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Four-Loop Singlet Contribution to the ρ Parameter

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Abstract

We compute the four-loop contribution to the electroweak ρ parameter induced by the singlet diagrams of the Z-boson self-energy. The numerical impact on the weak mixing angle and the W-boson mass is small.

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1 Introduction

The electroweak ρ parameter as introduced by Veltman [1] measures the relative strength of the charged and neutral current. It can be written as

$$\rho \;=\; 1 + \delta\rho\,, \tag{1}$$

with

$$\delta\rho \;=\; \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}\,. \tag{2}$$

$\Pi_{ZZ}(0)$ and $\Pi_{WW}(0)$ are the transverse parts of the W- and Z-boson self-energies evaluated for vanishing external momentum. The ρ parameter enters a variety of quantities which are determined from experiment with an enormous precision. In particular, it enters the relation between the W-boson mass, M_W, the fine structure constant, α, the Fermi constant, G_F, and the Z-boson mass, M_Z, which is given by [2]

$$M_W^2 \;=\; \frac{M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}\,M_Z^2 G_F\,(1-\Delta r)}}\right)\,. \tag{3}$$

The quantity Δr is conveniently parameterized in the form

$$\Delta r \;=\; \Delta\alpha - \frac{c_W^2}{s_W^2}\delta\rho + \Delta r^{\rm rem}\,, \tag{4}$$

with $c_W = M_W/M_Z$ and $s_W^2 = 1 - c_W^2$. $\Delta\alpha$ contains contributions from light fermions giving rise to a correction of about 6%. The leading corrections proportional to $G_F M_t^2$ are incorporated in $\delta\rho$ and amount at one-loop order to roughly -3% whereas the remaining part is small.

Eqs. (3) and (4) can be used to predict M_W where the formula

$$\delta M_W \;=\; \frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\,\delta\rho\,, \tag{5}$$

immediately accounts for the dominant shift in M_W due to the corrections to the ρ parameter. We can also look at the change of the effective leptonic weak mixing angle, $\sin^2\theta_{\rm eff}^{\rm lept}$, defined through the coupling of the Z-boson to leptons. The leading universal corrections originating from $\delta\rho$ can in analogy to Eq. (5) be written as

$$\delta\sin^2\theta_{\rm eff}^{\rm lept} \;=\; -\frac{c_W^2 s_W^2}{c_W^2 - s_W^2}\,\delta\rho\,. \tag{6}$$

Currently the uncertainties for M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$ are given by $\delta M_W = 34$ MeV and $\delta\sin^2\theta_{\rm eff}^{\rm lept} = 1.7\times10^{-4}$, respectively. However, a future linear collider running at the Z-boson pole, the so-called Giga-Z option, and around the W-pair threshold might reduce the uncertainties to $\delta M_W = 6$ MeV and $\delta\sin^2\theta_{\rm eff}^{\rm lept} = 1.3\times10^{-5}$ [3].

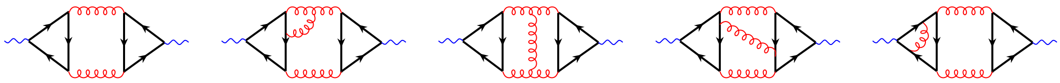

Figure 1: Sample three- and four-loop singlet diagrams contributing to the ρ parameter. In the fermion loops either top- or bottom-quarks are present.

The one-loop corrections to ρ have been computed in 1977 [1] and also the two-loop QCD corrections are known since almost 20 years [4, 5, 6]. Roughly 10 years ago the order $G_F M_t^2 \alpha_s^2$ QCD corrections [7, 8] constituted one of the first applications of the three-loop massive vacuum integrals. At three-loop order for the first time a new kind of Feynman graphs has to be considered, the so-called singlet diagrams as shown in Fig. 1 which only contribute to the Z-boson self-energy. They are characterized by the fact that in contrast to the non-singlet contribution the external Z-bosons couple to different fermion lines. We want to note that the singlet contribution forms a finite and gauge independent subset. At three-loop order it completely dominates the numerical corrections if the $\overline{\mathrm{MS}}$ definition is adopted for the top-quark mass. In the case of the pole mass definition the singlet part still amounts to about 30% of the total three-loop contribution. We want to mention that also two-loop [9, 10] and three-loop mixed electroweak/QCD [11] and even three-loop pure electroweak corrections [11] have been evaluated.

In this letter we consider the four-loop contribution to the ρ parameter originating from the singlet diagrams. In Fig. 1 some sample diagrams are shown. This constitutes one of the first applications of the four-loop vacuum master integrals evaluated recently in Ref. [12].

2 Technicalities

Since the boson self-energies have to be evaluated for zero external momentum, only the axial-vector part of the Z-boson correlator gives a non-zero contribution. Whereas for the non-singlet contribution the naive anti-commuting definition of γ_5 can be adopted, special care has to be taken in the singlet case. Actually, the definition of 't Hooft and Veltman [13] has to be adopted and additional counterterms have to be introduced in order to ensure the validity of the Ward identities. In the practical calculation we follow Ref. [14] and perform the following replacement in the axial-vector current

$$\gamma^\mu \gamma^5 \quad = \quad \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \, . \tag{7}$$

We pull out the ε-tensor from the actual integral and consider instead the completely antisymmetrized product of the three γ-matrices which can be written as

$$\gamma^{[\nu}\gamma^\rho\gamma^{\sigma]} \quad = \quad \frac{1}{2} \left(\gamma^\nu \gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho \gamma^\nu \right) \, . \tag{8}$$

As a consequence we have to deal with an object with six indices. Thus, for zero external momentum we obtain

$$\begin{aligned}
\Pi_{\rm ZZ} &= \frac{g_{\mu\mu'}}{4}\Pi^{\mu\mu'}_{\rm ZZ} \\
&= \frac{g_{\mu\mu'}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\mu'\nu'\rho'\sigma'}}{144}\Pi_{[\nu\rho\sigma][\nu'\rho'\sigma']} \\
&= -\frac{1}{24}\Pi^{[\nu\rho\sigma]}_{[\nu\rho\sigma]}\,.
\end{aligned} \tag{9}$$

In the practical calculation we consider the object $\Pi^{[\nu\rho\sigma]}_{[\nu\rho\sigma]}$ for which we also perform the renormalization as described in the following. Thus, in Eq. (9) the limit $D \to 4$ has been considered where $D = 4 - 2\epsilon$ is the space-time dimension.

The additional finite counterterm is only needed to one-loop order, since the singlet diagrams appear the first time at three-loop level. For each axial-vector vertex a factor [15, 14]

$$Z_5^s = 1 - C_F\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\,, \tag{10}$$

with $C_F = (N_c^2-1)/(2N_c)$ has to be considered. Furthermore, we have to consider the one-loop counterterms for the strong coupling constant and the top-quark mass defined by

$$\alpha_s^0 = Z_{\alpha_s}\alpha_s\,, \qquad m_t^0 = Z_m m_t\,, \tag{11}$$

where $m_t \equiv m_t(\mu)$ is renormalized in the $\overline{\rm MS}$ scheme. The renormalization constants are given by

$$\begin{aligned}
Z_{\alpha_s} &= 1 + \frac{1}{\epsilon}\left(-\frac{11}{12}C_A + \frac{1}{3}Tn_f\right)\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\,, \\
Z_m &= 1 - \frac{3}{4\epsilon}C_F\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\,,
\end{aligned} \tag{12}$$

with $C_A = N_c$ and $T = 1/2$. $n_f = 6$ is the number of active flavours. The transition to the pole mass is achieved via

$$m_t(\mu) = \left[1 + C_F\left(-1 - \frac{3}{4}\ln\frac{\mu^2}{M_t^2}\right)\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right]M_t\,. \tag{13}$$

We generate the Feynman diagrams with `QGRAF` [16] and adopt with the help of the packages `q2e` and `exp` [17, 18] the topologies and notation to the program performing the reduction of the four-loop vacuum diagrams [19]. As an output we obtain the corrections to the ρ parameter as a linear combination of several master integrals. All of them have been computed in Ref. [12].

It is interesting to note that some of the master integrals are multiplied by spurious poles of order $1/\epsilon^2$. As a consequence, for these the $\mathcal{O}(\epsilon)$ and even the $\mathcal{O}(\epsilon^2)$ contribution is needed. In the case of the master integral BB4 (which is the four-loop sunset vacuum bubble with one massless and four massive lines, see Eqs. (4.8) and (6.36) of Ref. [12]) it happens that the coefficient of order ϵ, which originally has only been evaluated in numerical form [12], enters the pole part of $\delta\rho$. Thus an analytical expression can be deduced (in Eq. (14) below denoted by $\mathtt{BB4}^{(1)}$) which perfectly agrees with the known numerical result. Furthermore, we have obtained an analytical expression for the coefficient of order ϵ^2, by combining the numerically known value with the basis of transcendentals known from an independent investigation [20]. These two coefficients read

$$
\begin{aligned}
\mathtt{BB4} &= J^4 \sum_{n\geq 0} \epsilon^n\, \mathtt{BB4}^{(n-4)}\,, \\
\mathtt{BB4}^{(1)} &= -\frac{1976975}{7776} + \frac{1792}{9}\zeta(3)\,, \\
\mathtt{BB4}^{(2)} &= -\frac{72443143}{46656} + \frac{47488}{27}\zeta(3) - \frac{8704}{3}\zeta(4) \\
&\quad + \frac{1024}{9}\ln^4 2 - \frac{2048}{3}\zeta(2)\ln^2 2 + \frac{8192}{3}a_4 \\
&= -1678.8869291077729634030303102679175091 51\ldots\,,
\end{aligned} \tag{14}
$$

where J is the one-loop tadpole, $\zeta(n)$ is Riemann's zeta function and

$$
a_4 \;=\; \mathrm{Li}_4(1/2) \;\approx\; 0.51747906167389938633\,. \tag{15}
$$

Let us mention that we performed the calculation using an arbitrary gauge parameter of the QCD gluon propagator, ξ. As expected the final result is independent of ξ even before inserting the values for the master integrals. This constitutes a nice check of our result.

3 Results and discussion

Let us in the following present our analytical result and discuss its numerical implications. For completeness we also repeat the QCD corrections up to three-loop order. For the $\overline{\mathrm{MS}}$ definition of the top-quark mass we obtain

$$
\begin{aligned}
\delta\rho^{\overline{\mathrm{MS}}} &= 3x_t\Bigg\{1 + \frac{\alpha_s}{4\pi}\left[8 - \frac{16}{3}\zeta(2) + 8\ln\frac{\mu^2}{m_t^2}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{26459}{81} - \frac{25064}{81}\zeta(2) - \frac{3560}{27}\zeta(3)\right. \\
&\quad + \frac{1144}{9}\zeta(4) - \frac{16}{9}B_4 - \frac{8}{9}D_3 + 882S_2 + n_f\left(-\frac{50}{3} + \frac{112}{9}\zeta(2) - \frac{64}{9}\zeta(3)\right) \\
&\quad - 56\zeta(3) + \left(\frac{668}{3} - \frac{304}{3}\zeta(2) + n_f\left(-\frac{88}{9} + \frac{32}{9}\zeta(2)\right)\right)\ln\frac{\mu^2}{m_t^2} \\
&\quad \left. + \left(76 - \frac{8}{3}n_f\right)\ln^2\frac{\mu^2}{m_t^2}\right] + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{256}{9} - 4528\zeta(3) + \frac{20816}{3}\zeta(4)\right. \\
&\quad \left. - \frac{2624}{9}\ln^4 2 + \frac{5248}{3}\zeta(2)\ln^2 2 - \frac{20992}{3}a_4 - 1232\zeta(3)\ln\frac{\mu^2}{m_t^2}\right] + \ldots\Bigg\}\,,
\end{aligned} \tag{16}
$$

where the "$-56\zeta(3)$" in the third line stems from the three-loop singlet diagram. In order α_s^3 only the singlet contribution is presented. Furthermore, we have

$$\begin{aligned}
x_t &= \frac{G_F m_t^2}{8\pi^2\sqrt{2}}\,, \\
S_2 &= \frac{4}{9\sqrt{3}}\mathrm{Im}(\mathrm{Li}_2(e^{i\pi/3})) \;\approx\; 0.26043413763216209896\,, \\
B_4 &= 16a_4 + \frac{2}{3}\ln^4 2 - 4\zeta(2)\ln^2 2 - \frac{13}{2}\zeta(4) \;\approx\; -1.7628000870737708641\,, \\
D_3 &= 6\zeta(3) - \frac{15}{4}\zeta(4) - 6[\mathrm{Im}(\mathrm{Li}_2(e^{i\pi/3}))]^2 \;\approx\; -3.0270094939876520198\,.
\end{aligned}$$

In Appendix A, we present the three- and four-loop result for the singlet contribution corresponding to Eq. (16) retaining, however, the colour factors C_F, C_A and T. With the help of Eq. (13) one obtains the singlet result in the on-shell scheme. Together with the non-singlet terms one gets

$$\begin{aligned}
\delta\rho^{\rm OS} &= 3X_t\Bigg\{1 + \frac{\alpha_s}{4\pi}\left[-\frac{8}{3} - \frac{16}{3}\zeta(2)\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\Bigg[\frac{314}{81} - \frac{26504}{81}\zeta(2) - \frac{3416}{27}\zeta(3) \\
&\quad - \frac{64}{3}\zeta(2)\ln 2 + \frac{1144}{9}\zeta(4) - \frac{16}{9}B_4 - \frac{8}{9}D_3 + 882S_2 + n_f\left(-\frac{8}{9} - \frac{208}{9}\zeta(2)\right. \\
&\quad \left.- \frac{64}{9}\zeta(3)\right) - 56\zeta(3) + \left(-\frac{88}{3} - \frac{176}{3}\zeta(2) + n_f\left(\frac{16}{9} + \frac{32}{9}\zeta(2)\right)\right)\ln\frac{\mu^2}{M_t^2}\Bigg] \\
&\quad + \left(\frac{\alpha_s}{4\pi}\right)^3\Bigg[\frac{256}{9} - \frac{11792}{3}\zeta(3) + \frac{20816}{3}\zeta(4) \\
&\quad - \frac{2624}{9}\ln^4 2 + \frac{5248}{3}\zeta(2)\ln^2 2 - \frac{20992}{3}a_4 - 784\zeta(3)\ln\frac{\mu^2}{M_t^2}\Bigg] + \ldots\Bigg\}\,, \qquad (17)
\end{aligned}$$

with $X_t = G_F M_t^2/(8\pi^2\sqrt{2})$.

Inserting the numerical values for the constants in Eqs. (16) and (17) and adopting $\mu = m_t$ and $\mu = M_t$, respectively, the numerical corrections read

$$\begin{aligned}
\delta\rho^{\overline{\rm MS}} &= 3x_t\left[1 - 0.19325\frac{\alpha_s}{\pi} + (-4.2072 + 0.23764)\left(\frac{\alpha_s}{\pi}\right)^2 - 3.2866\left(\frac{\alpha_s}{\pi}\right)^3\right], \\
\delta\rho^{\rm OS} &= 3X_t\left[1 - 2.8599\frac{\alpha_s}{\pi} + (-4.2072 - 10.387)\left(\frac{\alpha_s}{\pi}\right)^2 + 7.9326\left(\frac{\alpha_s}{\pi}\right)^3\right], \qquad (18)
\end{aligned}$$

where the three-loop contribution is split into the singlet (first number in round brackets) and the non-singlet piece. If we furthermore adopt $\alpha_s(m_t) = 0.108$ and $\alpha_s(M_t) = 0.107$, the expression for $\delta\rho$ looks like

$$\begin{aligned}
\delta\rho^{\overline{\rm MS}} &= 3x_t\Big(1 - 0.00664 - 0.00469 - 0.00013\Big)\,, \\
\delta\rho^{\rm OS} &= 3X_t\Big(1 - 0.09741 - 0.01693 + 0.00031\Big)\,, \qquad (19)
\end{aligned}$$

where the n$^{\text{th}}$ term inside the round brackets corresponds to the contribution of order $G_F M_t^2 \alpha_s^{(n-1)}$. One observes that the new four-loop singlet contribution is numerically small and amounts to about 3% of the three-loop result in the $\overline{\text{MS}}$ scheme and to less than 2% for on-shell top-quark masses. Note that the correction is positive in the on-shell and negative in the $\overline{\text{MS}}$ scheme.

It is interesting to mention that at three-loop order the singlet contribution completely dominates for $\overline{\text{MS}}$ top-quark masses and amounts to about 30% in the on-shell scheme. Thus, in case the same pattern also holds at four-loop order, the complete QCD corrections would be well under control. However, the numerical values in Eq. (18) suggest that for some reason the four-loop singlet contribution seems to be accidentally small.

Let us in the remaining part of this section briefly compare the numerical effect of the new terms with known corrections to $\delta\rho$. In the on-shell scheme the three-loop QCD corrections of order $\alpha_s^2 X_t$ lead to a shift of about -10 MeV in the W-boson mass and to $+5 \times 10^{-5}$ in the effective weak mixing angle. For Higgs-boson masses between 200 GeV and 300 GeV the three-loop corrections of order $\alpha_s X_t^2$ [11] have the opposite sign and with roughly half the magnitude they are still relevant for the precision to be reached at the Giga-Z option of a future e^+e^- linear collider. However, the pure electroweak corrections of order X_t^3 are very small and give rise to corrections well below 1 MeV for the shift in the W-boson mass. The same is true for the four-loop QCD singlet contributions considered in this letter.

In conclusion, we computed the four-loop singlet contribution to the ρ parameter which constitutes one of the first applications of the four-loop massive vacuum integrals to a physical quantity. The numerical size of the corrections turn out to be surprisingly small and lead to a shift in the W-boson mass below 1 MeV and to the effective weak mixing angle below 10^{-5}. This illustrates the good convergence properties of the perturbation theory and confirms the stable predictions based on the three-loop corrections. However, for a definite conclusion also the non-singlet contribution has to be evaluated.

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A Singlet contribution to the ρ parameter

In this Appendix we present the three- and four-loop singlet result expressed in terms of $C_A = N_c$, $C_F = (N_c^2 - 1)/(2N_c)$ and $T = 1/2$. Furthermore, we keep the label n_l which counts the number of massless quarks. The three-loop term can also be found in Ref. [21].

$$\delta\rho_{\text{sing}}^{\overline{\text{MS}}} = 3x_t \left(\frac{\alpha_s}{4\pi}\right)^2 C_F T \Bigg\{ -84\zeta(3) + \frac{\alpha_s}{4\pi}\Bigg[C_F\Bigg(-336\zeta(3) + 2400\zeta(4) - 128\ln^4 2$$
$$+ 768\zeta(2)\ln^2 2 - 3072 a_4 - 504\zeta(3)\ln\frac{\mu^2}{m_t^2}\Bigg) + C_A\Bigg(-\frac{7064}{3}\zeta(3)$$

$$
\begin{aligned}
&+3056\zeta(4)-\frac{320}{3}\ln^4 2+640\zeta(2)\ln^2 2-2560a_4-616\zeta(3)\ln\frac{\mu^2}{m_t^2}\Bigg)\\
&+n_lT\left(\frac{1120}{3}\zeta(3)-784\zeta(4)+\frac{64}{3}\ln^4 2-128\zeta(2)\ln^2 2+512a_4\right.\\
&\left.+224\zeta(3)\ln\frac{\mu^2}{m_t^2}\right)+T\left(\frac{256}{3}-\frac{1280}{3}\zeta(3)+224\zeta(3)\ln\frac{\mu^2}{m_t^2}\right)\Bigg]+\mathcal{O}(\alpha_s^2)\Bigg\}\,.
\end{aligned}
\tag{20}
$$

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