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Vacuum polarization in pQCD: first complete $O(\alpha_s^4)$ result*

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We report the results of the first complete QCD five-loop calculation, viz. analytical contribution of order $m_q^2\alpha_s^4$, to the (absorptive part of) the vacuum polarization function. The contribution directly leads to the knowledge of the the full $\mathcal{O}(\alpha_s^4 m_q^2/s)$ correction to the total cross-section of e^+e^- annihilation into hadrons. We compare our results with predictions following from various optimization schemes.

1. Introduction

Important physical observables like the famous ratio, $R(s) = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}$, and the decay rate of the Z boson are related to the vector and axial-vector current correlators (see, e.g. [1]). Perturbative QCD (pQCD) provides reliable predictions for $R(s)$ in the continuum, i.e. sufficiently above the respective quark threshold and the respective resonance region.

The current status of the pQCD predictions for $R(s)$ is as follows:

$\mathcal{O}(\alpha_s^2)$ order. The full knowledge of $R(s)$ with massive quarks at $\mathcal{O}(\alpha_s^2)$ is available[2,3]. The polarization function in order α_s^2 for massive quarks is therefore under full control.

$\mathcal{O}(\alpha_s^3)$ order. The massless result is known since long [4–6]. For high energy quark mass effects can be taken into account via the expansion in quark masses. This has been done for the quadratic and quartic terms in [7–9].

$\mathcal{O}(\alpha_s^4)$ order. At two previous workshops we have discussed the possibility to compute at least some contributions to $R(s)$ at this order [10,11] (which amounts to the knowledge of the polarization function at the record-breaking *five-loop* level). The first genuine QCD calculations of the polarization function at five loops have been reported in [12–14,11]; they have taken into account the gauge invariant subset of diagrams proportional to $a_s^4 n_f^3$ and $a_s^4 n_f^2$.

This talk presents complete results for the (diagonal) vector correlator at order $m_q^2\alpha_s^4$. We also compare our results with some of the available predictions following from various "optimization" schemes.

2. Calculation and results

Consider the two-point correlator of vector quark currents and the corresponding vacuum polarization function ($j_\mu^v = \overline{Q}\gamma_\mu Q$; Q is a quark field with mass m, all other $n_f - 1$ quarks are assumed to be massless)

$$\begin{aligned}\Pi_{\mu\nu}(q) &= i\int \mathrm{d}x e^{iqx}\langle 0|T[\ j_\mu^v(x) j_\nu^v(0)\ ]|0\rangle \\ &= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2). \qquad (1)\end{aligned}$$

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The physical observable $R(s)$ is related to $\Pi(q^2)$ by

$$R(s) = 12\pi \Im \Pi(q^2 + i\epsilon). \tag{2}$$

It is convenient to decompose $R(s)$ into the massless contribution and those quadratic in the quark mass as follows ($a_s = \alpha_s(\mu^2)/\pi$):

$$\begin{aligned} R(s) &= 3\left\{ r_0^V + \frac{m^2}{s} r_2^V \right\} + \ldots \\ &= 3\left\{ \sum_{i\geq 0} a_s^i \left(r_0^{V,i} + \frac{m^2}{s} r_2^{V,i} \right) \right\} + \ldots . \end{aligned}$$

For the calculation of $r_0^{V,4}$ one should compute the divergent parts of five-loop diagrams and finite parts of the four-loop ones [15]. Fortunately, as was first shown in Ref. [7], the quadratic mass correction $r_2^{V,4}$ can be obtained with the help of renormalization group methods *exclusively* from the four-loop function Π_2 defined as

$$\Pi = \Pi_0 + \frac{m^2}{-q^2}\Pi_2, \quad \Pi_2 = \frac{3}{16\pi^2} \sum_{i\geq 0} a_s^i k_2^{V,i}.$$

This is because the renormalization group equation for Π_2 can be presented in the form ($L = \ln \frac{\mu^2}{-q^2}$):

$$\frac{\partial}{\partial L}\Pi_2 = \left(-2\gamma_m(a_s) - \beta(a_s) a_s \frac{\partial}{\partial a_s} \right) \Pi_2 \tag{3}$$

The rhs of eq. (3) depends on the function Π_2 multiplied by *at least* one factor of α_s coming either from γ_m or from β, both of which are known to α_s^4. Thus, one needs only Π_2 to α_s^3 in order to find the derivative on the lhs of (3) to α_s^4. Once $\frac{\partial}{\partial L}\Pi_2$ is known, one can find the function Π_2 up to a constant contribution which has no effect on $R(s)$.

Using the new technique described in [16–20]. and the parallel version of FORM [21,22], we have computed the the function Π_2 at four loops.

The coefficients $k_2^{V,i}$ for $i = 1, 2$ are known from [23] while our result for $k_2^{V,3}$ reads

$$\begin{aligned} k_2^{V,3} = & \\ &+ n_f^2 \left[-\frac{5161}{1458} - \frac{8}{27}\zeta_3 \right] \\ &+ n_f \left[\frac{62893}{162} - \frac{4150}{243}\zeta_3 + \frac{424}{27}\zeta_3^2 + \frac{20}{3}\zeta_4 \right. \\ &\qquad \left. - \frac{28880}{243}\zeta_5 \right] + k_{2,0}^{V,3}, \end{aligned} \tag{4}$$

The last term

$$\begin{aligned} k_{2,0}^{V,3} &= -\frac{10499303}{1944} + \frac{66820}{81}\zeta_3 - \frac{7225}{27}\zeta_3^2 \\ &+ \frac{281390}{81}\zeta_5 - \frac{1027019}{648}\zeta_7 \end{aligned} \tag{5}$$

and allows the first evaluation of a five loop contribution to $R(s)$. Numerically,

$$\begin{aligned} \Pi_2 &= -8 - 21.333\, a_s + a_s^2\, (10.56\, n_f - 224.80) \\ &+ a_s^3\, (-3.896\, n_f^2 + 274.37\, n_f - 2791.81) \end{aligned} \tag{6}$$

Now the use of eqs. (2,3) yields the following results for the corresponding contribution to $R(s)$ ($a_s = a_s(s), \mu^2 = s$):

$$\begin{aligned} r_2^V &= 12\; a_s + a_s^2 \left\{ -\frac{13}{3}\, n_f + \frac{253}{2} \right\} \\ &+ a_s^3 \left\{ \left[\frac{125}{54} - \frac{1}{9}\pi^2 \right] n_f^2 + \left[-\frac{4846}{27} + \frac{17}{3}\pi^2 \right. \right. \\ &\qquad \left. - \frac{466}{27}\zeta_3 + \frac{1045}{27}\zeta_5 \right] n_f + 2442 \\ &\qquad \left. - \frac{285}{4}\pi^2 + \frac{490}{3}\zeta_3 - \frac{5225}{6}\zeta_5 \right\} \\ &+ a_s^4 \left\{ \left[-\frac{2705}{1944} + \frac{13}{108}\pi^2 \right] n_f^3 + \left[\frac{91943}{486} \right. \right. \\ &- \left. \frac{1121}{108}\pi^2 + \frac{53}{9}\zeta_3^2 + \frac{13}{324}\zeta_3 - \frac{3610}{81}\zeta_5 \right] n_f^2 \\ &+ \left. r_{2,1}^{V,4}\, n_f + r_{2,0}^{V,4} \right\}, \end{aligned} \tag{7}$$

$$\begin{aligned} r_{2,1}^{V,4} = & - \frac{5129327}{864} + \frac{19301}{72}\pi^2 - \frac{\pi^4}{9} \\ &+ \frac{126593}{648}\zeta_3 - \frac{15917}{72}\zeta_3^2 \\ &+ \frac{409585}{162}\zeta_5 - \frac{1027019}{1728}\zeta_7 \end{aligned} \tag{8}$$

$$\begin{aligned} r_{2,0}^{V,4} = & \frac{128033845}{2592} - \frac{68003}{32}\pi^2 - \frac{203227}{36}\zeta_3 \\ &+ \frac{296225}{144}\zeta_3^2 - \frac{2129485}{72}\zeta_5 \\ &+ \frac{42107779}{3456}\zeta_7 \end{aligned} \tag{9}$$

Finally, eqs. (7,9) in numerical form read

$$
\begin{aligned}
r_2^V &= 12\,a_s + a_s^2\,(-4.3333\,n_f + 126.5) \\
&+ \quad a_s^3\left(1.2182\,n_f^2 - 104.167\,n_f + 1032.14\right) \\
&+ \quad a_s^4\left(-0.20345\,n_f^3 + 49.0839\,n_f^2\right. \\
&\qquad - \quad \left. 1364.04\,n_f \;+ 6225.84\right). \qquad (10)
\end{aligned}
$$

3. Discussion

It is interesting to compare eqs. (5-10) with predictions made in [14] and based on the optimization methods FAC (Fastest Apparent Convergence) and PMS (Principle of Minimal Sensitivity) [24–28]. As in many cases FAC and PMS predictions are quite close [29] we will also use so-called NNA (Naive NonAbelianization) approach of [30].

We start from purely *Euclidian* quantity $k_2^{V,3}$. The predictions read (should be compared to the second line of eq. (6))

$$
\begin{aligned}
k_2^{V,3}(\mathrm{FAC}) &= -4.2\,n_f^2 + 277\,n_f - 2886 && (11)\\
k_2^{V,3}(\mathrm{PMS}) &= -4.2\,n_f^2 + 282\,n_f - 2916, && (12)\\
k_2^{V,3}(\mathrm{NNA}) &= -3.896\,n_f^2 + 129\,n_f - 1060. && (13)
\end{aligned}
$$

We observe a remarkably good agreement between FAC/PMS predictions and our result. In fact, the relative accuracy for all terms in eqs. (11,12) is better than 10%. Even more, for the calculationally most demanding contribution, namely, $k_{2,0}^{V,3}$, the relative error decreases up to a level of astonishingly small 5%! However, the observation can not be automatically translated to the essentially *Minkowskian* case of r_2^V. Indeed, for, say, $n_f = 4$ we get

$$
\begin{aligned}
r_2^V/12\ (\mathrm{exact}) &= a_s + 9.09722a_s^2 \\
&+ \quad 52.913a_s^3 + 128.499\,a_s^4, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
r_2^V/12\ (\mathrm{FAC}) &= a_s + 9.09722a_s^2 \\
&+ \quad 52.913a_s^3 + 177\,a_s^4, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
r_2^V/12\ (\mathrm{PMS}) &= a_s + 9.09722a_s^2 \\
&+ \quad 52.913\,a_s^3 + 193\,a_s^4, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
r_2^V/12\ (\mathrm{NNA}) &= a_s + 9.09722a_s^2 \\
&+ \quad 52.913\,a_s^3 - 754\,a_s^4. \qquad (17)
\end{aligned}
$$

Now the error in the predicted a_s^4 term increases to about 40%! The reason is sitting in the peculiar structure of the a_s^4 term in eqs. (14-17):

$$
-1311.03 - 0.515625\;k_{2,0}^{[V]3}. \qquad (18)
$$

Here the first term describes the "kinematical" effects due to the analytic continuation as well as the contributions coming from the interference between (of order a_s^3 and less) terms in the beta function, the quark anomalous dimension and Π_2 in eq. (3). Numerically $k_{2,0}^{[V]3}$ is about $3{\cdot}10^3$, hence there are strong cancellations in eq. (18) which lead to a significant loss in accuracy if the predicted value of $k_{2,0}^{[V]3}$ even slightly deviates from the exact result.

As for the NNA approach the corresponding prediction is in obvious conflict[3] with the result of the direct calculation already for the Euclidian quantity $k_2^{V,3}$.

4. Conclusions

We have presented the analytical calculation of contributions of order $m_q^2\alpha_s^4$ to $R(s)$. This constitutes the first full five loop contribution to the fundamental quantity $R(s)$, albeit only to its mass correction.

Predictions of FAC, PMS and NNA optimization methods are tested against the exact results. Good quantitative agreement is found for the the FAC/PMS predictions in the Euclidean region. The agreement deteriorates after analytical continuation to the physical Minkowskian region. NNA fails to predict the result.

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