Energy-square equations for relativistic closed systems

Hartmut Pilkuhn

Institut für Theoretische Teilchenphysik,

Universität, D-76128 Karlsruhe, Germany

Abstract

The most powerful relativistic two-body equations have only half as many components as their Dirac-Breit and Bethe-Salpeter counterparts and contain only the square of the total cms energy. In this letter, the "energy-square" theorem is extended to systems of $n > 2$ Dirac particles. The case $n = 3$ is discussed for baryons.

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I. INTRODUCTION

According to Einstein, a closed system of total energy K^0 moving with total momentum $\boldsymbol{K}$ satisfies $K^{02} - \boldsymbol{K}^2 = constant$ (units $c = \hbar = 1$ will be used). For a single atom in its ground state, the *constant* is M^2, M being the atomic mass. For all other atomic states, M^2 is better called E^2, E being the total cms energy. In relativistic quantum mechanics, it is customary to calculate E as the eigenvalue of a differential equation $H_D\psi_D = E\psi_D$. The index D stands for Dirac. H_D is the relativistic "Hamiltonian" for the system's constituents, which for a closed system includes the nucleus. It has been derived from quantum electrodynamics (QED) with ever increasing precision. Its crudest version is the Dirac-Coulomb Hamiltonian $H_{\rm DC}$, next comes the Dirac-Breit Hamiltonian $H_{\rm DB}$. The method is presently extended to mesons as quark-antiquark bound states and baryons as three-quark bound states, replacing QED by quantum chromodynamics (QCD). The results for the short-range interaction are encouraging.

Mainly with this extension in mind it is pointed out here that the above Einstein relation indicates a better approach to the fully relativistic quantum mechanics of closed systems. By reducing the S-matrix for the scattering of n fermions from $4^n \times 4^n$ components to $4^n \times 4^n/4$ components before including the interaction, one derives equations $h\psi = \frac{1}{2}E^2\psi$ of a new "hamiltonian" h, with a small residual E^2-dependence in some terms of h [1]. In the residual terms, one may approximate E^2 by M^2. From the point of view of QED and QCD, this "E^2-theorem" follows from CPT invariance. For every atom in a level E, there should exist an antiatom of exactly the same energy. In a formulation that dispenses with creation and annihilation operators for atoms and antiatoms, the antiatomic state appears as an atomic state of energy $-E$. Multilocal fields for composite systems are presently used in situations where their nonlocality is negligible, for example in the derivation of Bose-Einstein and Fermi-Dirac statistics for mesons and baryons. In particle theory, the theorem appeared first in a current algebra derivation of the Gell-Mann -Okubo mass formula for the light pseudoscalar mesons [2].

The problem is of course the construction of h from the reduced S-matrix. It has been solved for leptonium ($\mathrm{e}^-\mathrm{e}^+$ and $\mathrm{e}^-\mu^+$) [3,4]. For hydrogen, the inclusion of the proton's large anomalous magnetic moment in the relativistic proton current J_p^μ is a mere technicality. In this letter, we consider n point Dirac particles, including a "nucleus" of mass m_n and kinetic energy operator $\boldsymbol{\alpha}_n \boldsymbol{p}_n = \gamma_n^5 \boldsymbol{\sigma}_n \boldsymbol{p}_n$, where $\boldsymbol{\sigma}_i$ denotes the 2×2 Pauli matrix of particle i. We first show how $H_{\rm DC}$ can be converted to an equivalent h_C, for which the E^2-theorem is proven. The corresponding conversion of $H_{\rm DB}$ to h_B is straightforward, but the result is uninteresting in view of the cancellations expected from the reduced S-matrix approach. We shall thus only need the Gaunt part of the Breit operator between particles i and j, which combines with their Coulomb potential V_{ij} into $V_{ij}(1 - \gamma_i^5 \gamma_j^5 \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j)$. We shall also include a vector potential $\boldsymbol{A}(\boldsymbol{r})$, by taking the individual momentum operators as

$$\boldsymbol{\pi}_i = \boldsymbol{p}_i + q_i \boldsymbol{A}(t, \boldsymbol{r}_i), \quad \boldsymbol{p}_i = -i\nabla_i, \tag{1}$$

with $q_i = -e$ for electrons and $+Ze$ for the "nucleus". The extension to the radiation field $\boldsymbol{A}(t, \boldsymbol{r}) = e^{i\omega t} \boldsymbol{A}(\boldsymbol{r})$ shows that the time-dependent perturbation theory based on

$$i\partial_\tau \psi = h\psi, \quad \tau = t/E \equiv t_E \tag{2}$$

reproduces the relativistic atomic recoil in radiative decays, while the traditional $i\partial_t \psi_D = H_D \psi_D$ fails to do so [5]. After a summary of the two-fermion equation in section 3, the case $n = 3$ will be discussed in section 4.

The Dirac-Coulomb equation for n idealized Dirac particles including the nucleus can be put into the form

$$[\pi^0 - \sum_{i=1}^{n} (m_i \beta_i + \gamma_i^5 \boldsymbol{\sigma}_i \boldsymbol{\pi}_i)]\psi_D = 0, \quad \pi^0 = E - V_{tot}, \tag{3}$$

$$V_{tot} = \Sigma_{i<j} V_{ij}, \quad V_{ij} = q_i q_j / |\boldsymbol{r}_i - \boldsymbol{r}_j|, \tag{4}$$

with $\beta_i = \gamma_i^0$. For $n = 1$, one has $V_{tot} = 0$, and the cms condition $\Sigma_i \boldsymbol{p}_i = 0$ (for $\boldsymbol{A} = 0$) leads to $E_1 \psi_D = m_1 \beta_1 \psi_D$. The construction of h uses the chiral basis, in which γ^5 and β are diagonal and antidiagonal, respectively:

$$\psi_D = \begin{pmatrix} \psi_r \\ \psi_l \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

For a single free electron, elimination of the "lefthanded" two-component spinor ψ_l yields $E_1^2 \psi_r = m_1^2 \psi_r$. This trivial example shows how the construction of h halves the number of components of ψ_D.

II. THE ENERGY-SQUARE FORM OF THE DIRAC-COULOMB EQUATION

The spinor ψ_D of (3) may be decomposed into two spinors ψ and χ which are eigenstates of the total chirality operator:

$$\gamma^5_{tot} = \Pi^n_{i=1}\gamma^5_i. \tag{6}$$

$(\gamma^5_i)^2 = 1$ implies $(\gamma^5_{tot})^2 = 1$, such that its eigenvalues are $+1$ and -1:

$$\psi_D = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad \gamma^5_{tot}\psi = \psi, \quad \gamma^5_{tot}\chi = -\chi. \tag{7}$$

As each β_i anticommutes with γ^5_{tot}, it has only nondiagonal elements in this basis. With the abbreviation $m_+ = \Sigma_i m_i\beta_i$, (3) becomes

$$(\pi^0 - \Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i)\psi = m_+\chi, \quad (\pi^0 - \Sigma_i\gamma^5_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i)\chi = m_+\psi. \tag{8}$$

One of the γ^5_i may be eliminated using (7), say γ^5_n:

$$\gamma^5_n = \gamma^5_{tot}\gamma^5_e, \quad \gamma^5_e = \Pi^n_{i\neq n}\gamma^5_i. \tag{9}$$

Choosing n as the "nucleus" makes γ^5_e the total electronic chirality. Using now the first equation to eliminate χ, the second one produces

$$(\pi^0 - \Sigma\gamma^5_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i + \gamma^5_e\boldsymbol{\sigma}_n\boldsymbol{\pi}_n)m_+^{-1}(\pi^0 - \Sigma\gamma^5_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i - \gamma^5_e\boldsymbol{\sigma}_n\boldsymbol{\pi}_n)\psi = m_+\psi, \tag{10}$$

where the summation symbol refers only to the electrons.

The factor m_+^{-1} is removed by multiplying (10) by m_+,

$$(\pi^0 - \Sigma\gamma^{5\prime}_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i + \gamma^{5\prime}_e\boldsymbol{\sigma}_n\boldsymbol{\pi}_n)(\pi^0 - \Sigma\gamma^5_i\boldsymbol{\sigma}_i\boldsymbol{\pi}_i - \gamma^5_e\boldsymbol{\sigma}_n\boldsymbol{\pi}_n)\psi = m_+^2\psi, \tag{11}$$

$$\gamma^{5\prime} = m_+\gamma^5/m_+, \quad (\gamma^{5\prime})^2 = 1, \quad \gamma^{5\prime}_i\beta_i + \beta_i\gamma^{5\prime}_i = 0. \tag{12}$$

After a rescaling of variables, each bracket in (11) contains only odd powers of E:

$$\boldsymbol{r}_i = E\boldsymbol{r}_{iE}, \quad \boldsymbol{p}_i = \boldsymbol{p}_{iE}/E, \quad V_{tot} = E^{-1}V_{tot}(\boldsymbol{r}_{iE}). \tag{13}$$

The new variables $\boldsymbol{r}_{iE}$ will also appear in orthogonality relations. A constant magnetic field $\boldsymbol{B}$ provides $\boldsymbol{A} = \frac{1}{2}B\times\boldsymbol{r} = \frac{1}{2}E\boldsymbol{B}\times\boldsymbol{r}_E$. Clearly, the product of the two brackets contains only even powers of E, which proves the E^2-theorem for $H_{\rm DC}$. The addition of Breit operators does not alter the conclusion. In the new "hamiltonian" h of (11), terms of order V^2 will now also be neglected:

$$2h_C = m_+^2 + 2V_{tot,E} + \Sigma(\gamma_i^5 + \gamma_i^{5\prime})\boldsymbol{\sigma}_i\boldsymbol{\pi}_{iE} + (\gamma_e^5 - \gamma_e^{5\prime})\boldsymbol{\sigma}_n\boldsymbol{\pi}_{nE} + 2h(\boldsymbol{\pi}_i, \boldsymbol{\pi}_j, \boldsymbol{\pi}_n). \tag{14}$$

The last term is not written down here because of its length. It vanishes for $n = 2$ and $\boldsymbol{A} = 0$.

When particle n is a nucleus with mass $m_n \gg m_e$, one defines $E^2 = (m_n + E_e)^2$, E_e being the "electronic energy". Inserting

$$m_+^2 = m_n^2 + \Sigma_i m_i^2 + 2m_n\Sigma_i m_i\beta_n\beta_i + 2\Sigma_{i<j} m_i m_j \beta_i\beta_j, \tag{15}$$

one may cancel the m_n^2 in the equation $(2h - E^2)\psi = 0$, and often also neglect $m_i m_j = m_e^2$ relative to $m_n m_e$. The phase between ψ and χ in (7) may be fixed by $\beta_n\psi = \chi$, $\beta_n\chi = \psi$, which implies $\beta_n = 1$ in (15). The resulting $(m_+^2 - m_n^2)/2m_n$ is $\Sigma_i m_e\beta_i$, in agreement with the mass operator of the recoilless Dirac equation.

In the radiative decays of atoms or hadrons, the relativistic recoil appears automatically in the time-dependent perturbation theory based on (2) with $\boldsymbol{A} = e^{i\omega t}\boldsymbol{A}(\boldsymbol{r})$, using the unperturbed solution [5] $h^0(\tau) = exp\{iE^2\tau\}h(\tau = 0)$. If one wants to use standard orthogonality relations for h, one must replace E^2 by M^2 in the residual terms of h.

Relativistic recoil is small in atomic radiation but becomes appreciable in the hyperon decay $\Sigma^0 \to \Lambda\gamma$, and even enormous in the meson decay $\omega \to \pi^0\gamma$. It shows that the energy-square theorem must apply even when the interaction considered here is incomplete.

III. HYDROGENIC ATOMS

For $n = 2$, $q_1 = -e$, $q_2 = Ze$ gives $V_{tot} = V(r) = -Ze^2/r$, $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. One may also abbreviate $\gamma_e^5 = \gamma_1^5 \equiv \gamma^5$, and $\boldsymbol{\sigma}_1\boldsymbol{\pi}_1 \pm \boldsymbol{\sigma}_2\boldsymbol{\pi}_2 = \pi_\pm$. The resulting h is

$$2h = m_+^2 + 2EV + [E - V(1 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)]\gamma^5\pi_+ + \gamma^{5\prime}\pi_-[E - V(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)] + \gamma^{5\prime}\gamma^5\pi^+\pi^-. \tag{16}$$

For $\boldsymbol{A} = 0$, the cms condition $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$ allows one to put $\boldsymbol{p}_1 = \boldsymbol{p}$, $\boldsymbol{p}_2 = -\boldsymbol{p}$, which results in $\pi_+\pi_- = 0$. The term $\pi_- V(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) = \boldsymbol{p}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)V$ vanishes then also because of $(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) = 0$. The remainder gives, after a transformation $\psi = C_1\psi_1$,

$$h_1(\boldsymbol{A} = 0) = \tfrac{1}{2}m_+^2 + \gamma^5\boldsymbol{\sigma}_1\boldsymbol{p}_E + V_E + h_{hf}, \tag{17}$$

$$h_{hf} = i\gamma^5 \boldsymbol{\sigma}^\times V_E \boldsymbol{p}_E / E^2, \quad \sigma^\times = \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2. \tag{18}$$

An equivalent transformation $\psi = C_2\psi_2$ produces an h_2 which has $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ exchanged [6]. (17) is the complete hamiltonian to the order α^4. The V^2 expected from π^{02} in (11) is excluded, and so is the retardation part of the Breit operator. The remaining Gaunt interaction $-V\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2$ produces h_{hf}. It may be remembered that $V(1-\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)$ is the Fourier transform of the one-photon exchange amplitude in the cms, $T^{(1)} = \gamma_1^\mu g_{\mu\nu}\gamma_2^\nu/\boldsymbol{q}^2$, in which retardation is excluded by $q^0 = 0$. There exist of course two-photon exchange potentials and Bethe logarithms (calculated from time-ordered products of fields as usual), their largest contributions to E^2 being of the order $\alpha^5 \log\alpha$ and α^5.

Finally, a word about parity invariance. For $n = 2$, the transformation $\psi_D^P(\boldsymbol{r}_1, \boldsymbol{r}_2) = \beta_1\beta_2\psi_D(-\boldsymbol{r}_1, -\boldsymbol{r}_2)$ is consistent with the decomposition (7), as $\beta_1\beta_2$ commutes with $\gamma_1^5\gamma_2^5$. The decomposition is also parity invariant for $n = 4$, but not for $n = 1$ and $n = 3$. For such cases, the notation $\psi = \psi_r$, $\chi = \psi_l$ of the single-particle Dirac equation in the chiral basis (5) is appropriate. For $n = 1$, the parity transformation uses the linear relation between ψ_r and ψ_l. The corresponding relation exists also for the n-body Dirac-Coulomb equation (8) and its second-order form (11), but it disappears after the cancellation of $[V(...)]^2$. The parity transformation involves then two separate spinors ψ and χ, which are related only by the fact that χ satisfies the hermitian adjoint equation of ψ, with the same real eigenvalue. This feature appears also for $n = 2$ with a spinless particle 2; it caused considerable concern in an early paper [7].

IV. BARYONS

In the quark model, baryons are taken as three-fermion bound states. Their shortrange interactions should follow the QED formalism, with a one-gluon exchange potential V (including vacuum polarization) and the corresponding Gaunt interaction dominating at short distances. A more complete shortrange interaction will have to use the three-body S-matrix [8]. As $V + V_G$ is a sum of two-fermion operators, we assume instead the type of cancellations that occur for $n = 2$: No retardation, and no terms of order V^2. The Gaunt interaction may be included in the definition of π^0, similar to (14). A separate treatment of a "nucleus" is not necessary. With $\gamma_i^5\gamma_j^5\psi = \gamma_k^5\psi$ $(i \neq j \neq k)$ and $\gamma_i^5\gamma_j^5\chi = -\gamma_k^5\chi$, one must distinguish between a π_r^0 in the right bracket of (10), and a π_l^0 in the left one:

$$(\pi_l^0 - \Sigma_i \gamma_i^5 \boldsymbol{\sigma}_i \boldsymbol{\pi}_i) m_+^{-1} (\pi_r^0 - \Sigma_i \gamma_i^5 \boldsymbol{\sigma}_i \boldsymbol{\pi}_i)\psi = m_+ \psi, \tag{19}$$

$$\pi_{r,l}^0 = E - \Sigma_{i<j} V_{ij}(1 \mp \gamma_k^5 \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j). \tag{20}$$

In the form (11), π_l^0 has γ_k^5 replaced by $\gamma_k^{5\prime}$. With the above cancellations assumed, one obtains the "Coulomb-Gaunt" hamiltonian,

$$h_{CG} = h_C + \tfrac{1}{2}\Sigma_{i<j}[V_{ij,E}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\gamma_k^{5\prime}(1 - \Sigma_i\gamma_i^5\boldsymbol{\sigma}_i\boldsymbol{\pi}_{iE}/E^2) - (1 - \Sigma_i\gamma_i^{5\prime}\boldsymbol{\sigma}_i\boldsymbol{\pi}_{iE}/E^2)V_{ij}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\gamma_k^5]. \tag{21}$$

For $n = 2$, the transformation $\psi = C_1\psi_1$ was necessary to remove the pole of m_+^{-1} at $m_2 = m_1$, which occurs at $\beta = -1$. For $n = 3$, a corrsponding transformation is unnecessary, as $m_1(\beta_1 + \beta_2 + \beta_3) \neq 0$. The second operator in each bracket produces new hyperfine operators, which are not written out here:

$$h_{CG} = h_C + h'_G + h_{hf}, \quad h'_G = \Sigma_{i<j} V_{ij,E}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \tfrac{1}{2}(\gamma_k^{5\prime} - \gamma_k^5), \tag{22}$$

$$\tfrac{1}{2}(\gamma_k^{5\prime} - \gamma_k^5) = \gamma_k^5 m_k \beta_k (m_1\beta_1 + m_2\beta_2 + m_3\beta_3)^{-1}. \tag{23}$$

When k represents a heavy quark ($m_k \gg m_i, m_j$), h'_G is particularly large; it replaces essentially the Breit operator of the two light quarks.

V. REFERENCES

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