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Two-Loop Photonic Corrections to Massive Bhabha Scattering

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Abstract

We describe the details of the evaluation of the two-loop radiative photonic corrections to Bhabha scattering. The role of the corrections in the high-precision luminosity determination at present and future electron-positron colliders is discussed.

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1 Introduction

Electron-positron *Bhabha* scattering plays a special role in particle phenomenology. It provides a very efficient tool for luminosity determination at electron-positron colliders and thus it is crucial for extracting physics from the experimental data. Small angle Bhabha scattering has been particularly effective as a luminosity monitor at the energies of LEP and SLC because its cross section is large and QED dominated [1,2]. At a future International Linear Collider (ILC) the luminosity spectrum is not monochromatic due to beam-beam effects. Therefore measuring the cross section of the small angle Bhabha scattering alone is not sufficient, and the angular distribution of the large angle Bhabha scattering has been suggested for disentangling the luminosity spectrum [3,4]. Large angle Bhabha scattering is important also at colliders operating at a center of mass energy $\sqrt{s}$ of a few GeV, such as BABAR/PEP-II, BELLE/KEKB, BES/BEPC, KLOE/DAΦNE, and VEPP-2M, where it is used to measure the luminosity [5]. Since the accuracy of the theoretical evaluation of the Bhabha cross section directly affects the luminosity determination, remarkable efforts have been devoted to the study of the radiative corrections to this process (see [1] for an extensive list of references). Pure QED contributions are particularly important because they dominate the radiative corrections to the large angle scattering at intermediate energies 1-10 GeV and to the small angle scattering also at higher energies. The calculation of the QED radiative corrections to the Bhabha cross section is among the classical problems of perturbative quantum field theory with a long history. The first order corrections are well known (see [6,7] and references therein). To match the impressive experimental accuracy the complete second order QED effects have to be included on the theoretical side. The evaluation of the two-loop virtual corrections constitutes the main problem of the second order analysis. The complete two-loop virtual corrections to the scattering amplitudes in the massless electron approximation have been computed in Ref. [8], where dimensional regularization has been used for the infrared divergences. However, this approximation is not sufficient since one has to keep a nonvanishing electron mass to make the result compatible with available Monte Carlo event generators [1,5,9,10]. Recently an important class of the second order corrections, which include at least one closed fermion loop, has been obtained for a finite electron mass [11] including the soft photon bremsstrahlung [12]. A similar evaluation of the purely photonic two-loop corrections is a challenging problem at the limit of present computational techniques [13–15]. The most complete result available so far can be found in Ref. [16] where the contribution of double box diagrams is still missing. On the other hand in the energy range under consideration only the leading contribution in the small ratio m_e^2/s is of phenomenological relevance and should be retained in the theoretical evaluations. In this approximation all the two-loop corrections enhanced by a power of the large logarithm $\ln(m_e^2/s)$ are known so far for the small angle [17] and large angle [18,19] Bhabha scattering. The result for the nonlogarithmic contribution has been reported in a letter [20]. It was obtained by employing the general theory of infrared singularities in QED which allows to reduce the calculation in the small electron mass approximation to the analysis of a strictly massless scattering amplitude and the massive vector form factor. In the present paper we describe the details of this calculation. In the next section we outline the structure of the perturbative expansion for the Bhabha cross section. In Sect. 3 we consider the structure of the infrared logarithms and formulate the method of *infrared subtractions*. In Sect. 4 the explicit relation between the amplitudes of the massive and massless Bhabha scattering is established through the *infrared matching* procedure and the result for the two-loop corrections to the massive Bhabha scattering is obtained. Sect. 5 contains the numerical estimates and the summary.

2 Perturbative expansion of the cross section

We consider the phenomenologically interesting kinematical region $s,\ t,\ u \gg m_e^2$, where all the terms suppressed by the electron mass can be neglected. The perturbative expansion for the Bhabha cross section in the fine structure constant α is defined as follows

$$\sigma = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n \sigma^{(n)} \,. \tag{1}$$

In the small electron mass approximation the leading order differential cross section takes the form

$$\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}\Omega} = \frac{\alpha^2}{s}\left(\frac{1-x+x^2}{x}\right)^2 + \mathcal{O}(m_e^2/s)\,, \tag{2}$$

where $x = (1-\cos\theta)/2$ and θ is the scattering angle. The virtual corrections taken separately suffer from the *soft* divergences, which can be regulated *e.g.* by giving the photon a small auxiliary mass λ. These soft divergences are canceled in the inclusive cross section when one adds the photonic bremsstrahlung [21,22]. The standard generators to deal with the bremsstrahlung is to split it into a soft part which accounts for the emission of the photons with the energy below some cutoff $\varepsilon_{cut} \ll m_e$, and a hard part corresponding to the emission of the photons with the energy above ε_{cut}. The infrared finite hard part is then computed numerically using Monte-Carlo methods with physical cuts dictated by the experimental setup. At the same time the soft part is computed analytically and combined with the virtual corrections ensuring the cancellation of the singular dependence on λ. Thus in the first order we consider the sum of one-loop virtual correction and single soft photon emission

$$\delta^{(1)} = \delta_v^{(1)} + \delta_s^{(1)}\,, \tag{3}$$

where

$$\delta^{(n)} \equiv \frac{\mathrm{d}\sigma^{(n)}/\mathrm{d}\Omega}{\mathrm{d}\sigma^{(0)}/\mathrm{d}\Omega}\,, \tag{4}$$

and the expressions for $\delta_v^{(1)}$ and $\delta_s^{(1)}$ are given by Eqs. (43, 44) of the Appendix. Eq. (3) can be decomposed according to the asymptotic dependence on the electron mass

$$\frac{\mathrm{d}\sigma^{(1)}}{\mathrm{d}\sigma^{(0)}} = \delta_1^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(1)} + \mathcal{O}(m_e^2/s)\,. \tag{5}$$

For the pure photonic correction the coefficients $\delta_i^{(1)}$ read (see *e.g.* [18,19])

$$\begin{aligned}
\delta_1^{(1)} &= 4\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + 3\,, \\
\delta_0^{(1)} &= \left[-4 + 4\ln\left(\frac{x}{1-x}\right)\right]\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) - 4 - \frac{2}{3}\pi^2 - 2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(1-x) + f(x)\,,
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
f(x) &= (1-x+x^2)^{-2}\left\{\left(\frac{1}{3} - \frac{2}{3}x + \frac{9}{4}x^2 - \frac{13}{6}x^3 + \frac{4}{3}x^4\right)\pi^2 + \left(3 - 4x + \frac{9}{2}x^2 - \frac{3}{2}x^3\right)\right. \\
&\times \ln(x) + \left(\frac{3}{4}x - \frac{x^2}{4} - \frac{3}{4}x^3 + x^4\right)\ln^2(x) + \left[-\frac{1}{2}x - \frac{1}{2}x^3 + \left(2 - 4x + \frac{7}{2}x^2 - x^3\right)\right. \\
&\left.\left.\times \ln(x)\right]\ln(1-x) + \left(-1 + \frac{5}{2}x - \frac{7}{2}x^2 + \frac{5}{2}x^3 - x^4\right)\ln^2(1-x)\right\}\,,
\end{aligned} \tag{7}$$

$\mathrm{Li}_n(z)$ is the polylogarithm, $\varepsilon = \sqrt{s}/2$, ε_{cut} is the energy cut on the emitted soft photon. The second order correction can be represented as a sum of three terms

$$\delta^{(2)} = \delta^{(2)}_{vv} + \delta^{(2)}_{vs} + \delta^{(2)}_{ss} \tag{8}$$

which correspond to the two-loop virtual correction including the one-loop corrections to the amplitude square, one-loop virtual correction to single soft photon emission, and the double soft photon emission, respectively. In the small electron mass limit it has the following decomposition

$$\delta^{(2)} = \delta^{(2)}_2 \ln^2\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_1 \ln\left(\frac{s}{m_e^2}\right) + \delta^{(2)}_0 + \mathcal{O}(m_e^2/s)\,. \tag{9}$$

The pure photonic, *i.e.* without closed fermion loops, logarithmically enhanced contribution reads [18]

$$\begin{aligned}
\delta^{(2)}_2 &= 8\ln^2\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + 12\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + \frac{9}{2}\,, \\
\delta^{(2)}_1 &= \left[-16 + 16\ln\left(\frac{x}{1-x}\right)\right]\ln^2\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + \left[-28 - \frac{8}{3}\pi^2 + 12\ln\left(\frac{x}{1-x}\right) - 8\mathrm{Li}_2(x)\right. \\
&\left. +8\mathrm{Li}_2(1-x) + 4f(x)\right]\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) - \frac{93}{8} - \frac{5}{2}\pi^2 + 6\zeta(3) - 6\mathrm{Li}_2(x) \\
&+6\mathrm{Li}_2(1-x) + 3f(x)\,,
\end{aligned} \tag{10}$$

where $\zeta(3) = 1.202057\ldots$ is the value of the Riemann's zeta-function. In the rest of the paper we focus on the photonic contribution to $\delta^{(2)}_0$.

3 Structure of infrared logarithms

The general problem of the calculation of the small electron mass asymptotics of the corrections including the power-suppressed terms can systematically be solved within the expansion by regions approach [23,24]. We, however, are interested only in the leading order term. The leading order contribution in Eq. (9) contains the logarithmic terms, which become singular as m_e approaches zero revealing the collinear divergences regulated by the electron mass. In the massless limit both the collinear and the soft divergences can be treated by dimensional regularization as well. Here we should note that the collinear divergences in the massless approximation are also canceled in a cross section which is inclusive with respect to real photons and electron-positron pairs collinear to the initial or final state fermions [25]. This means that if an angular cut on the collinear emission is sufficiently large, $\theta_{cut} \gg \sqrt{m_e^2/s}$, the inclusive cross section is insensitive to the electron mass and can in principle be computed with $m_e = 0$ by using dimensional regularization for the infrared divergences for both virtual and real radiative corrections like it is done in the theory of QCD jets. However, as it has been mentioned above, all the available Monte Carlo event generators for Bhabha scattering with specific cuts on the photon bremsstrahlung dictated by the experimental setup employ a nonzero electron mass as an infrared regulator, which therefore has to be used also in the calculation of the virtual corrections. As far as the leading term in the small electron mass expansion is considered, the difference between the massive and the dimensionally regularized massless Bhabha scattering can be viewed as a difference between two regularization schemes for the infrared divergences. With the known massless two-loop result at hand, the calculation of the massive one is reduced to constructing the infrared matching term which relates two above regularization schemes. To perform the matching we develop the method of infrared subtractions which simplifies the calculation by fully exploiting the information on the general structure of infrared singularities in QED. The method was originally applied in Ref. [26] to the analysis of the two-loop corrections to the vector form factor in an Abelian gauge model with mass gap. Let $\mathcal{A}^{(2)}(m_e,\lambda)$ be the two-loop contribution to the massive electron-positron scattering amplitude with the photon mass used to regulate the soft divergences. The main idea of the method is to construct an auxiliary amplitude $\bar{\mathcal{A}}^{(2)}(m_e,\lambda)$, which has the same structure of the infrared singularities but is sufficiently simple to be evaluated at least in leading order in the small mass expansion. Then the difference $\mathcal{A}^{(2)}-\bar{\mathcal{A}}^{(2)}$ has a finite limit $\delta\mathcal{A}^{(2)}$ as $m_e,\ \lambda$ tend to zero. This quantity does not depend on the regularization scheme for $\mathcal{A}^{(2)}$ and $\bar{\mathcal{A}}^{(2)}$. It and can be evaluated by using dimensional regularization for each term and then taking the limit of four space-time dimensions. The full amplitude is given by a sum

$$\mathcal{A}^{(2)}(m_e,\lambda)=\bar{\mathcal{A}}^{(2)}(m_e,\lambda)+\delta\mathcal{A}^{(2)}+\mathcal{O}(m_e,\lambda)\,. \tag{11}$$

Thus the infrared divergences, which induce the asymptotic dependence of the virtual corrections on the electron and photon masses, are absorbed into the auxiliary amplitude while the technically most nontrivial calculation of the term $\delta\mathcal{A}^{(2)}$ is performed in the massless approximation. The matching of the massive and massless results is necessary only for the singular auxiliary amplitude. Note that the method does not require a diagram-by-diagram subtraction of the infrared divergences since only a general information on the infrared structure of the total two-loop correction is necessary to construct $\bar{\mathcal{A}}^{(2)}(m_e,\lambda)$. Our analysis is based on the following infrared properties of the corrections to the scattering amplitudes:

(i) exponentiation of the infrared logarithms [27–32];

(ii) factorization of the collinear logarithms into external legs [33];

(iii) nonrenormalization of the infrared exponents [28,30,31].

The first two properties are general and hold also for the closed fermion loop corrections and non-Abelian gauge theories. The last property is characteristic for the pure photonic corrections and plays a crucial role in our analysis. In Sects. 3.1-3.2 by means of (i)-(iii) we show that $\bar{\mathcal{A}}^{(2)}(m_e,\lambda)$ for the photonic contributions can be constructed of the two-loop corrections to the vector form factor and products of the one-loop corrections.

3.1 Vector form factor

The vector form factor $\mathcal{F}$ determines the electron scattering amplitude in an external field. It plays a special role since it is the simplest quantity which includes the complete information about the collinear logarithms, which is directly applicable to a process with an arbitrary number of electrons/positrons. Let us consider three different *Sudakov* asymptotic regimes:

(a) $m_e = \lambda = 0$;

(b) $|Q| \gg m_e \gg \lambda$;

(c) $|Q| \gg \lambda \gg m_e$;

where Q is the Euclidean momentum transfer. In case (a) the soft and collinear divergences are treated by dimensional regularization. In case (b) the collinear and soft divergences are regularized by m_e and λ, respectively. In case (c) the photon mass regulates both soft and collinear divergences. Though (c) has no direct application to QED, it is instructive to study yet another regularization scheme to get deeper insight into the general structure of infrared logarithms. We define the perturbative series for the form factor as follows: $\mathcal{F} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n f^{(n)}$. The one-loop coefficients read

$$\begin{aligned} f_a^{(1)} &= \left[-\frac{1}{2\epsilon^2} - \frac{3}{4\epsilon} - 2 + \frac{\pi^2}{24} + \left(-4 + \frac{\pi^2}{16} + \frac{7}{6}\zeta(3)\right)\epsilon + \left(-8 + \frac{\pi^2}{6} + \frac{7}{4}\zeta(3) + \frac{47}{2880}\pi^4\right)\right. \\ &\quad \left. \times \epsilon^2 \right] \left(\frac{\mu^2}{Q^2}\right)^{\epsilon}, \qquad (12) \\ f_b^{(1)} &= -\frac{1}{4}\ln^2\left(\frac{Q^2}{m_e^2}\right) + \left[\frac{1}{2}\ln\left(\frac{\lambda^2}{m_e^2}\right) + \frac{3}{4}\right]\ln\left(\frac{Q^2}{m_e^2}\right) - \frac{1}{2}\ln\left(\frac{\lambda^2}{m_e^2}\right) - 1 + \frac{\pi^2}{12} + \mathcal{O}(m_e^2, \lambda^2), \\ &\qquad (13) \\ f_c^{(1)} &= -\frac{1}{4}\ln^2\left(\frac{Q^2}{\lambda^2}\right) + \frac{3}{4}\ln\left(\frac{Q^2}{\lambda^2}\right) - \frac{7}{8} - \frac{\pi^2}{6} + \mathcal{O}(\lambda^2). \qquad (14) \end{aligned}$$

The asymptotic dependence of the form factor on Q in the Sudakov limit is governed by the evolution equation [30–32] which for the pure photonic contribution takes the form

$$\frac{\partial}{\partial \ln(Q^2)}\mathcal{F} = \left[-\frac{\alpha}{2\pi}\ln\left(Q^2\right) + \phi(m_e, \lambda, \epsilon, \alpha)\right]\mathcal{F}, \qquad (15)$$

where the anomalous dimension $\phi(m_e, \lambda, \epsilon, \alpha)$ is a series in α with the coefficients depending on the infrared regulators. We can write down the solution of Eq. (15) in the above three cases

$$\begin{aligned} \mathcal{F}_a &= (1 + \mathcal{O}(\alpha))\exp\left\{-\frac{\alpha}{4\pi}\left(\frac{2}{\epsilon^2} + (3 + \mathcal{O}(\alpha))\frac{1}{\epsilon}\right)\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right\}, \\ \mathcal{F}_b &= (1 + \mathcal{O}(\alpha))\exp\left\{\frac{\alpha}{4\pi}\left[-\ln^2\left(\frac{Q^2}{m_e^2}\right) + 2\left[\ln\left(\frac{Q^2}{m_e^2}\right) - 1\right]\ln\left(\frac{\lambda^2}{m_e^2}\right) + (3 + \mathcal{O}(\alpha))\right.\right. \end{aligned}$$

$$\times \ln\left(\frac{Q^2}{m_e^2}\right)\Big]\Big]\Big\} ,$$

$$\mathcal{F}_c = (1+\mathcal{O}(\alpha))\exp\left\{\frac{\alpha}{4\pi}\left[-\ln^2\left(\frac{Q^2}{\lambda^2}\right)+(3+\mathcal{O}(\alpha))\ln\left(\frac{Q^2}{\lambda^2}\right)\right]\right\} , \tag{16}$$

where the terms which receive the higher order corrections are explicitly shown. In derivation of Eq. (15) we have taken into account that in the case (b) also the logarithms of the photon mass exponentiate [28]

$$\mathcal{F}_b \propto \exp\left\{\frac{\alpha}{2\pi}\left[\ln\left(\frac{Q^2}{m_e^2}\right)-1\right]\ln\left(\frac{\lambda^2}{m_e^2}\right)\right\} . \tag{17}$$

The exponentiation of the "Sudakov" logarithms is a general property valid also for the corrections due to the fermion loops and for non-Abelian gauge theories. The exponent for the pure photonic corrections, however, has two distinguished properties. First, the double logarithmic term in the exponent is protected against the perturbative corrections. This fact is well known since the pioneering works [27,28,30,31]. A new observation is that beyond the first order in α the coefficients of the series for the single logarithmic term in the exponent are mass-independent and, therefore, should be the same in all three cases under consideration. The derivation of this result to a large extent repeats the proof of the nonrenormalization of the double logarithmic contribution. We refrain from giving the details of the derivation since it is not directly related to the subject of the present paper. We should emphasize that the above properties are not valid for the nonphotonic corrections, *e.g.* the second order single logarithmic contribution of the closed fermion loop to the exponent depends on the mass ratio λ/m_e and is different for the cases (b) and (c) [34].

Thus, as far as the photonic corrections are concerned, the perturbative series for the logarithm of the form factor beyond one loop includes only the first power of the logarithm with the universal coefficients. This means that the coefficients in the series for the form factor have the following structure

$$f^{(2)} = \frac{1}{2}\left(f^{(1)}\right)^2 + C^{(2)}\ln(Q^2) + \mathcal{O}(\ln^0(Q^2)) , \tag{18}$$

$$f^{(3)} = -\frac{1}{3}\left(f^{(1)}\right)^3 + f^{(2)}f^{(1)} + C^{(3)}\ln(Q^2) + \mathcal{O}(\ln^0(Q^2)) , \tag{19}$$

$$\ldots ,$$

where each $C^{(n)}$, $n > 1$ is equal for (a), (b), and (c). This prediction can be confronted with the explicit results for the two-loop corrections which are available in all three cases and read

$$f_a^{(2)} = \frac{1}{2}\left(f_a^{(1)}\right)^2 - \left(\frac{3}{32}-\frac{\pi^2}{8}+\frac{3}{2}\zeta(3)\right)\frac{1}{2\epsilon}\left(\frac{\mu^2}{Q^2}\right)^{2\epsilon} - \frac{1}{128} + \frac{29}{96}\pi^2 - \frac{15}{8}\zeta(3) - \frac{11}{720}\pi^4 , \tag{20}$$

$$
\begin{aligned}
f_b^{(2)} &= \frac{1}{2}\left(f_b^{(1)}\right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\ln\left(\frac{Q^2}{m_e^2}\right) + \frac{11}{8} + \frac{17}{32}\pi^2 - \frac{9}{4}\zeta(3) - \frac{2}{45}\pi^4 \\
&\quad - \frac{\pi^2\ln(2)}{2}\,, \qquad (21) \\
f_c^{(2)} &= \frac{1}{2}\left(f_c^{(1)}\right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\ln\left(\frac{Q^2}{\lambda^2}\right) + \frac{51}{128} + \frac{15}{16}\pi^2 + 5\zeta(3) - \frac{83}{360}\pi^4 \\
&\quad - \frac{2}{3}\pi^2\ln^2(2) + \frac{2}{3}\ln^4(2) + 16\,\mathrm{Li}_4\left(\frac{1}{2}\right)\,. \qquad (22)
\end{aligned}
$$

In the massless approximation the two-loop correction (20) has been known for a long time [35,36]. The two-loop correction (21) can be found in Ref. [37] as a specific limit of the result for an arbitrary momentum transfer. We independently cross-checked it by integrating the dispersion relation with the spectral density computed in Ref. [38]. To separate the closed fermion loop contribution included into the analysis [37,38] we used the result of Ref. [39]. The two-loop correction (22) has been obtained in Ref. [26]. As we see the two-loop corrections indeed have the universal logarithmic term corresponding to

$$
C^{(2)} = \frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\,. \qquad (23)
$$

By using the recent three-loop result [50] for the massless case we can completely predict the three-loop logarithmic corrections in massive cases, which are given by Eq. (19) with

$$
C^{(3)} = -\frac{29}{2} - 3\pi^2 - 68\zeta(3) + \frac{16}{3}\pi^2\zeta(3) + 240\zeta(5)\,. \qquad (24)
$$

3.2 Scattering amplitude

In the high energy limit the amplitude for the electron-positron scattering has two components corresponding to the scattering of particles of the same or opposite chirality. We can write the perturbative series for the amplitude as follows

$$
\mathcal{A} = \sum_{n=0}^{\infty}\left(\frac{\alpha}{\pi}\right)^n \mathcal{A}^{(n)}\,, \qquad \mathcal{A}^{(n)} = A^{(n)}\mathcal{A}^{(0)} \equiv \sum_{i=1}^{2} A_i^{(n)}\mathcal{A}_i^{(0)}\,, \qquad A_i^{(0)} = 1\,, \qquad (25)
$$

where $\mathcal{A}^{(0)}$ is a two component vector in the chiral basis corresponding to the tree amplitude. The collinear divergences are completely determined by external legs [33] and, therefore, are the same for the scattering amplitude and the square of the form factor. It is convenient to introduce a reduced amplitude

$$
\mathcal{A} = \mathcal{F}^2\tilde{\mathcal{A}}\,, \qquad (26)
$$

which is free of collinear divergences. It satisfies a linear differential equation [32,40] which for the photonic contribution takes the following form

$$
\frac{\partial}{\partial\ln\left(Q^2\right)}\tilde{\mathcal{A}} = -\frac{\alpha}{\pi}\ln\left(\frac{x}{1-x}\right)\tilde{\mathcal{A}}\,, \qquad (27)
$$

where the angular dependent anomalous dimension does not depend on chirality. The solution of Eq. (27) reads

$$
\begin{aligned}
\tilde{\mathcal{A}}|_{\lambda=0} &= \left(\mathcal{A}^{(0)}+\mathcal{O}(\alpha)\right)\exp\left[\frac{\alpha}{\pi}\ln\left(\frac{x}{1-x}\right)\frac{1}{\epsilon}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right], \\
\tilde{\mathcal{A}}|_{\lambda\neq 0} &= \left(\mathcal{A}^{(0)}+\mathcal{O}(\alpha)\right)\exp\left[-\frac{\alpha}{\pi}\ln\left(\frac{x}{1-x}\right)\ln\left(\frac{Q^2}{\lambda^2}\right)\right], \qquad (28)
\end{aligned}
$$

where the corrections in the prefactor of the exponent are different for different chiral components of the amplitude. There are no photonic corrections to the exponent in Eq. (28) [28]. Note that in the case (b) all the singular dependence of the corrections to the scattering amplitude on m_e is absorbed into the form factor. Eq. (28) implies that the logarithm of the reduced amplitude is finite beyond one loop and the coefficients of the series for each component of the chiral basis have the following structure

$$
\begin{aligned}
\tilde{A}_i^{(2)} &= \frac{1}{2}\left(\tilde{A}_i^{(1)}\right)^2+\mathcal{O}(\ln^0(Q^2)), \qquad (29)\\
\tilde{A}_i^{(3)} &= \frac{1}{6}\left(\tilde{A}_i^{(1)}\right)^3+\mathcal{O}(\ln^0(Q^2)). \qquad (30)\\
\ldots &.
\end{aligned}
$$

Now we can predict the singilar structure of the photonic corrections to the full amplitude to all orders and to construct the auxilary amplitude $\bar{A}^{(2)}$. In the two-loop approximation by using Eqs. (18, 29) we obtain

$$
\bar{A}_i^{(2)} = \frac{1}{2}\left(A_i^{(1)}\right)^2+2\left[f^{(2)}-\frac{1}{2}\left(f^{(1)}\right)^2\right]. \qquad (31)
$$

The expression (31) has all the necessary properties: it is composed of the one-loop corrections to the chiral amplitudes and the two-loop corrections to the form factor which are available in the massive case and it has the same structure of infrared divergences as the full amplitude.

4 Two-loop corrections to the massive Bhabha cross section

Once the result for the auxiliary amplitude in Eq. (11) is known, the problem is to evaluate the difference

$$
\delta A_i^{(2)} = A_i^{(2)}-\left[\frac{1}{2}\left(A_i^{(1)}\right)^2+2\left[f^{(2)}-\frac{1}{2}\left(f^{(1)}\right)^2\right]\right], \qquad (32)
$$

which matches the auxiliary and the full amplitudes. We should note that though the different terms on the right hand side of Eq. (32) taken separately are infrared divergent, their sum can be transformed into a sum of the convergent Feynman integrals. In similar way the pinch singularities disappear at the level of Feynman integrals in the two-loop corrections to the static potential after the proper infrared subtraction [41]. Thus, in principle, Eq. (32) does not need to be regularized. However, it is simpler to take the available results for the different terms of Eq. (32) in dimensional regularization and then to take the limit $d \to 4$ in the sum. The explicit result for the matrix elements of the two-loop and the tree amplitudes in dimensional regularization can be found in Ref. [8]. To disentangle the infrared divergences the authors of Ref. [8] used the formula suggested by S. Catani in Ref. [42]. In the next section we describe how this result can be matched to Eq. (32).

4.1 Two-loop infrared matching

In the case of pure photonic corrections the Catani formula for the structure of the infrared divergences takes the following form

$$\begin{aligned} \mathcal{A}^{(1)} &= \boldsymbol{I}^{(1)}\mathcal{A}^{(0)} + \mathcal{A}^{(1)}_{\text{fin}} \,, \\ \mathcal{A}^{(2)} &= \left[-\frac{1}{2}\left(\boldsymbol{I}^{(1)}\right)^2 + \boldsymbol{H}^{(2)}\right]\mathcal{A}^{(0)} + \boldsymbol{I}^{(1)}\mathcal{A}^{(1)} + \mathcal{A}^{(2)}_{\text{fin}} \,, \end{aligned} \tag{33}$$

where $A^{(n)}_{\text{fin}}$ are finite in the limit $\epsilon \to 0$ and the infrared divergences are described by the operators

$$\begin{aligned} \boldsymbol{I}^{(1)} &= \frac{e^{-\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{1}{\epsilon^2} + \frac{3}{2\epsilon}\right)\left[-\left(\frac{\mu^2}{-s}\right)^{\epsilon} - \left(\frac{\mu^2}{-t}\right)^{\epsilon} + \left(\frac{\mu^2}{-u}\right)^{\epsilon}\right] \,, \\ \boldsymbol{H}^{(2)} &= \frac{e^{-\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{\epsilon}\left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right)\left[-\left(\frac{\mu^2}{-s}\right)^{\epsilon} - \left(\frac{\mu^2}{-t}\right)^{\epsilon} + \left(\frac{\mu^2}{-u}\right)^{\epsilon}\right] \,. \end{aligned} \tag{34}$$

which are diagonal in the chiral basis. The form of the nonsingular terms in Eq. (34) is a matter of convention. We use the one of Refs. [8,42][1]. It is easy to check that the above expression is in full agreement with Eq. (31). Indeed, Eq. (33) is invariant under a redefinition

$$\begin{aligned} \boldsymbol{I}'^{(1)} &= \boldsymbol{I}^{(1)} + G \,, \qquad A'^{(1)}_{\text{fin}} = A^{(1)}_{\text{fin}} - G \,, \\ \boldsymbol{H}'^{(2)} &= \boldsymbol{H}^{(2)} + F \,, \qquad A'^{(2)}_{\text{fin}} = A^{(2)}_{\text{fin}} - \left(\frac{1}{2}G^2 + F\right) \,, \end{aligned} \tag{35}$$

where G and F are the two-component functions of x and ϵ which are regular at $\epsilon = 0$. By choosing

$$G = A^{(1)}_{\text{fin}} \,, \qquad F = 2f^{(2)} - (f^{(1)})^2 - \boldsymbol{H}^{(2)} \tag{36}$$

we reproduce the structure of Eqs. (31, 32) with

$$A^{(1)} = \boldsymbol{I}'^{(1)} \,, \qquad 2f^{(2)} - (f^{(1)})^2 = \boldsymbol{H}'^{(2)} \qquad \delta A^{(2)} = A'^{(2)}_{\text{fin}} \,. \tag{37}$$

[1]Our normalization of the operators differs from [8,42] by the overall factor 2 per loop.

Thus Eq. (33) can be considered as a direct consequence of the evolution equations (15, 27). Moreover, by analysing the evolution equations we can predict the form of the operator $\boldsymbol{H}^{(2)}$ for the scattering amplitude which was not determined in Ref. [42] but instead has been found by explicit calculation [8]. In fact the same is true for the four-quark amplitude where the Catani formula is a direct concequence of the non-Abelian evolution equations [32].

By using Eq. (36) one can transform the result of Ref. [8] for the matrix element of $\mathcal{A}^{(2)}_{\rm fin}$ into the one of $\delta\mathcal{A}^{(2)}$. However, we prefer to use directly the result of [8] for the finite part of the two-loop corrections and find the expressions for the operators $\boldsymbol{I}^{(1)}$ and $\boldsymbol{H}^{(2)}$ corresponding to the mass regularization of the infrared divergences. To perform this infrared matching let us first note that the primed operators defined through Eqs. (35, 36) have invariant meaning in contrast to the original definition (34). They correspond to the one-loop correction to the amplitude and two-loop correction to the logarithm of the form factor. Thus changing the regularization scheme for $\boldsymbol{I}'^{(1)}$ consists in replacing the one-loop massless result by the massive one. For $\boldsymbol{H}'^{(2)}$ the matching term which relates the dimensionally regularized and the massive result is given by twice the difference of the nonlogarithmic terms of Eq. (21) and Eq. (20), where the $(f^{(1)})^2/2$ contribution is subtracted. As we have already pointed out, the finite part $A'^{(2)}_{\rm fin} = \delta A^{(2)}$ does not depend on the regularization. Now we can perform the inverse transformation to the operators $\boldsymbol{I}^{(1)}$ and $\boldsymbol{H}^{(2)}$. Note that we are interested in the limit $d = 4$ and only need the value of the functions F and G at $\epsilon = 0$. The finite part of the two-loop correction to the amplitude $A^{(2)}_{\rm fin}$ does not change after these transformations while $\boldsymbol{I}^{(1)}$ and $\boldsymbol{H}^{(2)}$ become logarithmic functions of electron and photon masses

$$
\begin{aligned}
\boldsymbol{I}^{(1)} &= -\frac{1}{2}\ln^2\left(\frac{s}{m_e^2}\right) + \left[\ln\left(\frac{\lambda^2}{m_e^2}\right) + \frac{3}{2} - \ln\left(\frac{x}{1-x}\right) + i\pi\right]\ln\left(\frac{s}{m_e^2}\right) + \Big[-1 \\
&\quad + \ln\left(\frac{x}{1-x}\right) - i\pi\Big]\ln\left(\frac{\lambda^2}{m_e^2}\right) + 2 - \frac{2}{3}\pi^2 + \frac{3}{2}\ln\left(\frac{x}{1-x}\right) - \frac{1}{2}\ln^2(x) \\
&\quad + \frac{1}{2}\ln^2(1-x) - \frac{3}{2}i\pi\,, \\
\boldsymbol{H}^{(2)} &= \left(\frac{3}{16} - \frac{\pi^2}{4} + 3\zeta(3)\right)\left[\ln\left(\frac{s}{m_e^2}\right) + \ln\left(\frac{x}{1-x}\right) - i\pi\right] + \frac{177}{64} + \frac{11}{24}\pi^2 \\
&\quad - \frac{3}{4}\zeta(3) - \frac{7}{120}\pi^4 - \pi^2\ln(2)\,.
\end{aligned}
\tag{38}
$$

By plugging Eq. (38) into Eq. (33) we reproduce the known result for the logarithmic corrections to the amplitudes. Beside the logarithmic correction the operators (38) produce a nonlogarithmic contribution in Eq. (33) which can be considered as the matching term between the dimensionally regularized and the massive result for the amplitudes. Note that in Ref. [8] the explicit result is given only for the matrix elements of the two-loop and the tree amplitudes rather than the expressions for the chiral amplitudes. This, however, is sufficient for the matching because the operators Eq. (34) are diagonal in the chiral basis.

4.2 The result

Now we are in a position to derive the result for the second order correction to the cross section of the massive Bhabha scattering. It can be split into three parts:

(i) the corrections involving the soft real emission;

(ii) the interference of the one-loop corrections to the amplitudes;

(iii) the interference of the two-loop corrections and the tree amplitudes.

The soft photon emission is known to factorize and the corresponding second order corrections to the cross section introduced in Eq. (8) are of the following form

$$\delta_{vs}^{(2)} = \delta_v^{(1)}\delta_s^{(1)}\,, \qquad \delta_{ss}^{(2)} = \frac{1}{2}\delta_s^{(1)\,2}\,. \tag{39}$$

With the known one-loop corrections to the chiral amplitudes at hand (see *e.g.* Ref. [43]) it is straightforward to obtain the corresponding interference term in the cross section, Eq. (45) of the Appendix. The derivation of the contribution (iii) has been described in the previous section. Collecting all the contributions we obtain the result for the nonlogarithmic photonic correction which is given by Eq. (46) of the Appendix.

In the limit of small scattering angles the virtual corrections to the cross section are completely determined by the corrections to the electron and positron form factors in the t-channel amplitude [17,44]. We check that our result for the the virtual corrections in the the limit $x \to 0$ reduces to

$$\delta_{vv}^{(2)}\Big|_{x\to 0} = 6\left(f_b^{(1)}\right)^2 + 4f_b^{(2)} + \mathcal{O}(x)\,, \tag{40}$$

where $f_b^{(n)}$ are given by Eqs. (13, 21) with $Q^2 = xs$. This agrees with the asymptotic small angle expression given in [17] which is quite a nontrivial check of our result. The two-loop nonlogarithmic corrections to the cross section in the small angle limit becomes

$$\begin{aligned}\delta_0^{(2)}\Big|_{x\to 0} \;=\; & \left[8\ln^2\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + 12\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + \frac{9}{2}\right]\ln^2(x) + \left[-16\ln^2\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) - 28\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right)\right. \\ & \left. -\frac{93}{8} - \frac{\pi^2}{2} + 6\zeta(3)\right]\ln(x) + 8\ln^2\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + 16\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) + C\,,\end{aligned} \tag{41}$$

where

$$C = \frac{27}{2} + \frac{17}{8}\pi^2 - 9\zeta(3) - \frac{8}{45}\pi^4 - 2\pi^2\ln(2) = -0.744199\ldots\,. \tag{42}$$

Note that our result is not valid for very small scattering angles corresponding to $x \lesssim m_e^2/s$ and for almost backward scattering corresponding to $1 - x \lesssim m_e^2/s$, where the power-suppressed terms of the form m_e^2/t and m_e^2/u become important.

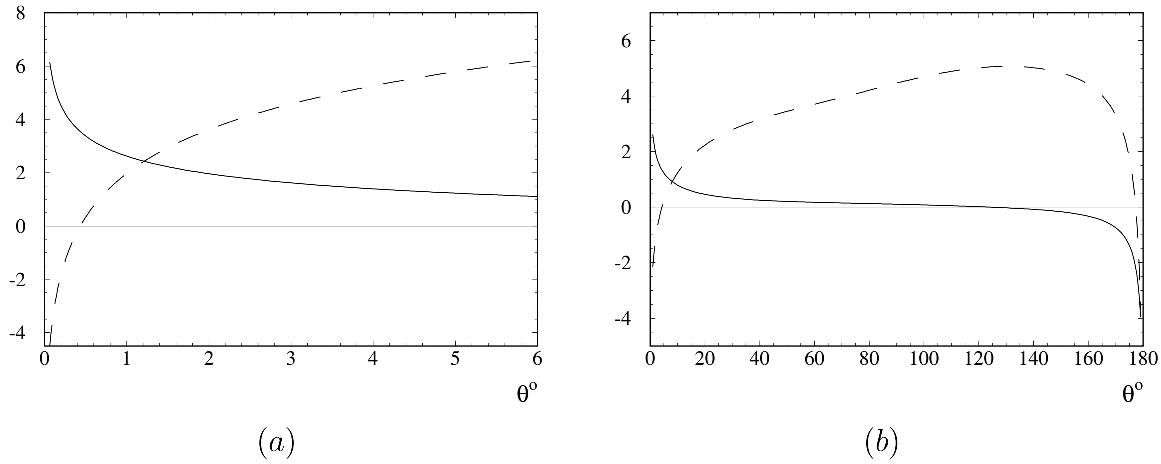


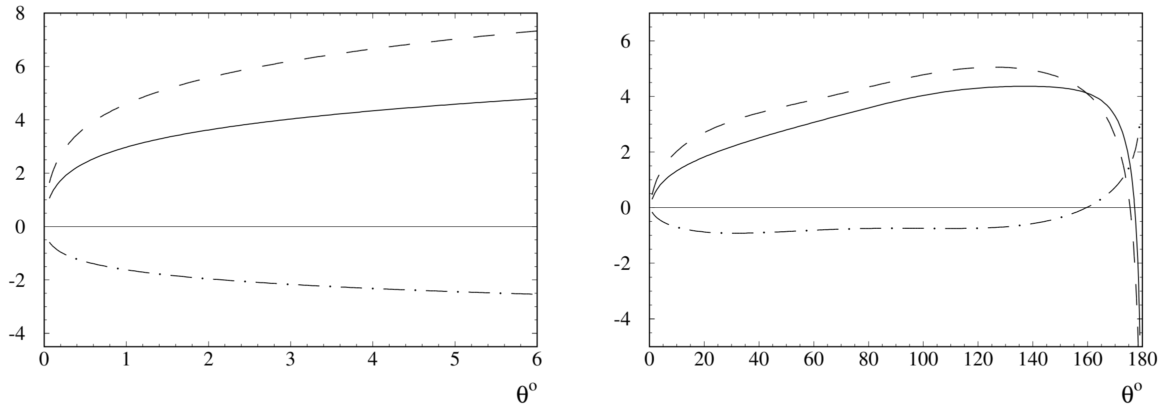
Figure 1: (a) Logarithmically enhanced (dashed line) and nonlogarithmic (solid line) second order corrections to the differential cross section of the small angle Bhabha scattering as functions of the scattering angle for $\sqrt{s} = 100$ GeV and $\ln(\varepsilon_{cut}/\varepsilon) = 0$, in permill. ($b$) The same as (a) but for the large angle Bhabha scattering and $\sqrt{s} = 1$ GeV.

5 Numerical estimates and summary

Let us now discuss the phenomenological relevance of our result. The determination of the luminosity for the GigaZ option of ILC is most demanding with respect to the theoretical predictions for the small angle Bhabha scattering. It requires the accuracy at the level of 0.1 permill [4]. At the same time the low-energy experiments aimed at the determination of the hadronic vacuum polarization contribution through the measurement of $\sigma(e^+e^- \to \text{ hadrons})$ require about one permill accuracy of the large angle Bhabha cross section. Such a high accuracy is necessary to reduce theoretical uncertainty due to the hadronic vacuum polarization contribution to the muon anomalous magnetic moment and to the value of the QED coupling constant at Z peak (see *e.g.* Ref. [45]). Neither of the existing Monte-Carlo event generators for small [1,2] and large angle [5,9,10] Bhabha scattering, which as yet do not incorporate the complete second order QED corrections, can guarantee the required precision.

The second order photonic corrections to the differential cross section $(\alpha/\pi)^2 \mathrm{d}\sigma^{(2)}/\mathrm{d}\sigma^{(0)}$ are plotted as functions of the scattering angle for the small angle Bhabha scattering at $\sqrt{s} = 100$ GeV on Fig. (1a) and for the large angle Bhabha scattering at $\sqrt{s} = 1$ GeV on Fig. (1b). We separate the logarithmically enhanced corrections given by the first two terms of Eq. (9) and the nonlogarithmic contribution given by the last term of this equation. All the terms involving a power of the logarithm $\ln(\varepsilon_{cut}/\varepsilon)$ are excluded from the numerical estimates because the corresponding contribution critically depends on the event selection algorithm and cannot be unambiguously estimated without imposing specific cuts on the photon bremsstrahlung.

We observe that for scattering angles $\theta \lesssim 18^o$ and $\theta \gtrsim 166^o$ the nonlogarithmic contribution



(a) (b)

Figure 2: (a) Potonic (dashed line), fermionic (dot-dashed line), and total (solid line) second order corrections to the differential cross section of the small angle Bhabha scattering as functions of the scattering angle for $\sqrt{s} = 100$ GeV and $\ln(\varepsilon_{cut}/\varepsilon) = \ln(\varepsilon_{cut}^{e^+e^-}/\varepsilon) = 0$, in permill. (b) The same as (a) but for the large angle Bhabha scattering and $\sqrt{s} = 1$ GeV.

exceeds a benchmark of 0.5 permill which makes it relevant for the luminosity determination at the low-energy electron-positron colliders. Furthermore, the nonlogarithmic contribution exceeds the logarithmically enhanced one for $\theta \lesssim 8^o$ at $\sqrt{s} = 1$ GeV and for $\theta \lesssim 1^o$ at $\sqrt{s} = 100$ GeV. However, in the limit of small scattering angles the dominant nonlogarithmic contributions enhanced by a power of $\ln(x)$, which are important for luminosity determination at LEP, directly follow from the logarithmic approximation and are already known [17]. The new x-independent contribution to the small angle cross section is represented by the constant C in Eq. (41). Numerically it amounts of approximately -0.04 permill and is relevant for the GigaZ accuracy.

To get the total second order correction (without the hard bremsstrahlung) our result should be combined with the fermionic contribution. In Fig. (2) we plot the second order photonic contribution against the fermionic one. The fermionic contribution incorporates the second order corrections with at least one closed fermion loop including the single soft photon emission [12] and the contribution due to the emission of the soft real electron-positron pair of the energy below a cutoff $\varepsilon_{cut}^{e^+e^-} \ll s$. The latter has been computed in Ref. [46] in the logarithmic approximation and cancels the artificial $\ln^3(m_e^2/s)$ term of the closed fermion loop contribution.

Finally, we would like to mention the electroweak corrections to Bhabha scattering which can be important at the considered level of accuracy. The one-loop correction is well known [43]. For the large angle scattering above the electroweak scale, however, the two-loop electroweak corrections could be important. In the case of $e^+e^- \to \mu^+\mu^-$ annihilation the corrections enhanced in the high energy limit by a power of the large logarithm $\ln(M^2/s)$, where M stands for W or Z boson mass, have been computed in [34,47–49]. They dominate the electroweak corrections for the energies $\sqrt{s}\gtrsim 500$ GeV characteristic to ILC. Due to the strong numerical cancellations between the terms with different powers of the large logarithm the total electroweak logarithmic two-loop contribution does not exceed a few permill in this energy region. This analysis can be generalized to the large angle Bhabha scattering by adding the t-channel contribution.

To conclude, we have derived the two-loop radiative photonic corrections to Bhabha scattering in the leading order of the small electron mass expansion up to nonlogarithmic term. The nonlogarithmic contribution has been found numerically important for the practically interesting range of scattering angles. Together with the result of Ref. [11,12] for the fermion loop corrections our result gives a complete expression for the two-loop virtual corrections. It should be incorporated into the Monte Carlo event generators to match the demands of the present and future electron-positron colliders for the accuracy of the luminosity determination.

Acknowledgments

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Appendix

The one-loop virtual photonic correction to the normalized cross section reads

$$\begin{aligned}\delta_v^{(1)} &= -\ln^2\left(\frac{s}{m_e^2}\right) + \left[2\ln\left(\frac{\lambda^2}{m_e^2}\right) + 3 - 2\ln\left(\frac{x}{1-x}\right)\right]\ln\left(\frac{s}{m_e^2}\right) + \left[-2 + 2\ln\left(\frac{x}{1-x}\right)\right] \\ &\quad \times \ln\left(\frac{\lambda^2}{m_e^2}\right) - 4 - \ln^2(x) + \ln^2(1-x) + f(x)\,, \end{aligned} \tag{43}$$

where $f(x)$ is given by Eq. (7). The correction due to the single soft photon emission reads

$$\begin{aligned}\delta_s^{(1)} &= \ln^2\left(\frac{s}{m_e^2}\right) + \left[4\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) - 2\ln\left(\frac{\lambda^2}{m_e^2}\right) + 2\ln\left(\frac{x}{1-x}\right)\right]\ln\left(\frac{s}{m_e^2}\right) + \Bigg[-2 \\ &\quad +2\ln\left(\frac{x}{1-x}\right)\Bigg]\left[2\ln\left(\frac{\varepsilon_{cut}}{\varepsilon}\right) - \ln\left(\frac{\lambda^2}{m_e^2}\right)\right] - \frac{2}{3}\pi^2 + \ln^2(x) - \ln^2(1-x) \\ &\quad -2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(1-x)\,. \end{aligned} \tag{44}$$

The second order contribution to the cross section due to the interference of the one-loop virtual corrections to the amplitude reads

$$\delta_{vv}^{(1\times1)} = 4 + \Bigl(1-x+x^2\Bigr)^{-2}\Biggl(\biggl(-\frac{2}{3}+\frac{4}{3}x-\frac{13}{4}x^2+\frac{17}{6}x^3-\frac{5}{12}x^4\biggr)\pi^2+\biggl(\frac{1}{36}-\frac{x}{18}+\frac{13}{24}x^2-\frac{49}{72}x^3 +\frac{137}{288}x^4\biggr)\pi^4+\biggl[-6+8x-9x^2+3x^3+\biggl(\frac{1}{2}+\frac{5}{6}x-\frac{x^2}{8}+\frac{5}{8}x^3-3x^4\biggr)\pi^2\biggr]\ln(x)+\biggl[\frac{17}{4}-7x +\frac{31}{4}x^2-\frac{5}{2}x^3+\biggl(\frac{5}{6}-\frac{x}{24}+\frac{x^2}{12}+\frac{x^3}{12}+\frac{17}{16}x^4\biggr)\pi^2\biggr]\ln^2(x)+\biggl(-\frac{3}{2}+\frac{25}{8}x-2x^2-\frac{x^3}{8}\biggr)\ln^3(x) +\biggl(\frac{1}{4}-\frac{7}{8}x+\frac{33}{32}x^2-\frac{x^3}{8}+\frac{x^4}{32}\biggr)\ln^4(x)+\biggl\{x+x^3+\biggl(\frac{x}{6}+\frac{3}{2}x^2-\frac{101}{24}x^3+3x^4\biggr)\pi^2+\biggl[-4 +\frac{29}{4}x-\frac{29}{4}x^2+2x^3+\biggl(\frac{1}{3}+\frac{x}{3}-\frac{7}{6}x^2+\frac{7}{3}x^3-\frac{17}{8}x^4\biggr)\pi^2\biggr]\ln(x)+\biggl(3-\frac{15}{4}x+\frac{3}{2}x^2+\frac{3}{8}x^3\biggr) \times\ln^2(x)+\biggl(-1+\frac{11}{4}x-\frac{9}{4}x^2+\frac{x^3}{2}-\frac{x^4}{8}\biggr)\ln^3(x)\biggr\}\ln(1-x)+\biggl[-x+\frac{5}{4}x^2-x^3+\biggl(\frac{1}{8}-\frac{5}{12}x +\frac{37}{24}x^2-\frac{23}{12}x^3+\frac{17}{16}x^4\biggr)\pi^2+\biggl(\frac{x}{4}+\frac{3}{8}x^2-\frac{3}{8}x^3\biggr)\ln(x)+\biggl(1-\frac{9}{4}x+\frac{15}{8}x^2-\frac{3}{4}x^3+\frac{3}{16}x^4\biggr) \times\ln^2(x)\biggr]\ln^2(1-x)+\biggl[\frac{x}{8}-\frac{x^2}{2}+\frac{x^3}{8}+\biggl(\frac{x}{2}-\frac{3}{4}x^2+\frac{x^3}{2}-\frac{x^4}{8}\biggr)\ln(x)\biggr]\ln^3(1-x)+\biggl(\frac{1}{32} -\frac{x}{8}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{32}\biggr)\ln^4(1-x)\;, \tag{45}$$

where the trivial terms proportional to $\ln(\lambda^2/m_e^2)$ and $\ln(m_e^2/s)$ are omitted.

The total second order nonlogarithmic photonic contribution to the normalized cross section reads

$$\delta_0^{(2)} = 8\mathcal{L}_\varepsilon^2 + \Bigl(1-x+x^2\Bigr)^{-2}\biggl[\biggl(\frac{4}{3}-\frac{8}{3}x-x^2+\frac{10}{3}x^3-\frac{8}{3}x^4\biggr)\pi^2+\Bigl(-12+16x-18x^2+6x^3\Bigr)\ln(x) +\Bigl(2x+2x^3\Bigr)\ln(1-x)+\Bigl(-3x+x^2+3x^3-4x^4\Bigr)\ln^2(x)+\Bigl(-8+16x-14x^2+4x^3\Bigr)\ln(x) \times\ln(1-x)+\Bigl(4-10x+14x^2-10x^3+4x^4\Bigr)\ln^2(1-x)+\Bigl(1-x+x^2\Bigr)^2(16+8\mathrm{Li}_2(x) -8\mathrm{Li}_2(1-x))\biggr]\mathcal{L}_\varepsilon+\frac{27}{2}-2\pi^2\ln(2)+\Bigl(1-x+x^2\Bigr)^{-2}\Biggl(\biggl(\frac{83}{24}-\frac{125}{24}x+\frac{13}{4}x^2+\frac{19}{24}x^3-\frac{25}{24}x^4\biggr) \times\pi^2+\biggl(-9+\frac{43}{2}x-34x^2+22x^3-9x^4\biggr)\zeta(3)+\biggl(-\frac{11}{90}-\frac{5}{24}x+\frac{29}{180}x^2+\frac{23}{180}x^3-\frac{49}{480}x^4\biggr)\pi^4 +\biggl[-\frac{93}{8}+\frac{231}{16}x-\frac{279}{16}x^2+\frac{93}{16}x^3+\biggl(-\frac{3}{2}+\frac{13}{4}x-\frac{7}{12}x^2-\frac{11}{8}x^3\biggr)\pi^2+\Bigl(12-12x+8x^2 -x^3\Bigr)\zeta(3)\biggr]\ln(x)+\biggl[\frac{9}{2}-\frac{43}{8}x+\frac{17}{8}x^2+\frac{29}{8}x^3-\frac{9}{2}x^4+\biggl(\frac{x}{4}+\frac{x^2}{2}+\frac{5}{24}x^3+\frac{19}{48}x^4\biggr)\pi^2\biggr]\ln^2(x) +\biggl(\frac{67}{24}x-\frac{5}{4}x^2-\frac{2}{3}x^3\biggr)\ln^3(x)+\biggl(\frac{7}{48}x+\frac{5}{96}x^2-\frac{x^3}{12}+\frac{43}{96}x^4\biggr)\ln^4(x)+\biggl\{3x+3x^3+\biggl(\frac{7}{6}x$$

$$
\begin{aligned}
&-\frac{73}{24}x^2+\frac{15}{8}x^3\Big)\pi^2+\Big(-6+6x-x^2-4x^3\Big)\zeta(3)+\Big[-8+\frac{21}{2}x-\frac{45}{4}x^2+x^4+\Big(1-\frac{x}{6}+\frac{x^2}{12}\\
&-\frac{x^3}{3}-\frac{x^4}{8}\Big)\pi^2\Big]\ln(x)+\Big(6-11x+\frac{35}{4}x^2-\frac{15}{8}x^3\Big)\ln^2(x)+\Big(\frac{2}{3}+\frac{x}{12}-\frac{x^3}{3}+\frac{5}{24}x^4\Big)\ln^3(x)\Big\}\\
&\times\ln(1-x)+\Big[\frac{7}{2}-6x+\frac{45}{4}x^2-6x^3+\frac{7}{2}x^4+\Big(-\frac{17}{24}+\frac{7}{6}x-\frac{25}{24}x^2-\frac{13}{48}x^4\Big)\pi^2+\Big(-3+\frac{23}{4}x\\
&-\frac{23}{4}x^2+\frac{9}{8}x^3\Big)\ln(x)+\Big(\frac{7}{2}-\frac{41}{8}x+\frac{31}{8}x^2+\frac{3}{8}x^3-\frac{13}{16}x^4\Big)\ln^2(x)\Big]\ln^2(1-x)+\Big[\frac{3}{8}x+\frac{1}{6}x^2\\
&+\frac{3}{8}x^3+\Big(-4+\frac{29}{6}x-\frac{49}{12}x^2+\frac{5}{6}x^3+\frac{7}{8}x^4\Big)\ln(x)\Big]\ln^3(1-x)+\Big(\frac{1}{32}-\frac{3}{4}x+\frac{71}{48}x^2-\frac{29}{24}x^3\\
&+\frac{9}{32}x^4\Big)\ln^4(1-x)+\Big\{8-16x+24x^2-16x^3+8x^4+\Big(\frac{7}{3}-3x+\frac{3}{4}x^2+\frac{5}{6}x^3-\frac{2}{3}x^4\Big)\pi^2\\
&+\Big[-6+\frac{11}{2}x-4x^2+x^3+\Big(2-\frac{11}{4}x+\frac{7}{4}x^2+\frac{x^3}{4}-x^4\Big)\ln(x)\Big]\ln(x)+\Big[\frac{3}{2}x-\frac{x^2}{4}+x^3\\
&+\Big(-4+9x-\frac{15}{2}x^2+2x^3\Big)\ln(x)+\Big(-1-\frac{7}{2}x+\frac{25}{4}x^2-5x^3+2x^4\Big)\ln(1-x)\Big]\ln(1-x)+\Big(2\\
&-4x+6x^2-4x^3+2x^4\Big)\mathrm{Li}_2(x)\Big\}\mathrm{Li}_2(x)+\Big\{-8+16x-24x^2+16x^3-8x^4+\Big[-\frac{2}{3}+\frac{4}{3}x\\
&+\frac{x^2}{2}-\frac{5}{3}x^3+\frac{2}{3}x^4\Big]\pi^2+\Big[6-8x+9x^2-3x^3+\Big(\frac{3}{2}x-\frac{x^2}{2}-\frac{3}{2}x^3+2x^4\Big)\ln(x)\Big]\ln(x)+\Big[-x\\
&-\frac{x^2}{4}-\frac{x^3}{2}+\Big(10-14x+9x^2\Big)\ln(x)+\Big(-8+11x-\frac{31}{4}x^2+\frac{x^3}{2}+x^4\Big)\ln(1-x)\Big]\ln(1-x)\\
&+\Big(-4+8x-12x^2+8x^3-4x^4\Big)\mathrm{Li}_2(x)+\Big(2-4x+6x^2-4x^3+2x^4\Big)\mathrm{Li}_2(1-x)\Big\}\mathrm{Li}_2(1-x)\\
&+\Big[\frac{5}{2}x-5x^2+2x^3+\Big(-4-x+x^2+2x^3-2x^4\Big)\ln(x)+(6-6x+x^2+4x^3)\ln(1-x)\Big]\mathrm{Li}_3(x)\\
&+\Big[\frac{x}{2}-\frac{x^3}{2}+(-6+5x+3x^2-5x^3)\ln(x)+\Big(6-10x+10x^3-6x^4\Big)\ln(1-x)\Big]\mathrm{Li}_3(1-x)\\
&+\Big(-2+\frac{17}{2}x-\frac{17}{2}x^3+2x^4\Big)\mathrm{Li}_4(x)+\Big(7x-\frac{9}{2}x^2-4x^3+6x^4\Big)\mathrm{Li}_4(1-x)+\Big(-6+4x\\
&+\frac{9}{2}x^2-7x^3\Big)\mathrm{Li}_4\Big(-\frac{x}{1-x}\Big)\Big),
\end{aligned}
\tag{46}
$$

where $\mathcal{L}_\varepsilon = [1-\ln(x/(1-x))]\ln(\varepsilon_{cut}/\varepsilon)$.

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