge group with the coupling  $t_W g$ .

The procedure of separating the QED contribution can naturally be formulated in terms of the functions parameterizing the solution of the hard evolution equation for the four-fermion amplitude. The functions  $\gamma(\alpha)$ ,  $\zeta(\alpha)$ , and  $\chi(\alpha)$  are mass-independent. Therefore, these functions parameterizing the electroweak logarithms can be obtained by subtracting the QED quantities from the result for the unbroken symmetry phase to all orders in the coupling constants (*cf.* Eq. (29)) without referring to the simplified model. To the order of interest they can be found in [10] or easily derived from the results of the previous sections. For example, after the subtraction, the  $\beta_0 = 0$  part of the matrix  $\chi^{(2)}$  for  $I, J = L$  takes the form

$$\begin{aligned} \chi_{\lambda\lambda}^{(2)}|_{\beta_0=0} &= \left( -\frac{20}{9}N_g + \frac{130}{9} - \frac{2\pi^2}{3} \right) \left[ -4(\ln(x_+) + i\pi) + 2\ln\left(\frac{x_+}{x_-}\right) \right] \\ &+ \left[ -\left(\frac{100}{27}N_g + \frac{4}{9}\right)t_W^4 Y_{f'} Y_f + \frac{640}{27}N_g s_W^4 Q_{f'} Q_f \right] \ln\left(\frac{x_+}{x_-}\right), \end{aligned}$$

$$\begin{aligned}
\chi_{\lambda d}^{(2)}|_{\beta_0=0} &= \left(-\frac{5}{3}N_g + \frac{65}{6} - \frac{\pi^2}{2}\right) \ln\left(\frac{x_+}{x_-}\right), \\
\chi_{d\lambda}^{(2)}|_{\beta_0=0} &= \left(-\frac{80}{9}N_g + \frac{520}{9} - \frac{8\pi^2}{3}\right) \ln\left(\frac{x_+}{x_-}\right), \\
\chi_{dd}^{(2)}|_{\beta_0=0} &= \left[-\left(\frac{100}{27}N_g + \frac{4}{9}\right)t_W^4 Y_{f'} Y_f + \frac{640}{27}N_g s_W^4 Q_{f'} Q_f\right] \ln\left(\frac{x_+}{x_-}\right). \tag{50}
\end{aligned}$$

For  $I$  or/and  $J = R$  it is reduced to

$$\chi^{(2)}|_{\beta_0=0} = \left[-\left(\frac{100}{27}N_g + \frac{4}{9}\right)t_W^4 Y_{f'} Y_f + \frac{640}{27}N_g s_W^4 Q_{f'} Q_f\right] \ln\left(\frac{x_+}{x_-}\right). \tag{51}$$

Note that no QED subtraction is necessary for  $\chi^{(2)}|_{\beta_0}$  because of the specific normalization of the QED factor  $\mathcal{U}$ . Thus  $\xi^{(2)}$  is the only two-loop coefficient at the order of interest which is sensitive to fine details of the gauge boson mass generation. We evaluate it approximately by using the above simplified model with the Higgs boson of zero hypercharge and of the mass  $M_H = M_{W,Z}$ . In this model the interference diagrams including the heavy  $SU_L(2)$  and the light hypercharge  $U(1)$  gauge bosons are identical to the ones of the  $U(1) \times U(1)$  model discussed in Sect. 2.2 and the corresponding contribution to  $\xi^{(2)}$  vanishes. At the same time the pure non-Abelian contribution to  $\xi^{(2)}$  is given by Eq. (19). Collecting all the pieces for the two-loop contribution to the scheme-independent sum  $\zeta(\alpha) + \xi(\alpha)$  in the simplified model we obtain

$$\begin{aligned}
\zeta^{(2)} + \xi^{(2)} &= \left[-\frac{34}{3}N_g + \frac{749}{12} + \frac{43}{18}\pi^2 - \frac{176}{3}\zeta(3) + 5\sqrt{3}\pi + \frac{26}{3}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right)\right. \\
&\quad \left.+ (3 - 4\pi^2 + 48\zeta(3))t_W^2 \frac{Y_f^2}{4}\right] T_f(T_f + 1), \tag{52}
\end{aligned}$$

where  $T_f = 0$  and  $1/2$  for the right- and left-handed fermions, respectively.

## 4.2 Two-loop virtual corrections

The expressions for the two-loop corrections to the electroweak amplitudes is obtained by projecting the result of Sect. 3 on a relevant initial/final state with the proper assignment of isospin/hypercharge. For example, the projection of the basis (31) on the states corresponding to the neutral current processes reads  $\mathcal{A}_{IJ}^\lambda \rightarrow T_f^3 T_{f'}^3 \mathcal{A}_{IJ}^{f'f}$ ,  $\mathcal{A}_{IJ}^d \rightarrow \mathcal{A}_{IJ}^{f'f}$ . The only complication in combinatorics is related to the fact that now we are having different gauge groups for the fermions of different chirality. To take into account the light fermion contribution one has to replace  $n_f \rightarrow 2N_g$  for the  $SU(2)_L$  and  $n_f \rightarrow 5N_g/3$  for the hypercharge  $U(1)$  gauge group, where  $N_g = 3$  stands for the number of generations. The result for the linear-logarithmic corrections is given in the following approximation:

- (i) the fermion coupling to the Higgs boson is neglected, *i.e.* all the fermions are considered to be massless;



(ii) the  $W - Z$  mass splitting is neglected;

(iii) the approximation (52) is used for the two-loop coefficient  $\zeta^{(2)} + \xi^{(2)}$ .

The accuracy of the approximation is discussed in Sect. (4.5). The result for the  $n$ -loop correction to the amplitude (46) can be decomposed as

$$A^{(n)} = \mathcal{A}_{LL}^{(n)} + \mathcal{A}_{NLL}^{(n)} + \mathcal{A}_{NNLL}^{(n)} + \dots \quad (53)$$

Explicit expressions for the corrections up to  $\mathcal{A}_{NNLL}^{(2)}$  can be found in [9, 10]. The two-loop  $N^3LL$  term is new. For convenience we split it in parts as follows:

$$\mathcal{A}_{N^3LL}^{(2)} = \sum_{i=1}^6 \Delta_i \mathcal{A}_{N^3LL}^{(2)}. \quad (54)$$

The correction corresponding to the  $\xi^{(2)}$ ,  $\zeta^{(2)}$  and  $F_0^{(1)}\zeta^{(1)}$  terms of Eq. (43) is

$$\begin{aligned} a^2 \Delta_1 \mathcal{A}_{N^3LL}^{(2)} = & \frac{ig^2}{s} \sum_{I,J=L,R} \left\{ \left[ \left( -\frac{34}{3}N_g + \frac{749}{12} + \frac{43}{18}\pi^2 - \frac{176}{3}\zeta(3) + 5\sqrt{3}\pi \right. \right. \right. \\ & \left. \left. + \frac{26}{3}\sqrt{3}\text{Cl}_2\left(\frac{\pi}{3}\right) + (3 - 4\pi^2 + 48\zeta(3))t_W^2 \frac{Y_f^2}{4} \right] \times T_f(T_f + 1) + (f \leftrightarrow f') \right] \\ & - \left( \frac{21}{2} + 2\pi^2 \right) \left[ T_f(T_f + 1) + t_W^2 \frac{Y_f^2}{4} - s_W^2 Q_f^2 + (f \leftrightarrow f') \right] \\ & \left. \times \left[ T_f(T_f + 1) + t_W^2 \frac{Y_f^2}{4} + (f \leftrightarrow f') \right] \right\} \left( T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) l(Q^2) a \mathcal{A}_{IJ}^{f'f}, \quad (55) \end{aligned}$$

where  $a = g^2/16\pi^2$  and  $l(Q^2) = a \ln(Q^2/M^2)$ . The correction corresponding to the  $\chi^{(2)}$  term of Eq. (43) for vanishing beta-function reads

$$\begin{aligned} a^2 \Delta_2 \mathcal{A}_{N^3LL}^{(2)} = & \frac{ig^2}{s} \sum_{I,J=L,R} \left\{ \left( -\frac{20}{9}N_g + \frac{130}{9} - \frac{2\pi^2}{3} \right) \left[ \left( -4(\ln(x_+) + i\pi) \right. \right. \right. \\ & \left. \left. + \ln\left(\frac{x_+}{x_-}\right) (2 + t_W^2 Y_{f'} Y_f) \right) T_{f'}^3 T_f^3 + \frac{3}{4} \ln\left(\frac{x_+}{x_-}\right) \delta_{IL} \delta_{JL} \right] + \ln\left(\frac{x_+}{x_-}\right) \left[ -\left(\frac{100}{27}N_g + \frac{4}{9}\right) \right. \right. \\ & \left. \left. \times t_W^4 Y_{f'} Y_f + \frac{640}{27} N_g s_W^4 Q_{f'} Q_f \right] \left( T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \right\} l(Q^2) a \mathcal{A}_{IJ}^{f'f}, \quad (56) \end{aligned}$$

while the  $\beta_0$  part of  $\chi^{(2)}$  term gives

$$\begin{aligned} a^2 \Delta_3 \mathcal{A}_{N^3LL}^{(2)} = & \frac{ig^2}{s} \sum_{I,J=L,R} \left\{ \left[ \left( \frac{32}{9}N_g - 7 \right) T_{f'}^3 T_f^3 + \left( \frac{20}{9}N_g + \frac{1}{6} \right) t_W^4 \frac{Y_{f'} Y_f}{4} \right] t_W^2 \frac{Y_{f'} Y_f}{4} \right. \\ & \left. \times \left[ f(x_+, x_-) (\delta_{IR} \delta_{JR} + \delta_{IL} \delta_{JL}) - f(x_-, x_+) (\delta_{IR} \delta_{JL} + \delta_{IL} \delta_{JR}) \right] \right. \\ & - \left[ \left( \frac{80}{27}N_g^2 - \frac{46}{3}N_g - \frac{86}{27} \right) T_{f'}^3 T_f^3 + \left( \frac{2000}{243}N_g^2 + \frac{130}{81}N_g + \frac{2}{27} \right) t_W^6 \frac{Y_{f'} Y_f}{4} \right] + \left( \frac{4}{3}N_g - \frac{43}{6} \right) \\ & \left. \times \left[ \left( \frac{1}{2}f(x_+, x_-) + \frac{170}{9} + 2\pi^2 \right) T_{f'}^3 T_f^3 + \frac{3}{16}f(x_+, x_-) \delta_{IL} \delta_{JL} \right] \right\} l(Q^2) a \mathcal{A}_{IJ}^{f'f}, \quad (57) \end{aligned}$$

where  $\delta_{IJ} = 1$  for  $I = J$  and zero otherwise. Note that our analysis implies the  $\overline{MS}$  coupling constants renormalized at the scale  $M$  in the one-loop result with the exception of the one power of the coupling constants originating from the Born amplitude (46) renormalized at the scale  $Q$ . The correction corresponding to the  $\zeta^{(1)}\text{Re}[\tilde{A}_0^{(1)}]$  term of Eq. (43) is

$$\begin{aligned}
a^2 \Delta_4 \mathcal{A}_{N^3 LL}^{(2)} &= \frac{ig^2}{s} \sum_{I,J=L,R} 3 \left[ T_f(T_f + 1) + t_W^2 \frac{Y_f^2}{4} - s_W^2 Q_f^2 + (f \leftrightarrow f') \right] \left\{ t_W^2 \frac{Y_{f'} Y_f}{4} \right. \\
&\times \left( 2T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \left[ f(x_+, x_-) (\delta_{IR} \delta_{JR} + \delta_{IL} \delta_{JL}) - f(x_-, x_+) (\delta_{IR} \delta_{JL} + \delta_{IL} \delta_{JR}) \right] \\
&- \left[ \left( \frac{20}{9} N_g + \frac{4}{9} \right) T_{f'}^3 T_f^3 + \left( \frac{100}{27} N_g + \frac{4}{9} \right) t_W^4 \frac{Y_{f'} Y_f}{4} \right] + \left( \frac{1}{2} f(x_+, x_-) + \frac{170}{9} + 2\pi^2 \right) T_{f'}^3 T_f^3 \\
&\left. + \frac{3}{16} f(x_+, x_-) \delta_{IL} \delta_{JL} \right\} l(Q^2) a \mathcal{A}_{IJ}^{f'f}. \tag{58}
\end{aligned}$$

The correction corresponding to the  $\chi^{(1)}\text{Re}[\tilde{A}_0^{(1)}]$  term of Eq. (43) is

$$\begin{aligned}
a^2 \Delta_5 \mathcal{A}_{N^3 LL}^{(2)} &= \frac{ig^2}{s} \sum_{I,J=L,R} \left\{ \left[ \left( -8 \ln(x_+) + i\pi \right) + \ln \left( \frac{x_+}{x_-} \right) \left( 4 + t_W^2 Y_{f'} Y_f \right) \right] T_{f'}^3 T_f^3 \right. \\
&+ \frac{3}{2} \ln \left( \frac{x_+}{x_-} \right) \delta_{IL} \delta_{JL} + \ln \left( \frac{x_+}{x_-} \right) \left( t_W^2 Y_{f'} Y_f - 4s_W^2 Q_{f'} Q_f \right) \left( 2T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \left. \right] t_W^2 \frac{Y_{f'} Y_f}{4} \\
&\times \left[ f(x_+, x_-) (\delta_{IR} \delta_{JR} + \delta_{IL} \delta_{JL}) - f(x_-, x_+) (\delta_{IR} \delta_{JL} + \delta_{IL} \delta_{JR}) \right] \\
&+ \left[ \left( \frac{1}{2} f(x_+, x_-) - \frac{20}{9} N_g + \frac{166}{9} + 2\pi^2 \right) \left( -4 \ln(x_+) + i\pi \right) + 2 \ln \left( \frac{x_+}{x_-} \right) \right] \\
&+ \ln \left( \frac{x_+}{x_-} \right) \left( - \left( \frac{100}{27} N_g + \frac{4}{9} \right) t_W^4 Y_{f'} Y_f + \frac{3}{4} f(x_+, x_-) \right) \left. \right] T_{f'}^3 T_f^3 \\
&+ \left( \frac{3}{8} f(x_+, x_-) - \frac{5}{3} N_g + \frac{83}{6} + \frac{3}{2} \pi^2 \right) \ln \left( \frac{x_+}{x_-} \right) \delta_{IL} \delta_{JL} \\
&+ \left[ \left( \frac{1}{2} f(x_+, x_-) - \frac{20}{9} N_g + \frac{166}{9} + 2\pi^2 \right) T_{f'}^3 T_f^3 - \left( \frac{100}{27} N_g + \frac{4}{9} \right) t_W^4 \frac{Y_{f'} Y_f}{4} \right. \\
&\left. + \frac{3}{16} f(x_+, x_-) \delta_{IL} \delta_{JL} \right] \ln \left( \frac{x_+}{x_-} \right) \left[ t_W^2 Y_{f'} Y_f - 4s_W^2 Q_{f'} Q_f \right] \left. \right\} l(Q^2) a \mathcal{A}_{IJ}^{f'f}. \tag{59}
\end{aligned}$$

The correction corresponding to the  $\chi^{(1)}F_0^{(1)}$  term of Eq. (43) reads

$$\begin{aligned}
a^2 \Delta_6 \mathcal{A}_{N^3 LL}^{(2)} &= \frac{ig^2}{s} \sum_{I,J=L,R} - \left\{ \left[ \left( -4 \ln(x_+) + i\pi \right) + \ln \left( \frac{x_+}{x_-} \right) \left( 2 + t_W^2 Y_{f'} Y_f \right) \right] T_{f'}^3 T_f^3 \right. \\
&+ \frac{3}{4} \ln \left( \frac{x_+}{x_-} \right) \delta_{IL} \delta_{JL} \left. \right] + \ln \left( \frac{x_+}{x_-} \right) \left( t_W^2 Y_{f'} Y_f - 4s_W^2 Q_{f'} Q_f \right) \left( T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \left. \right\} \\
&\times \left( \frac{7}{2} + \frac{2}{3} \pi^2 \right) \left[ T_f(T_f + 1) + t_W^2 \frac{Y_f^2}{4} + (f \leftrightarrow f') \right] l(Q^2) a \mathcal{A}_{IJ}^{f'f}. \tag{60}
\end{aligned}$$

The above expressions are derived for Euclidean positive  $Q^2$ . For the annihilation process one has to make the analytical continuation to the Minkowskian region of negative  $Q^2 = -(s+i0)$  and it is natural to normalize the QED factor to  $\mathcal{U}|_{s=M^2} = 1$  at the Minkowskian point  $s = M^2$  rather than at  $Q^2 = M^2$ . If we use this normalization condition then  $\text{Re} [U_0^{(1)}] = -\pi^2(Q_f^2 + Q_{f'}^2)$  and no QED subtraction is necessary for (the real part of) the amplitude at  $s = M^2$ .

### 4.3 The effect of $W - Z$ mass splitting

The NLL and NNLL result for the electroweak corrections has been obtained in Refs. [9,10] neglecting the  $Z - W$  boson mass splitting which is suppressed by a small factor  $\delta_M = 1 - M_W^2/M_Z^2 \approx 0.2$ . The effect of the mass splitting in the NLL and NNLL terms can, however, be comparable to the N<sup>3</sup>LL contribution and should be taken into account at this accuracy level. In the NLL and NNLL approximation the mass splitting can easily be taken into account through the modification of the one-loop parameters  $\xi^{(1)}$ ,  $F_0^{(1)}$ , and  $\tilde{\mathcal{A}}_0^{(1)}$ . Let us take  $M_W$  as the argument of the logarithms. Then the mass splitting corrections originate from the  $Z$  boson contribution. From the explicit result for the one-loop diagrams in the leading order in  $s_W^2$  and<sup>2</sup>  $\delta_M \propto s_W^2$  we obtain

$$\begin{aligned}\delta\xi^{(1)} &= \frac{2}{3}T_f(T_f + 1)\delta_M, \\ \delta F_0^{(1)} &= -T_f(T_f + 1)\delta_M, \\ \delta\tilde{\mathcal{A}}_0^{(1)} &= -4\ln\left(\frac{x_+}{x_-}\right)T_{f'}^3T_f^3\left(T_{f'}^3T_f^3 + t_W^2\frac{Y_{f'}Y_f}{4}\right)\delta_M\mathcal{A}_{LL}^{f'f}.\end{aligned}\quad (61)$$

By using Eq. (61) it is straightforward to get the leading effect of the mass splitting in one and two loops through the NNLL approximation. The corrections to the amplitudes read

$$\begin{aligned}a\mathcal{A}^{(1)}\Big|_{\delta_M} &= \frac{ig^2}{s}\sum_{I,J=L,R}\left[[T_f(T_f + 1) + (f \leftrightarrow f')]\left(-a + \frac{2}{3}l(s)\right) - 4T_{f'}^3T_f^3\ln\left(\frac{x_+}{x_-}\right)a\right] \\ &\quad \times\left(T_{f'}^3T_f^3 + t_W^2\frac{Y_{f'}Y_f}{4}\right)\delta_M\mathcal{A}_{IJ}^{f'f}, \\ a^2\mathcal{A}^{(2)}\Big|_{\delta_M} &= \frac{ig^2}{s}\sum_{I,J=L,R}\left\{\left[[T_f(T_f + 1) + (f \leftrightarrow f')]\left(-\frac{2}{3}L(s)l(s) + 3l^2(s)\right)\right.\right. \\ &\quad \left.+ 4T_{f'}^3T_f^3\ln\left(\frac{x_+}{x_-}\right)l^2(s)\right]\left[T_f(T_f + 1) + t_W^2\frac{Y_f^2}{4} - s_W^2Q_f^2 + (f \leftrightarrow f')\right] \\ &\quad \times\left(T_{f'}^3T_f^3 + t_W^2\frac{Y_{f'}Y_f}{4}\right) + \left[\left(-4(\ln(x_+) + i\pi) + \ln\left(\frac{x_+}{x_-}\right)\right)(2 + t_W^2Y_{f'}Y_f)\right]T_{f'}^3T_f^3 \\ &\quad \left.+ \frac{3}{4}\ln\left(\frac{x_+}{x_-}\right)\delta_{IL}\delta_{JL} + \ln\left(\frac{x_+}{x_-}\right)(t_W^2Y_{f'}Y_f - 4s_W^2Q_{f'}Q_f)\right\}\end{aligned}$$

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<sup>2</sup>We do not express  $\delta_M$  through  $s_w^2$  to emphasize the origin of the corrections.

$$\times \left( T_{f'}^3 T_f^3 + t_W^2 \frac{Y_{f'} Y_f}{4} \right) \left] [T_f(T_f + 1) + (f \leftrightarrow f')] \frac{2}{3} l^2(s) \right\} \delta_M \mathcal{A}_{IJ}^{f'f}, \quad (62)$$

which completes the result of Refs. [9, 10]. Note that the exponentiation of the mass splitting contribution due to  $\delta\xi^{(1)}$  has been observed first by the explicit calculation in NLL approximation [20].

#### 4.4 Semi-inclusive cross sections

To get the infrared safe result one has to take into account the real photon emission in an inclusive way. In practice, the massive gauge bosons are supposed to be detected as separate particles. Thus it is of little physical sense to treat the hard photons with energies far larger than  $M_{Z,W}$  separately because of gauge symmetry restoration. In particular, the radiation of the hard real photons is not of the Poisson type because of its non-Abelian  $SU(2)_L$  component. Therefore, we restrict the analysis to semi-inclusive cross sections with the real emission only of photons with energies far smaller than  $M_{Z,W}$ , which is of pure QED nature. To derive the result for such a cross section one has to add to the expressions given above the QED corrections due to the real photon emission and the *pure* QED virtual corrections which are determined for  $m_f \ll \lambda \ll M$ , where  $m_f$  is a light fermion mass, by Eqs. (48). However, in standard QED applications a nonzero mass  $m_f$  is kept for an on-shell fermion which regularizes the collinear divergences. To derive the QED factor for  $\lambda$  far less than the fermion mass  $\lambda \ll m_f \ll M$  one has to change the kernel of the infrared evolution equation and match the new solution to Eq. (48) at the point  $\lambda = m_f$ . For several light flavors of significantly different masses the matching is necessary when  $\lambda$  crosses the value of each fermion mass. The sum of the virtual and real QED corrections to the cross section gives a factor which depends on  $s$ , the fermion masses, and the experimental cuts, but not on  $M_{Z,W}$ . The detailed analysis of the QED corrections goes beyond the scope of the present paper. In the single flavor case the second order QED corrections including the soft real emission are known beyond the logarithmic approximation (see [55] and references therein).

#### 4.5 Numerical estimates

With the expressions for the chiral amplitudes at hand, we can compute the total two-loop logarithmic virtual corrections to the basic observables for the neutral current four-fermion processes. Let us consider the total cross sections of the quark-antiquark/ $\mu^+ \mu^-$  production in  $e^+ e^-$  annihilation. In one and two loops the logarithmic corrections read

$$\begin{aligned} R(e^+ e^- \rightarrow Q\bar{Q}) &= 1 - 1.66 L(s) + 5.60 l(s) - 8.39 a \\ &\quad + 1.93 L^2(s) - 11.28 L(s) l(s) + 33.79 l^2(s) - 60.87 l(s) a, \\ R(e^+ e^- \rightarrow q\bar{q}) &= 1 - 2.18 L(s) + 20.94 l(s) - 35.07 a \\ &\quad + 2.79 L^2(s) - 51.98 L(s) l(s) + 321.20 l^2(s) - 757.35 l(s) a, \\ R(e^+ e^- \rightarrow \mu^+ \mu^-) &= 1 - 1.39 L(s) + 10.35 l(s) - 21.26 a \\ &\quad + 1.42 L^2(s) - 20.33 L(s) l(s) + 112.57 l^2(s) - 312.90 l(s) a, \end{aligned} \quad (63)$$

where  $L(s) = a \ln^2(s/M^2)$ ,  $Q = u, c, t$ ,  $q = d, s, b$ ,  $R(e^+e^- \rightarrow \mu^+\mu^-) = \sigma/\sigma_B(e^+e^- \rightarrow \mu^+\mu^-)$  and so on. The  $\overline{MS}$  couplings in the Born cross section are renormalized at  $\sqrt{s}$ . Numerically, we have  $L(s) = 0.07$  (0.11) and  $l(s) = 0.014$  (0.017) for  $\sqrt{s} = 1$  TeV and 2 TeV, respectively. Here  $M = M_W$  has been chosen for the infrared cutoff and  $a = 2.69 \cdot 10^{-3}$ ,  $s_W^2 = 0.231$  for the  $\overline{MS}$  couplings renormalized at the gauge boson mass. The complete one-loop corrections are known exactly and we have included the dominant one-loop terms in Eqs. (63)–(67) to demonstrate the structure of the expansion rather than for precise numerical estimates though the above expressions approximate the exact one-loop result with 1% accuracy in the TeV region. In Eq. (63) we included the leading correction in the  $W - Z$  mass difference through NNLL approximation. The coefficient of the linear logarithm is computed in the approximation of Sect. (4.1). In the case of a quark-antiquark final state the strong interaction could also produce logarithmically growing terms. For massless quarks the complete  $\mathcal{O}(\alpha_s^2)$  corrections including the bremsstrahlung effects can be found in [35]. Note that Eqs. (63) are symmetric under exchange of the initial and final state fermions and, therefore, also applicable to the Drell-Yan processes at hadron colliders. In this case the two-loop QCD corrections are given in [36]. For the total cross sections of the four-quark electroweak processes we obtain

$$\begin{aligned}
R(Q'\bar{Q}' \rightarrow Q\bar{Q}) &= 1 - 2.07 L(s) + 19.03 l(s) - 32.63 a \\
&\quad + 2.67 L^2(s) - 46.64 L(s) l(s) + 278.94 l^2(s) - 666.05 l(s) a, \\
R(Q\bar{Q} \rightarrow q\bar{q}) &= 1 - 2.56 L(s) + 8.49 l(s) - 11.94 a \\
&\quad + 3.53 L^2(s) - 20.39 L(s) l(s) + 65.20 l^2(s) - 91.92 l(s) a, \\
R(q'\bar{q}' \rightarrow q\bar{q}) &= 1 - 2.87 L(s) + 25.63 l(s) - 38.89 a \\
&\quad + 4.20 L^2(s) - 71.87 L(s) l(s) + 423.61 l^2(s) - 919.35 l(s) a. \quad (64)
\end{aligned}$$

Our results can easily be generalized to  $f\bar{f} \rightarrow f\bar{f}$  processes with identical quarks by including the  $t$ -channel contribution.

For  $e^+e^-$  annihilation we also give a numerical estimate of corrections to the cross section asymmetries. In the case of the forward-backward asymmetry  $A^{FB}$  (the difference of the cross section integrated over forward and backward hemispheres with respect to the electron beam direction divided by the total cross section) we get

$$\begin{aligned}
R^{FB}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 0.09 L(s) - 1.22 l(s) + 1.77 a \\
&\quad + 0.12 L^2(s) + 0.59 L(s) l(s) - 1.65 l^2(s) + 3.39 l(s) a, \\
R^{FB}(e^+e^- \rightarrow q\bar{q}) &= 1 - 0.14 L(s) + 7.17 l(s) - 10.84 a \\
&\quad + 0.02 L^2(s) - 1.32 L(s) l(s) - 33.07 l^2(s) + 93.44 l(s) a, \\
R^{FB}(e^+e^- \rightarrow \mu^+\mu^-) &= 1 - 0.04 L(s) + 5.50 l(s) - 14.43 a \\
&\quad + 0.27 L^2(s) - 6.43 L(s) l(s) + 22.91 l^2(s) - 33.11 l(s) a, \quad (65)
\end{aligned}$$

where  $R^{FB} = A^{FB}/A_B^{FB}$ . For the left-right asymmetry  $A^{LR}$  (the difference of the cross sections of the production of left- and right-handed particles divided by the total cross

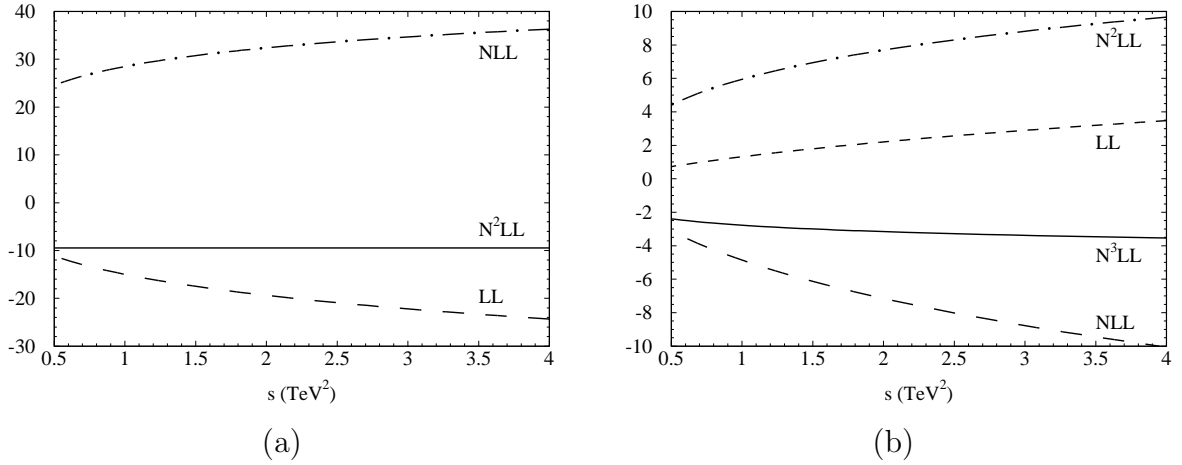


Figure 1: Separate logarithmic contributions to  $R(e^+e^- \rightarrow q\bar{q})$  in % to the Born approximation: (a) the one-loop LL ( $\ln^2(s/M^2)$ , long-dashed line), NLL ( $\ln^1(s/M^2)$ , dot-dashed line) and  $N^2$ LL ( $\ln^0(s/M^2)$ , solid line) terms; (b) the two-loop LL ( $\ln^4(s/M^2)$ , short-dashed line), NLL ( $\ln^3(s/M^2)$ , long-dashed line), NNLL ( $\ln^2(s/M^2)$ , dot-dashed line) and  $N^3$ LL ( $\ln^1(s/M^2)$ , solid line) terms.

section) we obtain in the same notations

$$\begin{aligned}
R^{LR}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 4.48 L(s) + 17.51 l(s) - 13.16 a \\
&\quad - 1.16 L^2(s) + 15.66 L(s) l(s) - 43.50 l^2(s) + 44.05 l(s) a, \\
R^{LR}(e^+e^- \rightarrow q\bar{q}) &= 1 - 1.12 L(s) + 12.05 l(s) - 16.44 a \\
&\quad - 0.81 L^2(s) + 18.02 L(s) l(s) - 130.74 l^2(s) + 278.71 l(s) a, \\
R^{LR}(e^+e^- \rightarrow \mu^+\mu^-) &= 1 - 13.24 L(s) + 116.58 l(s) - 148.42 a \\
&\quad - 0.79 L^2(s) + 23.68 L(s) l(s) - 155.46 l^2(s) - 116.67 l(s) a.
\end{aligned} \tag{66}$$

Finally, for the left-right asymmetry  $\tilde{A}^{LR}$  (the difference of the cross sections for the left- and right-handed initial state particles divided by the total cross section) which differs from  $A^{LR}$  for the quark-antiquark final state we have

$$\begin{aligned}
\tilde{R}^{LR}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 2.75 L(s) + 10.60 l(s) - 9.05 a \\
&\quad - 0.91 L^2(s) + 11.16 L(s) l(s) - 33.49 l^2(s) + 28.28 l(s) a, \\
\tilde{R}^{LR}(e^+e^- \rightarrow q\bar{q}) &= 1 - 1.07 L(s) + 11.75 l(s) - 16.21 a \\
&\quad - 0.77 L^2(s) + 17.06 L(s) l(s) - 125.18 l^2(s) + 267.60 l(s) a.
\end{aligned} \tag{67}$$

The numerical structure of the corrections in the case of  $e^+e^-$  annihilation is shown in Figs. 1-3. In Fig. 1 the values of different logarithmic contributions to  $R(e^+e^- \rightarrow q\bar{q})$  are

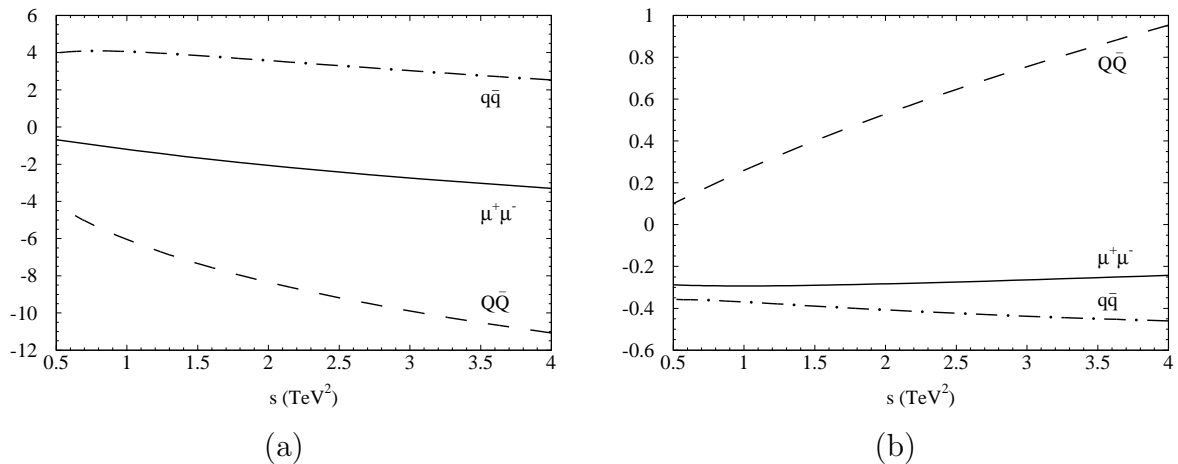


Figure 2: The total logarithmic corrections to  $R(e^+e^- \rightarrow Q\bar{Q})$  (dashed line),  $R(e^+e^- \rightarrow q\bar{q})$  (dot-dashed line) and  $R(e^+e^- \rightarrow \mu^+\mu^-)$  (solid line) in % to the Born approximation: (a) the one-loop correction up to N<sup>2</sup>LL term; (b) the two-loop correction up to N<sup>3</sup>LL term.

plotted separately as functions of  $s$ . In the 1 – 2 TeV region the two-loop LL, NLL, NNLL and N<sup>3</sup>LL corrections to the cross sections can be as large as 1 – 4%, 5 – 10%, 5 – 10% and 2 – 3% respectively. The two-loop logarithmic terms have a sign-alternating structure resulting in significant cancellations typical for the Sudakov limit [10, 16, 21]. Although the individual logarithmic contributions can be as large as 10%, their sum does not exceed 1% in absolute value at energies below 2 TeV for all the cross sections (see Fig. 2). In the region of a few TeV the corrections do not reach the double-logarithmic asymptotics. The quartic, cubic and quadratic logarithms are comparable in magnitude. Then the logarithmic expansion starts to converge. Still, the linear-logarithmic contribution must be included to reduce the theoretical uncertainty below 1%. The two-loop logarithmic corrections to the asymmetries also amount up to 1% in absolute value at energies below 2 TeV with the only exception of  $R^{LR}(e^+e^- \rightarrow \mu^+\mu^-)$  (see Fig. 3).

Let us discuss the accuracy of our result. On the basis of the explicit evaluation of the light fermion/scalar [16] and the Abelian contribution [21] we estimate the uncalculated two-loop nonlogarithmic term to be at a few permill level. The power-suppressed terms do not exceed a permill in magnitude for  $\sqrt{s} > 500$  GeV as well [16]. By comparing our numerical estimates to the equal mass approximation given in the Appendix we find that the leading effect of the  $W - Z$  mass splitting results in a variation of at most 5% of the coefficients of the two-loop cubic and quadratic logarithms. Thus the expansion in the  $W - Z$  mass difference converges well for these coefficients and the leading correction term taken into account in our evaluation is sufficient for a permill accuracy of the cross sections. The main uncertainty of our result for the two-loop single logarithmic contribution is due to our approximation for the  $\xi^{(2)}$  coefficient. Neglecting the gauge boson mixing effects, which are suppressed by a factor of

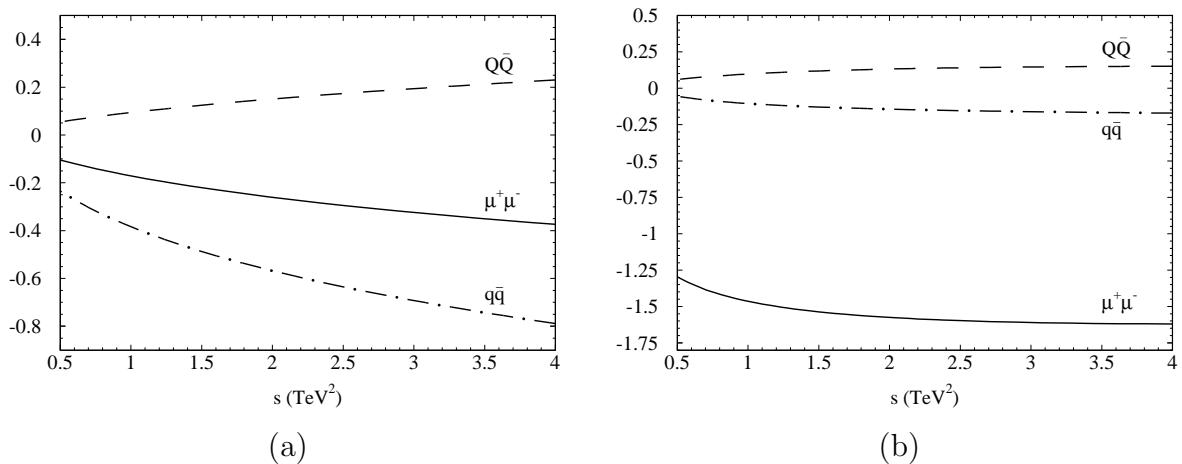


Figure 3: (a) The total logarithmic two-loop corrections to the forward-backward asymmetry  $R^{FB}(e^+e^- \rightarrow Q\bar{Q})$  (dashed line),  $R^{FB}(e^+e^- \rightarrow q\bar{q})$  (dot-dashed line) and  $R^{FB}(e^+e^- \rightarrow \mu^+\mu^-)$  (solid line) in % to the Born approximation. (b) The same as (a) but for the left-right asymmetry  $\tilde{R}^{LR}$ .

$\sin^2 \theta_W$ , brings an error of 20% to the coefficient of the two-loop single logarithm. Neglecting the difference between the Higgs and gauge boson masses does not lead to a numerically important error because the scalar boson contribution is relatively small. By comparing the results for  $M_H = M_{W,Z}$  and  $M_H \ll M_{W,Z}$  we estimate the corresponding uncertainty in the coefficient of the two-loop single logarithm to be about 5%. The same estimate is true for the uncertainty due to the top quark mass effect on the  $t\bar{t}$  virtual pair contribution. Hence for the production of light fermions our formulae are supposed to approximate the exact coefficients of the two-loop linear logarithms with approximately 20% accuracy which results in a few permill uncertainty in the cross sections. By adding up the errors from different sources in quadrature we find the total uncertainty of the cross section to be in a few permill – one percent range depending on the process. Thus we get an accurate estimate of the two-loop correction, which is sufficient for practical applications to the future collider physics. The only essential deviation of the exact two-loop logarithmic contributions from our result Eqs. (63–67) for the production of the third generation quarks is due to the large top quark Yukawa coupling. The corresponding corrections are known to NLL approximation and can numerically be as important as the generic non-Yukawa ones [11, 13, 56, 57].

## 5 Summary

In the present paper we have derived the analytical result for the two-loop logarithmic corrections to the vector form factor and the four-fermion cross sections in the spontaneously broken  $SU_L(2)$  model by combining the explicit calculation and the evolution equation ap-



proach. We have completed the analysis of the dominant logarithmically enhanced two-loop electroweak corrections to the basic observables of the neutral current four-fermion processes at high energy. The two-loop linear-logarithmic contribution has been obtained with an estimated accuracy of 20%. The  $W - Z$  mass splitting effect neglected in [9,10] has been taken into account through the NNLL approximation. Our result for the two-loop logarithmic corrections along with the known exact one-loop expressions approximates the cross sections with an accuracy of a few permill, which is sufficient for practical applications to the future collider physics. The general approach for the calculation of the logarithmically enhanced corrections developed in the present paper can also be applied to the analysis of the gauge boson production and the supersymmetric extensions of the standard model where only the NLL approximation is available so far [18,19].

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## Appendix

The NNLL result for the cross sections and asymmetries in the approximation  $M_Z = M_W$  can be obtained from the expressions of Ref. [10]<sup>3</sup>

$$\begin{aligned}
 R(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 1.66 L(s) + 5.31 l(s) - 8.36 a \\
 &\quad + 1.93 L^2(s) - 10.59 L(s)l(s) + 31.40 l^2(s), \\
 R(e^+e^- \rightarrow q\bar{q}) &= 1 - 2.18 L(s) + 20.58 l(s) - 34.02 a \\
 &\quad + 2.79 L^2(s) - 51.04 L(s)l(s) + 309.34 l^2(s), \\
 R(e^+e^- \rightarrow \mu^+\mu^-) &= 1 - 1.39 L(s) + 10.12 l(s) - 20.61 a \\
 &\quad + 1.42 L^2(s) - 19.81 L(s)l(s) + 107.03 l^2(s), \tag{68}
 \end{aligned}$$

$$\begin{aligned}
 R^{FB}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 0.09 L(s) - 1.23 l(s) + 1.47 a \\
 &\quad + 0.12 L^2(s) + 0.64 L(s)l(s) - 1.40 l^2(s), \\
 R^{FB}(e^+e^- \rightarrow q\bar{q}) &= 1 - 0.14 L(s) + 7.15 l(s) - 10.43 a \\
 &\quad + 0.02 L^2(s) - 1.31 L(s)l(s) - 33.46 l^2(s), \\
 R^{FB}(e^+e^- \rightarrow \mu^+\mu^-) &= 1 - 0.04 L(s) + 5.49 l(s) - 14.03 a \\
 &\quad + 0.27 L^2(s) - 6.32 L(s)l(s) + 21.01 l^2(s), \tag{69}
 \end{aligned}$$

$$\begin{aligned}
 R^{LR}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 4.48 L(s) + 16.66 l(s) - 13.28 a \\
 &\quad - 1.16 L^2(s) + 15.21 L(s)l(s) - 41.79 l^2(s), \\
 R^{LR}(e^+e^- \rightarrow q\bar{q}) &= 1 - 1.12 L(s) + 11.86 l(s) - 15.83 a \\
 &\quad - 0.81 L^2(s) + 17.74 L(s)l(s) - 127.05 l^2(s), \\
 R^{LR}(e^+e^- \rightarrow \mu^+\mu^-) &= 1 - 13.24 L(s) + 113.77 l(s) - 139.94 a \\
 &\quad - 0.79 L^2(s) + 23.34 L(s)l(s) - 155.36 l^2(s). \tag{70}
 \end{aligned}$$

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<sup>3</sup>Throughout Sect 4. of Ref. [10] the terms with the factor  $(aN_g + b)t_W^2$ , where  $a$  and  $b$  stand for some constants, should be multiplied by an extra  $t_W^2$ , and the terms with the factor  $N_g s_W^2$  should be multiplied by an extra  $s_W^2$ . This results in a small change of the numerical estimates.

$$\begin{aligned}
\tilde{R}^{LR}(e^+e^- \rightarrow Q\bar{Q}) &= 1 - 2.75 L(s) + 10.07 l(s) - 9.02 a \\
&\quad - 0.91 L^2(s) + 10.80 L(s)l(s) - 32.10 l^2(s), \\
\tilde{R}^{LR}(e^+e^- \rightarrow q\bar{q}) &= 1 - 1.07 L(s) + 11.56 l(s) - 15.60 a \\
&\quad - 0.77 L^2(s) + 16.78 L(s)l(s) - 121.56 l^2(s). \tag{71}
\end{aligned}$$