TTP05-04

Two-Loop Electroweak Logarithms

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We present the complete analytical result for the two-loop logarithmically enhanced contributions to the high energy asymptotic behavior of the vector form factor and the four-fermion cross section in a spontaneously broken $SU(2)$ gauge model. On the basis of this result we derive the dominant two-loop electroweak corrections to the neutral current four-fermion processes at high energies.

PACS numbers: 12.15.Lk

Recently a new wave of interest to the Sudakov asymptotic regime [1, 2] has been triggered by the study of higher-order corrections to electroweak processes at high energies [3–12]. Experimental and theoretical studies of electroweak interactions have traditionally explored the range from very low energies, *e.g.* through parity violation in atoms, up to energies comparable to the masses of the W- and Z-bosons, *e.g.* at the LEP or the Tevatron. The advent of multi-TeV colliders like the LHC during the present decade or a future linear electron-positron collider will give access to a completely new energy domain. Once the characteristic energies $\sqrt{s}$ are far larger than the masses of the W- and Z-bosons, $M_{W,Z}$, exclusive reactions like electron-positron (or quark-antiquark) annihilation into a pair of fermions or gauge bosons will receive virtual corrections enhanced by powers of the large *electroweak* logarithm $\ln\left(s/M_{W,Z}^2\right)$. The leading double-logarithmic corrections may well amount to ten or even twenty percent in one-loop approximation and reach several percent in two-loop approximation. Moreover, in the TeV region, the subleading logarithms turn out to be equally important [6, 10]. One percent accuracy of the theoretical estimates for the cross sections necessary for the search of new physics beyond the standard model can be guaranteed only by including *all* the logarithmic two-loop corrections.

The direct calculation of the two-loop electroweak corrections is an extremely challenging theoretical problem at the limit of available computational techniques even in the high energy limit. However, the asymptotic high energy behavior of the amplitudes is governed by evolution equations which turn out to be a powerful tool in the analysis of the logarithmic corrections. In Ref. [5] the leading logarithmic (LL) electroweak corrections have been obtained to all orders of perturbative expansion within the infrared evolution equation approach. This analysis has been extended to the NLL and N^{2}LL approximation [24] in Refs. [6, 10] by combining the hard and infrared evolution equations. Starting with the N^{3}LL approximation the corrections become sensitive to fine details of the gauge boson mass generation and the analysis is complicated by the presence of the mass gap and mixing in the gauge sector. In Ref. [12] the general matching procedure has been formulated which relates theories with and without mass gap, thus setting the stage for the calculation of the logarithmically enhanced two-loop corrections to electroweak processes. In this Letter the analysis of Ref. [12] will be completed. We first present explicit analytical results for the two-loop logarithmic corrections to the vector form factor and four-fermion cross section in the spontaneously broken $SU(2)$ model with the gauge and Higgs bosons of the same mass M and six doublets of left-handed massless fermions inspired by the standard model. Then we proceed along the line of Ref. [12] and derive the numerical results for the dominant two-loop electroweak corrections to the cross sections of the neutral current four-fermion processes in the full $SU_L(2) \times U(1)$ theory with light fermions. We neglect the fermion mass effects which can be important for the top and bottom quark production.

The vector form factor $\mathcal{F}$ determines the fermion scattering amplitude in an external Abelian field. It plays a special role since it is the simplest quantity which includes the complete information about the universal *collinear* logarithms directly applicable to any process with an arbitrary number of fermions. Let us write the perturbative expansion for the form factor as $\mathcal{F} = \sum_n \left(\frac{\alpha}{4\pi}\right)^n f^{(n)}$, where $f^{(0)} = 1$ corresponds to the Born approximation and the coupling constant α is renormalized at the scale M according to $\overline{\rm MS}$ prescription. In the $SU(2)$ model the one-loop coefficient $f^{(1)}$ in the Sudakov limit $M/Q \to 0$ can easily be obtained from the known $U(1)$ result (see *e.g.* [12]) by multiplying with the group factor 3/4. For the two-loop logarithmic contribution of the virtual gauge and Higgs bosons we find by explicit calculation

$$f^{(2)} = \frac{9}{32}\bar{\mathcal{L}}^4 + \frac{5}{48}\bar{\mathcal{L}}^3 - \left(\frac{691}{48} - \frac{7}{8}\pi^2\right)\bar{\mathcal{L}}^2 + \left(\frac{167}{4}\right.$$

$$-\frac{11}{24}\pi^2 - \frac{61}{2}\zeta_3 + \frac{15}{4}\sqrt{3}\pi + \frac{13}{2}\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg)\bar{\mathcal{L}}$$
$$+\mathcal{O}(\bar{\mathcal{L}}^0)\,. \qquad (1)$$

where $\bar{\mathcal{L}} = \ln\left(Q^2/M^2\right)$, Q is the Euclidean momentum transfer, all power-suppressed terms are neglected, $\zeta_3 = 1.202057\ldots$ and $\mathrm{Cl}_2(\pi/3) = 1.014942\ldots$ are the values of the Riemann's zeta-function and the Clausen function, respectively. In Eq. (1) we do not include the contribution due to the virtual fermion loop computed in [11]. The Abelian contribution to Eq. (1) has been evaluated in Ref. [12]. For the calculation of the leading power behavior of the two-loop on-shell vertex diagrams with two massive propagators in the Sudakov limit we used the expansion by regions approach [13] (for the application to the Sudakov form factor see also [6]). The method is based on the separation of the contributions of the dynamical modes characteristic for the Sudakov limit [14] in dimensional regularization. Our result for the hard modes agrees with the dimensionally regularized massless result of Ref. [15]. The result for the coefficients of the quartic, cubic and quadratic logarithms in Eq. (1) is in full agreement with the predictions of the evolution equation approach [10]. In particular, they are not sensitive to details of the gauge boson mass generation. This is not true for the coefficient of the linear-logarithmic term which depends *e.g.* on the Higgs boson mass. For example, in the (hypothetical) case of a light Higgs boson with mass $M_H \ll M$ the coefficient of the linear logarithm in Eq. (1) becomes

$$\frac{333}{8} - \frac{11}{48}\pi^2 - \frac{61}{2}\zeta_3 + \frac{33}{8}\sqrt{3}\pi + \frac{21}{4}\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\,. \qquad (2)$$

Let us now consider the four-fermion process $f\bar{f} \to f'\bar{f}'$. We define the perturbative expansion for the corresponding normalized total cross section as follows: $\mathcal{R} \equiv \sigma/\sigma_{\mathrm{Born}} = \sum_n \left(\frac{\alpha}{4\pi}\right)^n r^{(n)}$, $r^{(0)} = 1$, where the coupling constant in the Born cross section is renormalized at the scale $\sqrt{s}$ while the series in α is renormalized at the scale M. The one-loop coefficient $r^{(1)}$ in the Sudakov limit can be found in Ref. [10]. The four-fermion amplitude can be decomposed into (the square of) the form factor and a *reduced* amplitude [6, 10]. The latter carries all the Lorentz and isospin indices and does not contain collinear logarithms. The logarithmic corrections to the reduced amplitude are obtained by solving a renormalization group like equation [16]. The corresponding two-loop anomalous dimensions can be extracted from the existing massless QCD calculations [17–19] (see [10, 20]), or obtained within the Wilson line approach [21]. By combining Eq. (1) with the result for the reduced amplitude and integrating the cross section over the production angle we obtain the two-loop logarithmic contribution

$$r_+^{(2)} = \frac{9}{2}\mathcal{L}^4 - \frac{449}{6}\mathcal{L}^3 + \left(\frac{4855}{18} + \frac{37}{3}\pi^2\right)\mathcal{L}^2 + \Bigg(\frac{48049}{216}$$
$$-\frac{1679}{18}\pi^2 - 122\zeta_3 + 15\sqrt{3}\pi + 26\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg)\mathcal{L}$$
$$+\mathcal{O}(\mathcal{L}^0)\,, \qquad (3)$$

and

$$r_-^{(2)} = \frac{9}{2}\mathcal{L}^4 - \frac{125}{6}\mathcal{L}^3 - \left(\frac{799}{9} - \frac{37}{3}\pi^2\right)\mathcal{L}^2 + \Bigg(\frac{38005}{216}$$
$$-\frac{383}{18}\pi^2 - 122\zeta_3 + 15\sqrt{3}\pi + 26\sqrt{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\Bigg)\mathcal{L}$$
$$+\mathcal{O}(\mathcal{L}^0)\,, \qquad (4)$$

for the initial and final state fermions of the same or opposite isospin, respectively. In Eqs. (3, 4) the virtual fermion loop contribution is included and $\mathcal{L} = \ln(s/M^2)$ is real in the physical region of positive $s = -Q^2$. The coefficients of the quartic, cubic and quadratic logarithms in Eqs. (3, 4) are already given in Ref. [10], the linear-logarithmic term is new.

The main distinction of the analysis in the standard electroweak model with the spontaneously broken $SU_L(2)\times U(1)$ gauge group from the pure $SU_L(2)$ case is the presence of the massless photon which results in infrared divergences in fully exclusive cross sections. The divergences are cancelled in cross sections which are inclusive with respect to the soft photon bremsstrahlung. Besides the electroweak logarithms the inclusive cross sections get logarithmic corrections of the form $\ln(s/\varepsilon_{cut}^2)$ and $\ln(s/m^2)$ where m is an initial or final state fermion mass and ε_{cut} is the soft photon energy cut. In the case $m,\ \varepsilon_{cut} \ll M_{W,Z}$ these logarithms are of pure QED nature and are known to factorize. Note that the two-loop pure QED corrections to the four-fermion cross section are known even beyond the logarithmic approximation (see [22] and references therein). Within the evolution equation approach [5] it has been found [10] that the electroweak and QED logarithms up to the N^{2}LL approximation can be disentangled by means of the following two-step procedure: (i) the corrections are evaluated using the fields of the unbroken symmetry phase with all the gauge bosons of the same mass $M \approx M_{Z,W}$; (ii) the QED contribution with an auxiliary photon mass M is factorized leaving the pure electroweak logarithms. This reduces the calculation of the two-loop electroweak logarithms up to the quadratic term to a problem with a single mass parameter. Then the effect of the $Z-W$ boson mass splitting can systematically be taken into account within an expansion around the equal mass approximation [12]. In general the above two-step procedure is not valid in the N^{3}LL approximation which is sensitive to fine details of the gauge boson mass generation. For the exact calculation of the coefficient of the two-loop linear-logarithmic term one has to use the true mass eigenstates of the standard model. The evaluation of the corrections in this case becomes a very complicated multiscale problem. The analysis, however, is drastically simplified in a model with a Higgs boson of zero hypercharge. In this model the mixing is absent and the above two-step procedure can be applied to disentangle all the two-loop logarithms of the $SU_L(2)$ gauge boson mass from the infrared logarithms associated with the massless hypercharge gauge boson [12]. With the result for the $SU_L(2)$ model presented in this Letter at hand we are able to complete the analysis of the two-loop logarithmic corrections in the simplified model. In the standard model the mixing of the gauge bosons results in a linear-logarithmic contribution which is not accounted for in this approximation. It is, however, suppressed by a small factor $\sin^2\theta_W \approx 0.2$, with θ_W being the Weinberg angle. Therefore, the approximation gives an estimate of the coefficient in front of the linear electroweak logarithm with 20% accuracy.

Let $\mathcal{R}_{ff'} \equiv \sigma/\sigma_{em}$ be the normalized total cross section of the $f\bar{f}$ annihilation into a $f'\bar{f}'$ pair. Here σ_{em} stands for the cross section which incorporates the pure QED radiative corrections and is free of the electroweak logarithms. It is convenient to normalize σ_{em} so that the virtual QED corrections vanish at $m = 0,\ s = \lambda^2$, where λ is the auxiliary photon mass, and to use the electroweak coupling constants renormalized at the scale $\sqrt{s}$ in the Born approximation [10]. In the standard electroweak model the perturbative expansion involves two parameters: the $SU_L(2)$ coupling constant α and the $U(1)$ hypercharge coupling constant α'. For convenience we eliminate the latter by means of the relation $\alpha' = \tan^2\theta_W\,\alpha$ and define the one-parameter series for the cross section $\mathcal{R}_{ff'} = \sum_n \left(\frac{\alpha}{4\pi}\right)^n r^{(n)}_{ff'}$, $r^{(0)}_{ff'} = 1$, in terms of the $\overline{\rm MS}$ $SU_L(2)$ coupling renormalized at the scale of the gauge boson mass. The complete one-loop result for the cross section is known exactly (see *e.g.* Ref. [23] and references therein). For the two-loop logarithmic corrections to the phenomenologically interesting processes we obtain the following numerical approximation:

$$
\begin{aligned}
r^{(2)}_{lQ} &= 1.93\,\mathcal{L}^4 - 11.28\,\mathcal{L}^3 + 33.79\,\mathcal{L}^2 - 60.87\,\mathcal{L}\,,\\
r^{(2)}_{lq} &= 2.79\,\mathcal{L}^4 - 51.98\,\mathcal{L}^3 + 321.20\,\mathcal{L}^2 - 757.35\,\mathcal{L}\,,\\
r^{(2)}_{Qq} &= 3.53\,\mathcal{L}^4 - 20.39\,\mathcal{L}^3 + 65.20\,\mathcal{L}^2 - 91.92\,\mathcal{L}\,,\\
r^{(2)}_{ll'} &= 1.42\,\mathcal{L}^4 - 20.33\,\mathcal{L}^3 + 112.57\,\mathcal{L}^2 - 312.90\,\mathcal{L}\,,\\
r^{(2)}_{QQ'} &= 2.67\,\mathcal{L}^4 - 46.64\,\mathcal{L}^3 + 278.94\,\mathcal{L}^2 - 666.05\,\mathcal{L}\,,\\
r^{(2)}_{qq'} &= 4.20\,\mathcal{L}^4 - 71.87\,\mathcal{L}^3 + 423.61\,\mathcal{L}^2 - 919.35\,\mathcal{L}\,,
\end{aligned}
\tag{5}
$$

where $\mathcal{L} = \ln\left(s/M_W^2\right)$, l stands for a charged lepton, Q and q stand for the u, c, t and d, s, b quarks, respectively. Note that the result is symmetric under exchange of the initial and final state fermions and can easily be generalized to $f\bar{f} \to f\bar{f}$ processes by including the t channel contribution which goes beyond the scope of this Letter. In Eq. (5) we use the value $\sin^2\theta_W = 0.231$ corresponding to the $\overline{\rm MS}$ coupling constants renormalized at the scale M_Z. The coefficients of the cubic and quadratic logarithms in the two-loop corrections to the cross sections of e^+e^- annihilation have been computed in Refs. [6, 10] neglecting the $W - Z$ boson mass difference [25]. In Eq. (5) we included the leading correction in the mass difference $1 - M_W/M_Z$ to these coefficients. The coefficient of the linear logarithm is computed in the approximation described above.

For the case of e^+e^- annihilation the size of the corrections is shown in Figs. 1, 2. In Fig. 1 the values of different two-loop logarithmic contributions to $\mathcal{R}_{lq}$ are plotted separately as functions of s for $\alpha = 3.38 \cdot 10^{-2}$. The two-loop logarithmic terms have a sign-alternating structure resulting in significant cancellations. Although the individual logarithmic contributions can be as large as 10%, their sum does not exceed 1% at energies be-

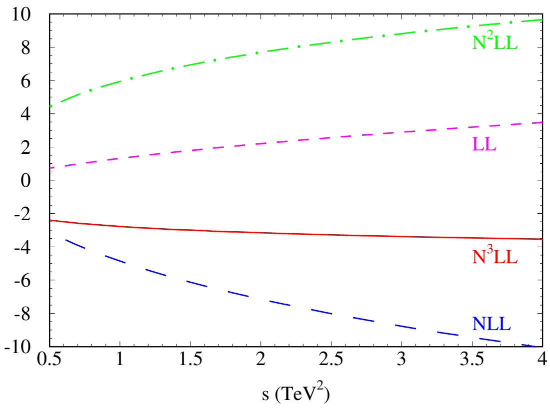

FIG. 1: The LL (short-dashed line), NLL (long-dashed line), NNLL (dot-dashed line) and N^{3}LL (solid line) two-loop electroweak corrections to $\mathcal{R}_{lq}$ in percent.

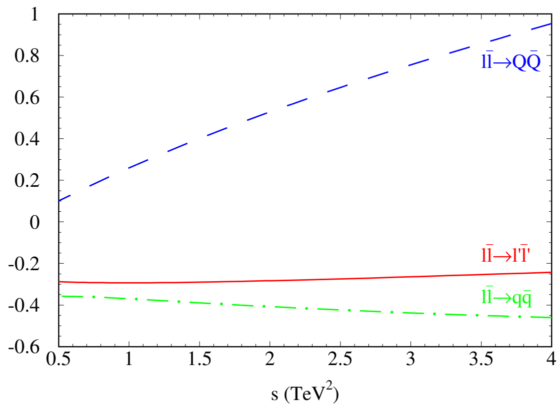

FIG. 2: The total electroweak logarithmic two-loop corrections to $\mathcal{R}_{lQ}$ (dashed line), $\mathcal{R}_{lq}$ (dot-dashed line) and $\mathcal{R}_{ll'}$ (solid line) in percent.

low 2 TeV for all the cross sections (see Fig. 2). In the region of a few TeV the corrections do not reach the double-logarithmic asymptotics. The quartic, cubic and quadratic logarithms are comparable in magnitude. The linear-logarithmic term is a few times smaller than the quadratic logarithm which is in agreement with the prediction of Ref. [10] for the structure of the two-loop corrections and justifies neglecting the nonlogarithmic contribution. Still, the linear-logarithmic contribution amounts to a few percent and must be included to reduce the theoretical uncertainty below 1%.

Let us discuss the accuracy of our result. On the basis of the explicit evaluation of the light fermion/scalar [11] and the Abelian contribution [12] we estimate the uncalculated two-loop nonlogarithmic term to few permill. For $\sqrt{s} > 500$ GeV the power-suppressed terms do not exceed a permill in magnitude as well [11]. The leading effect of the $W - Z$ mass splitting results in a variation of the coefficients of the two-loop cubic and quadratic logarithms of at most 5%. Thus the expansion in the $W - Z$ mass difference converges well for these coefficients and the leading correction term taken into account in our evaluation is sufficient for a permill accuracy of the cross sections. Neglecting the gauge boson mixing effects, which are suppressed by a factor of $\sin^2\theta_W$, induces an error of 20% in the coefficient of the two-loop single logarithm. Neglecting the difference between the Higgs and gauge boson masses leads to a variation of the linear logarithmic coefficient of at most 5% since the scalar boson contribution is relatively small. The same is true for the uncertainty due to the top quark mass effect on the $t\bar{t}$ virtual pair contribution. Hence for the production of light fermions our formulae are supposed to approximate the exact coefficients of the two-loop linear logarithms with approximately 20% accuracy leading to a few permill uncertainty in the cross sections. By adding up the errors from different sources in quadrature we find the total uncertainty of the cross section to be from a few permill up to one percent, depending on the process. This result should be sufficient for all practical applications to collider physics. The only essential deviation of the exact two-loop logarithmic contributions from our result is relevant for the production of the third generation quarks and is due to the large top quark Yukawa coupling. The corresponding corrections are known to NLL approximation and can numerically be as important as the generic non-Yukawa ones [7].

To conclude, we have derived the analytical result for the two-loop logarithmic corrections to the vector form factor and four-fermion cross section in the spontaneously broken $SU_L(2)$ model. We have also obtained the dominant two-loop electroweak corrections to neutral current four-fermion processes, which are crucial for the high-precision physics at the LHC and the next generation of linear colliders.

We thank the authors of Refs. [18, 20, 21] for very usefull communications. The work of J.H.K and A.A.P. was supported in part by BMBF Grant No. 05HT4VKA/3 and SFB/TR9. The work of V.A.S. was supported in part by the Russian Foundation for Basic Research through project 05-02-17645 and DFG Grant No. Ha 202/110-1.

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[24] N^{m}LL stands for the corrections of the form $\alpha^n \ln^{2n-m}(s)$ for an arbitrary n.

[25] Throughout Sect 4. of Ref. [10] the terms with the factor $(aN_g + b)t_W^2$, where a and b stand for some constants, should be multiplied by an extra t_W^2, and the terms with the factor $N_g s_W^2$ should be multiplied by s_W^2. This results in a small change of the numerical estimates.