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An algorithm for the high-energy expansion of multi-loop diagrams to next-to-leading logarithmic accuracy

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Abstract:
We present an algorithm to compute arbitrary multi-loop massive Feynman diagrams in the region where the typical energy scale $\sqrt{s}$ is much larger than the typical mass scale M, i.e. $s \gg M^2$, while various different energy and mass parameters may be present. In this region we perform an asymptotic expansion and, using sector decomposition, we extract the leading contributions resulting from ultraviolet and mass singularities, which consist of large logarithms $\ln(s/M^2)$ and $1/\varepsilon$ poles in $D = 4 - 2\varepsilon$ dimensions. To next-to-leading accuracy, at L loops all terms of the form $\alpha^L \varepsilon^{-k} \ln^j(s/M^2)$ with $j + k = 2L$ and $j + k = 2L - 1$ are taken into account. This algorithm permits, in particular, to compute higher-order next-to-leading logarithmic electroweak corrections for processes involving various kinematical invariants of the order of hundreds of GeV and masses $M_\mathrm{W} \sim M_\mathrm{Z} \sim M_\mathrm{H} \sim m_\mathrm{t}$ of the order of the electroweak scale, in the approximation where the masses of the light fermions are neglected.

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1 Introduction

With the advent of colliders in the TeV energy range, multi-loop calculations, both in QCD and in the electroweak theory, will play an increasingly important role in order to provide theoretical predictions of sufficiently high accuracy. On the one hand, this is due to the high level of experimental precision, especially at the linear collider [1–3]. On the other hand, the size of radiative corrections grows with the energy.

This latter feature is particularly important for the electroweak corrections [4], since at TeV colliders the energy scale $\sqrt{s}$ starts to be much larger than the weak-boson mass scale $M = M_{\mathrm{W}} \sim M_{\mathrm{Z}}$. For $M^2 \ll s$, i.e. if one performs an asymptotic expansion in M^2/s, the electroweak corrections assume the form of a tower of large logarithms,

$$\alpha^L \sum_{j=0}^{2L} a_j \ln^j \left(\frac{s}{M^2}\right), \tag{1}$$

at L-loop level. These logarithms originate from ultraviolet (UV) and mass singularities. Above the electroweak scale, such logarithms start to grow and become the leading contribution to the electroweak corrections. At $\sqrt{s} = 1\,\mathrm{TeV}$ they give rise to corrections of tens of per cent at one loop and a few per cent at two loops. Thus, in order to reach the per-mille level accuracy envisaged by the linear collider, it becomes mandatory to compute electroweak radiative corrections up to the two-loop level.

Despite of recent progress in this direction [5], the exact analytical computation of two-loop electroweak corrections for processes involving many (more than three) external legs and various internal masses is out of sight. One is therefore forced either to adopt a numerical approach, as for instance in Refs. [6,7], or to consider an asymptotic expansion of the type (1), as we do in the present paper.

The asymptotic behaviour of higher-order electroweak corrections has received a lot of interest in the recent years [8–21]. On the one hand, resummation prescriptions have been proposed [8–12], which predict the leading logarithms (LLs) and next-to-leading logarithms (NLLs), i.e. terms with $j = 2L$ and $2L - 1$ in (1), for arbitrary processes and also the next-to-next-to-leading logarithms (NNLLs), with $j = 2L-2$, for $2 \to 2$ massless fermionic processes [13]. On the other hand, in order to check these resummations, which rely on the assumption that various aspects of the symmetry-breaking mechanism can be neglected in the high-energy limit, explicit diagrammatic two-loop calculations have been performed. For the LLs [15–17] as well as for the angular-dependent subset of the NLLs [18] such checks have been already completed and the exponentiation predicted in Refs. [10–12] has been confirmed for arbitrary processes. At (or beyond) the level of the NLLs only very few calculations exist. In the electroweak Standard Model, a calculation of a massless fermionic form factor [19] confirmed the resummation prescriptions of Refs. [11,12] up to the NLLs. The evaluation of the complete tower of logarithms contributing to this form factor in the special case of a $\mathrm{U}(1) \times \mathrm{U}(1)$ theory with a massive and a massless gauge boson [20,21] was found to be in agreement with the resummation prescriptions of Ref. [13] at the NNLL level.

For general processes, the terms with $j = 2L-1$ in the asymptotic expansion (1) remain largely untested and those for $j \leq 2L-2$ almost completely unknown. It is important to further investigate the behaviour of these subleading logarithms since, depending on the process, their phenomenological impact at the TeV scale might be comparable with that of the LLs, as was found in Refs. [11,13,20,21] for processes involving massless fermions.

In this paper we present an algorithm to compute the LLs and NLLs in a fully automated way. This algorithm, which has already been used in Refs. [18,19], applies to arbitrary multi-loop Feynman diagrams involving various energy parameters $s_j \sim s$ and masses $M_k \sim M$ of the same order. In the asymptotic region $s \gg M^2$, the coefficients of the LLs and NLLs are computed analytically as a function of the ratios s_j/s and M_k/M. The mass singularities originating from soft/collinear massless particles (photons, gluons, or fermions) as well as the UV singularities are regulated dimensionally and appear as $1/\varepsilon$ poles in $D = 4 - 2\varepsilon$ dimensions.

The algorithm is based on the so-called sector decomposition, a technique that permits to separate overlapping UV or mass singularities in Feynman-parameter (FP) integrals. Originally, this method was introduced in order to deal with UV singularities [22]. Later, it turned out to be applicable to mass singularities [23,24], and a first general algorithm to isolate $1/\varepsilon$ poles in massless loop integrals has been formulated in Ref. [7]. The algorithm that we present here permits to extract combinations (products) of $1/\varepsilon$ poles and mass-singular logarithms of s/M^2, which arise in the case where both massless and massive particles are present.

The input is an arbitrary multi-loop tensor integral, which is first translated into a FP integral. Depending on the location of UV and mass singularities in FP space, the integration range is decomposed into sectors, and each sector is remapped into the original integration range. This can be done iteratively until, in each sector, all singularities arise when individual FPs tend to zero and these FPs can be factorized. As a consequence, a simple power counting permits to determine the degree of singularity, i.e. the power of logarithms and $1/\varepsilon$ poles resulting from each sector. One can thus easily select those sectors that are responsible for the leading and next-to-leading singularities. Furthermore, those FP integrations that lead to singularities have a standard structure and can be computed in analytic form.

The coefficients of the resulting logarithms and $1/\varepsilon$ poles are represented as integrals over the remaining FPs. As a result of sector decomposition, these last integrations are free of singularities and, within the Euclidean region, the integrands are smooth functions which can be integrated numerically. Moreover, the coefficients of the leading and next-to-leading singularities are very simple integrals and in all examples we considered up to now (two-loop diagrams with up to 4 external legs and various internal masses) they could be solved in analytic form by standard computer algebra programs such as MATHEMATICA.

The paper is organized as follows: in Sect. 2 we describe the basics of the high-energy limit and the logarithmic approximation. In Sect. 3 we review the general expressions for multi-loop FP integrals and their UV and mass singularities and sketch the idea of the sector decomposition. The sector decomposition and the extraction of the singularities of massless diagrams is presented in Sect. 4 and the treatment of massive diagrams in Sect. 5. The applicability of the proposed method is discussed in Sect. 6. Some notations and conventions and several results for relevant massless and massive integrals are listed in the appendices.

2 High-energy limit and logarithmic approximation

The computation of higher-order electroweak corrections is a multi-scale problem. Typically there are various energy parameters $s_1, s_2, \ldots$, which correspond to the kinematical invariants of the process, masses of the order of the electroweak scale $M = M_\mathrm{W} \sim M_\mathrm{Z} \sim M_\mathrm{H} \sim m_\mathrm{t}$, various light-fermion masses $m_f \ll M$ and a fictitious infinitesimal photon mass to regulate infrared singularities. If all light fermions (including b quarks) and photons are treated as massless, then all particle masses are either zero or of the order M. Assuming this approach, in the present paper we consider the kinematical regime where all energy parameters are much higher than the electroweak scale. To be general, we consider multi-loop Feynman diagrams characterized by the hierarchy

$$s \sim |s_1| \sim |s_2| \sim \ldots \sim |s_J| \gg M^2 \sim M_1^2 \sim M_2^2 \sim \ldots \sim M_K^2 \tag{2}$$

of energy parameters and masses, i.e. we assume that there are only two different scales $s > 0$ and $M^2 > 0$. The external particles can be either on-shell with masses of the order M or zero, or off-shell with invariant masses of the order s. In this regime, we perform an asymptotic expansion in the small mass-to-energy ratio

$$w = \frac{M^2}{s} \ll 1, \tag{3}$$

treating the ratios s_j/s and M_k/M as constants in the limit $w \to 0$. The leading terms of this expansion are divergent. On the one hand there are mass singularities associated to massive particles which appear as logarithms of w. On the other hand, mass singularities from massless particles and UV singularities appear as $1/\varepsilon$ poles in $D = 4 - 2\varepsilon$ dimensions. The leading terms of the asymptotic expansion of L-loop Feynman integrals[1] assume the form of a double series in ε and $\ln(w)$,

$$A(w) = w^d \sum_{j=0}^{2L} \sum_{k=-j}^{\infty} a_{j,k}\, \varepsilon^k \ln^{j+k}(w) + \mathcal{O}(w^{d+1}). \tag{4}$$

In general, the loop integrals can give rise to non-logarithmic mass singularities, i.e. d can be negative. However, in spontaneously broken gauge theories such singularities are compensated by mass terms or coupling constants proportional to masses in such a way that the resulting contributions are at most logarithmically divergent. In the present paper, we compute loop integrals restricting ourselves to the leading terms of order w^d whereas contributions of order w^{d+1}, which are suppressed by powers of masses, are neglected.

The various terms in (4) are classified according to their degree of singularity j, which corresponds to the total power of logarithms and $1/\varepsilon$ poles. For a fixed value of j we compute all terms of the order

$$\varepsilon^{-j}, \varepsilon^{-j+1} \ln(w), \varepsilon^{-j+2} \ln^2(w), \ldots, \ln^j(w), \varepsilon \ln^{j+1}(w), \ldots, \tag{5}$$

up to the needed order in ε. At L-loop level, all terms of the type (5) with $j = 2L, 2L-1, 2L-2, \ldots$, are collectively denoted as leading logarithms (LLs), next-to-leading

[1] We assume that overall factors with mass dimension have been factorized such that the amplitude $A(w)$ is dimensionless.

logarithms (NLLs), next-to-next-to-leading logarithms (NNLLs), and so on. For instance, with this convention, the logarithms of order $\ln^{2L}(w)$ as well as the poles of order ε^{-2L} are denoted as LLs. In the present paper we adopt a NLL approximation, i.e. we include all contribution of the type (5) with $j \geq 2L-1$. To denote this approximation we use the symbol $\stackrel{\mathrm{NLL}}{=}$.

Since there are different energies and masses, the coefficients of the expansion (4) depend on their ratios, s_j/s and M_k/M. This dependence is taken into account and permits, for instance, to compute electroweak corrections without assuming that $M_\mathrm{W} = M_\mathrm{Z} = M_\mathrm{H} = m_\mathrm{t}$ and/or including angular-dependent corrections.

3 Feynman-parameter integrals

In this section we review the well-known technique [25] to translate loop integrals into FP integrals. Then we discuss the origin of UV and mass singularities in FP space.

3.1 Derivation of the Feynman-parameter representation

Let us consider a generic L-loop Feynman integral in D dimensions with I internal lines, E external lines, and m loop momenta in the numerator,

$$G_0 = \int \prod_{l=1}^{L} \mathrm{d}\tilde{q}_l \, \frac{q_{a_1}^{\mu_1} \cdots q_{a_m}^{\mu_m}}{\prod_{s=1}^{I} D_s^{n_s}} \, R_{\mu_1 \ldots \mu_m}, \qquad \mathrm{d}\tilde{q}_l = \mu^{(4-D)} \frac{\mathrm{d}^D q_l}{(2\pi)^D}. \tag{6}$$

The propagators $D_s^{-1} = (r_s^2 - m_s^2 + \mathrm{i}\epsilon)^{-1}$ with infinitesimal positive imaginary part ϵ, which are raised to integer powers $n_s \geq 1$, are determined by the internal momenta r_s, which consist of linear combinations of loop momenta $q = (q_1, \ldots, q_L)$ and external momenta $p = (p_1, \ldots, p_E)$, depending on the topology. The loop momenta occurring in the numerator are contracted with a projector $R_{\mu_1 \ldots \mu_m}$, which depends in general on the external momenta. In the following, we choose to split off a universal factor $(4\pi\mu^2/s)^{(2-D/2)L}$ from the L-loop Feynman integral in order to simplify the notation. This is equivalent to the choice $\mu^2 = s/(4\pi)$. This factor can be trivially restored at the end if needed.

In order to combine the products of propagators into a single denominator one introduces FPs by means of the well-known identity

$$\frac{1}{\prod_{s=1}^{I} D_s^{n_s}} = \frac{\Gamma(N)}{\prod_{s=1}^{I} \Gamma(n_s)} \int_0^1 \mathrm{d}^I \vec{\alpha} \; \delta\Big(1 - \sum_{s=1}^{I} \alpha_s\Big) \frac{\prod_{s=1}^{I} \alpha_s^{n_s - 1}}{\Big[\sum_{s=1}^{I} \alpha_s D_s\Big]^N} \tag{7}$$

with $N = \sum_{s=1}^{I} n_s$. In order to perform the integration over the loop momenta q, we eliminate the linear terms from the quadratic form in the denominator, so that

$$\sum_{s=1}^{I} \alpha_s D_s = \sum_{i,j=1}^{L} q_i M_{ij} q_l - 2 \sum_{i=1}^{L} q_i Q_i - J + \mathrm{i}\epsilon = \sum_{i,j=1}^{L} k_i M_{ij} k_l - \Delta + \mathrm{i}\epsilon. \tag{8}$$

This is easily obtained by a shift of the loop momenta with

$$k_l = q_l - v_l, \qquad v_l = \sum_{i=1}^{L} M_{li}^{-1} Q_i, \qquad \Delta = J + \sum_{i,j=1}^{L} Q_i M_{ij}^{-1} Q_j. \tag{9}$$

The resulting FP integral consists of a sum of terms with different numbers of loop momenta in the numerator. Only terms with an even number of loop momenta do not vanish, i.e. terms of the type

$$G = \frac{\Gamma(N)}{\prod_{s=1}^{I}\Gamma(n_s)} \int_0^1 \mathrm{d}^I\vec{\alpha}\,\delta\Big(1-\sum_{s=1}^{I}\alpha_s\Big) \prod_{s=1}^{I}\alpha_s^{n_s-1}$$
$$\times \int \prod_{l=1}^{L}\mathrm{d}\tilde{k}_l \,\frac{k_{a_1}^{\mu_1}\cdots k_{a_{2p}}^{\mu_{2p}}}{\left[k^{\mathrm{T}}Mk-\Delta+\mathrm{i}\epsilon\right]^N}\, v_{a_{2p+1}}^{\mu_{2p+1}}\cdots v_{a_m}^{\mu_m}\, R_{\mu_1\ldots\mu_m} \tag{10}$$

with $0 \le 2p \le m$. The integration over the loop momenta yields

$$\Gamma(N)\int \prod_{l=1}^{L}\mathrm{d}\tilde{k}_l\,\frac{k_{a_1}^{\mu_1}\ldots k_{a_{2p}}^{\mu_{2p}}}{\left[k^{\mathrm{T}}Mk-\Delta+\mathrm{i}\epsilon\right]^N} =$$
$$= (-1)^{N-p}\left[\frac{\mathrm{i}s^{2-D/2}}{(4\pi)^2}\right]^L \Gamma\left(N-\frac{LD}{2}-p\right)\frac{(\det M)^{-D/2}}{(\Delta-\mathrm{i}\epsilon)^{N-LD/2-p}}\,\gamma_{a_1\ldots a_{2p}}^{\mu_1\ldots\mu_{2p}}(M^{-1}), \tag{11}$$

where $\gamma(M^{-1}) = 1$ for $p = 0$, whereas for $p \ge 1$

$$\gamma_{a_1\ldots a_{2p}}^{\mu_1\ldots\mu_{2p}}(M^{-1}) = \frac{1}{2^p}\sum_{\substack{i_1\cdots i_p\\ j_1\cdots j_p}}\prod_{k=1}^{p}\left(M^{-1}\right)_{a_{i_k}a_{j_k}} g^{\mu_{i_k}\mu_{j_k}} \tag{12}$$

with the sum running over all $(2p)!/(p!\,2^p)$ pairings $(i_1, j_1), \ldots, (i_p, j_p)$ obtained from the set $\{1, 2, \ldots, 2p\}$. As a result we are left with a FP integral of the form

$$G = \int_0^1 \mathrm{d}^I\vec{\alpha}\;\delta\Big(1-\sum_{s=1}^{I}\alpha_s\Big)\,\hat{g}(\vec{\alpha}). \tag{13}$$

In order to isolate the inverse powers of $\det M$ which originate from M^{-1}, we write the integrand $\hat{g}(\vec{\alpha})$ in terms of[2]

$$\mathcal{U} = \det M, \qquad \tilde{M}_{ij}^{-1} = M_{ij}^{-1}\det M, \qquad \mathcal{F} = s^{-1}\Delta\det M - \mathrm{i}\epsilon, \qquad w_i = v_i \det M, \tag{14}$$

resulting in

$$\hat{g}(\vec{\alpha}) = \Gamma(e+L\varepsilon)\frac{\mathcal{N}(\vec{\alpha})}{[\mathcal{F}(\vec{\alpha})]^{e+L\varepsilon}\,[\mathcal{U}(\vec{\alpha})]^{f-(L+1)\varepsilon}} \tag{15}$$

in $D = 4 - 2\varepsilon$ dimensions, with $e = N - 2L - p$, $f = 2(L + 1) - N + m$ and

$$\mathcal{N}(\vec{\alpha}) = (-1)^{N-p}s^{-e}\left[\frac{\mathrm{i}}{(4\pi)^2}\right]^L \prod_{s=1}^{I}\frac{\alpha_s^{n_s-1}}{\Gamma(n_s)}\,\gamma_{a_1\ldots a_{2p}}^{\mu_1\ldots\mu_{2p}}(\tilde{M}^{-1})\,w_{a_{2p+1}}^{\mu_{2p+1}}\cdots w_{a_m}^{\mu_m}\,R_{\mu_1\ldots\mu_m}. \tag{16}$$

[2]Note that we factorize the energy scale s in order to define $\mathcal{F}$ as a dimensionless function.

3.2 Ultraviolet and mass singularities

As discussed in Sect. 2, the large logarithmic contributions that dominate the high-energy expansion of loop diagrams originate from UV and mass singularities. Let us now discuss the origin of such singularities in the FP integrals (13)–(16).

The overall gamma function $\Gamma(e+\varepsilon)$ in (15) represents an obvious source of UV singularities if $e \leq 0$. In addition, there are less trivial UV and mass singularities that originate from end-point singularities of the integrand. Let us focus on this kind of divergences, which result from the zeros of the polynomials $\mathcal{U}$ and/or $\mathcal{F}$ when their exponents, f and e, are positive.

The zeros of the function $\mathcal{U}$, which are independent of the external masses and momenta, are associated to UV subdivergences. Instead, the zeros of the function $\mathcal{F}$ can always be avoided by choosing non-zero internal and external masses and thus correspond to mass singularities. These latter, as is well known, originate from soft and/or collinear regions in momentum space.

It is useful to recall that the functions $\mathcal{U}$ and $\mathcal{F}$, which are independent of the loop momenta occurring in the numerator of (6), are determined by the topology of the corresponding Feynman diagram [25]. First, $\mathcal{U}$ is an homogeneous polynomial of degree L in the FPs,

$$\mathcal{U}(\vec{\alpha}) = \sum_{\mathcal{T}} \prod_{\substack{k=1 \\ i_k \notin \mathcal{T}}}^{L} \alpha_{i_k}, \tag{17}$$

where the sum runs over all trees $\mathcal{T}$ of the Feynman diagram. A tree is obtained by cutting L lines $(i_1, \ldots, i_L)$ of the diagram such that all vertices remain connected. Second, $\mathcal{F}$ is a homogeneous polynomial of degree $L+1$ in the FPs,

$$s\mathcal{F}(\vec{\alpha}) = -\sum_{\mathcal{C}} s_{\mathcal{C}} \prod_{\substack{k=1 \\ i_k \in \mathcal{C}}}^{L+1} \alpha_{i_k} + \mathcal{U}(\vec{\alpha}) \sum_{s=1}^{I} \alpha_s m_s^2 - \mathrm{i}\epsilon, \tag{18}$$

where the first sum runs over all cuts $\mathcal{C}$ of the Feynman diagram. A cut is a set of $L+1$ lines $(i_1, \ldots, i_{L+1})$ that when cut divide the diagram in two connected subdiagrams. Finally,

$$s_{\mathcal{C}} = \left(\sum_{k=1}^{L+1} \tilde{r}_{i_k} \right)^2 \quad \text{with} \quad \tilde{r}_{i_k} = \pm r_{i_k} \tag{19}$$

denotes the squared total momentum[3] flowing through the cut $\mathcal{C}$.

All coefficients of the polynomial $\mathcal{U}$ are non-negative. This implies that the UV subdivergences, which result from the zeros of this polynomial, originate only from regions of the type

$$\{\vec{\alpha}\,|\,\alpha_{j_1} = \ldots = \alpha_{j_n} = 0\} \quad \text{with} \quad 1 \leq n \leq I-1, \tag{20}$$

where one or more FPs vanish. Here $\{j_1, \ldots, j_n\}$ is a subset of $\{1, \ldots, I\}$. Mass singularities originate from the zeros of the polynomial $\mathcal{F}$ in the limit of vanishing internal and external masses, where $\mathcal{F}$ depends only on the invariants $s_{\mathcal{C}}$ that are not squares of on-shell

[3] The signs in (19) must be chosen in such a way that all momenta $\tilde{r}_{i_k}$ have the same direction with respect to the cut $\mathcal{C}$.

external momenta, i.e. the invariants s_j. In the case where these invariants are negative ($s_j < 0$) all coefficients of the polynomial (18) are non-negative for vanishing internal and external masses, and mass singularities only originate from regions of the type (20). This is valid for arbitrary, i.e. also positive, values of the invariants s_j since mass singularities appear independently of the directions of the external momenta [26] and thus of the values (and the signs) of these invariants.

3.3 Subtraction of overlapping singularities

Using the fact that all singularities we are interested in originate from regions of the type (20), we can isolate them at the level of the integrand and simplify their computation. This can be done with simple subtractions that split the integrand into singular and non-singular parts in such a way that the integration of the singular part is drastically simplified.

Let us first illustrate how to proceed in the simple case of a one-dimensional integral assuming that the integrand has the form $1/[\alpha f_0(\alpha)]$ with a polynomial f_0 such that $f_0(\alpha) > 0$ for $0 \leq \alpha \leq 1$. Here, the logarithmic singularity can be isolated by means of a simple subtraction at $\alpha = 0$:

$$\int_0^1 \frac{\mathrm{d}\alpha}{\alpha} \frac{1}{f_0(\alpha)} = \underbrace{\int_0^1 \frac{\mathrm{d}\alpha}{\alpha} \frac{1}{f_0(0)}}_{\text{singular}} + \underbrace{\int_0^1 \frac{\mathrm{d}\alpha}{\alpha} \left[\frac{1}{f_0(\alpha)} - \frac{1}{f_0(0)}\right]}_{\text{finite}}. \tag{21}$$

As a result, the singular part, which requires the introduction of an appropriate regularization prescription, becomes much simpler than the original integral. Also the remaining non-singular part is simplified since the integrand is a smooth function and can be integrated, either analytically or numerically, in the absence of a regulator.

In the case of massive integrals, the mass-to-energy ratio $w = M^2/s$ plays the role of the regulator and the mass-singular logarithms that arise in the high-energy limit, $w \ll 1$, can be isolated by means of a subtraction similar to (21). If we assume that the integrand has the form $1/[\alpha f_0(\alpha) + w f_1(\alpha)]$, with polynomials f_0 and f_1 such that $f_i(\alpha) > 0$ for $0 \leq \alpha \leq 1$, then we can write

$$\int_0^1 \frac{\mathrm{d}\alpha}{\alpha f_0(\alpha) + w f_1(\alpha)} = \int_0^1 \frac{\mathrm{d}\alpha}{\alpha f_0(0) + w f_1(0)} + \int_0^1 \frac{\mathrm{d}\alpha}{\alpha} \left[\frac{1}{f_0(\alpha)} - \frac{1}{f_0(0)}\right] + \mathcal{O}(w), \tag{22}$$

and the mass-suppressed terms of order w can be neglected.

In order to be able to perform the above subtractions, it is crucial that the polynomial which leads to the singularity is written in the form $\alpha f_0(\alpha)$, where the integration variable α, which is responsible for the singularity at $\alpha = 0$, is *factorized*. In the above one-dimensional examples this is always possible, since any polynomial with a zero at $\alpha = 0$ can be written in this form. Instead, in the case of multi-dimensional FP integrals, the situation is complicated by the presence of so-called *overlapping* singularities, which originate only from regions where different FPs vanish simultaneously, i.e. regions of the type (20) with $n > 1$. The polynomials that are responsible for these overlapping singularities can be written in the form $\sum_{k=1}^{n} \alpha_{j_k} f_{0,k}(\vec{\alpha})$ and the FPs α_{j_k} cannot be factorized.

This difficulty can be overcome by means of the so-called sector-decomposition technique, which consists of a decomposition of the integration range into various sectors followed by a remapping of each sector into the original integration range. As we show in the following two sections, the sectors can be chosen in such a way that the resulting integrals are of the form

$$\int_0^1 \frac{\mathrm{d}\vec{\alpha}}{\left(\prod_j \alpha_j\right) f_0(\vec{\alpha}) + w f_1(\vec{\alpha})}, \tag{23}$$

where all FPs which are responsible for the singularities are factorized.

4 Sector decomposition of massless diagrams

The sector decomposition of the FP integrals (13)–(16) consists of two steps. First a primary sector decomposition is performed that permits to eliminate the δ-function in such a way that the regions (20), which give rise to singularities, remain unchanged. In the second step, an iterated sector decomposition that permits to factorize all FPs that lead to singularities as discussed in the previous section is carried out. At the end of the sector decomposition those integrations that lead to $1/\varepsilon$ poles and logarithms can be performed by means of standard formulas, which we derive in the appendices, using recursive subtractions of the type (21) and (22).

In this section we describe the sector-decomposition technique to extract $1/\varepsilon$ poles from massless[4] diagrams [7]. A sector-decomposition algorithm to extract logarithmic singularities from massive diagrams is presented in Sect. 5.

4.1 Primary sector decomposition

As discussed in Sect. 3.3, our capability to isolate and simplify the singularities relies on the fact that they originate from regions of the type (20). Therefore, when one eliminates the δ-function from the FP integrals (13)–(16) by performing the first integration, care must be taken that the structure of the singular regions remains unchanged.[5] To this end, one can decompose the $(I-1)$-dimensional hyperplane that is defined by the δ-function, into I primary sectors $\mathcal{P}_1, \ldots, \mathcal{P}_I$, defined as $\mathcal{P}_i = \{\vec{\alpha}\,|\,\alpha_k \leq \alpha_i \text{ for } 1 \leq k \leq I$ and $\sum_{j=1}^I \alpha_j = 1\}$, such that the sector $\mathcal{P}_i$ does not contain the hyperplane $\{\vec{\alpha}|\alpha_i = 0\}$. This yields

$$G = \sum_{i=1}^I G_i \qquad \text{with} \quad G_i = \int_0^1 \mathrm{d}\alpha_i \int_0^{\alpha_i} \prod_{\substack{k=1\\k\neq i}}^I \mathrm{d}\alpha_k\, \delta\left(1 - \sum_{j=1}^I \alpha_j\right) \alpha_i^{-I} \hat{g}\left(\frac{\vec{\alpha}}{\alpha_i}\right), \tag{24}$$

where we used the fact that $\hat{g}$ is a homogeneous function of degree $-I$ in the integration variables, i.e. $\hat{g}(\lambda\vec{\alpha}) = \lambda^{-I}\hat{g}(\vec{\alpha})$ as can be easily seen from (7). Now, each sector $\mathcal{P}_i$ can be mapped to the $(I-1)$-dimensional hypercube by means of the variable transformation

$$\alpha_k = \begin{cases} \alpha_i \eta_k & \text{if} \quad 1 \leq k < i \\ \alpha_i \eta_{k-1} & \text{if} \quad i < k \leq I \end{cases} \qquad \text{with} \quad \mathrm{d}^I \vec{\alpha} = \mathrm{d}\alpha_i\, \alpha_i^{I-1} \mathrm{d}^{I-1}\vec{\eta}. \tag{25}$$

[4] This technique is applicable also to massive diagrams, however, it permits to isolate only the $1/\varepsilon$ poles and not the mass-singular logarithms.

[5] In particular, one cannot integrate one of the FPs, let's say α_i, down to $\alpha_i = 0$. Otherwise, a singularity that might be located in the hyperplane $\{\vec{\alpha}|\alpha_i = 0\}$ would be shifted to the hyperplane $\{\vec{\alpha}|\sum_{j\neq i} \alpha_j = 1\}$.

Then, eliminating the δ-function through the α_i-integration one obtains

$$G_i = \int_0^1 \frac{\mathrm{d}\alpha_i}{\alpha_i} \int_0^1 \mathrm{d}^{I-1}\vec{\eta}\, \delta\left(1-\alpha_i\left(1+\sum_{j=1}^{I-1}\eta_j\right)\right) \hat{g}_i(\vec{\eta}) = \int_0^1 \mathrm{d}^{I-1}\vec{\eta}\, \hat{g}_i(\vec{\eta}), \qquad (26)$$

where

$$\hat{g}_i(\vec{\eta}) = \hat{g}\left(\frac{\vec{\alpha}}{\alpha_i}\right) = \hat{g}(\eta_1, \dots, \eta_{i-1}, 1, \eta_i, \dots, \eta_{I-1}), \qquad (27)$$

and using analogous definitions for $\mathcal{U}_i$, $\mathcal{F}_i$ and $\mathcal{N}_i$,

$$\hat{g}_i(\vec{\eta}) = \Gamma(e+L\varepsilon) \frac{\mathcal{N}_i(\vec{\eta})}{[\mathcal{F}_i(\vec{\eta})]^{e+L\varepsilon}\, [\mathcal{U}_i(\vec{\eta})]^{f-(L+1)\varepsilon}}. \qquad (28)$$

It is easy to see that this primary sector decomposition does not modify the structure of the singular regions (20). Indeed, each hyperplane $\{\vec{\alpha}|\alpha_k = 0 \text{ and } \sum_{j=1}^{I}\alpha_j = 1\}$, which can give rise to singularities, has been divided into the sectors $\mathcal{P}_i$ with $i \neq k$, and there it has been mapped to the hyperplanes $\{\vec{\eta}|\eta_l = \alpha_k/\alpha_i = 0\}$ with $l = k$ for $i > k$ and $l = k-1$ for $i < k$. Therefore, all singularities still originate from regions of the type $\{\vec{\eta}\,|\,\eta_{j_1} = \ldots = \eta_{j_n} = 0\}$ with $1 \leq n \leq I-1$.

4.2 Iterated decomposition of UV and mass singularities

Now we perform an iterative decomposition of the primary-sector integrals (26) into sums of subsector integrals, where all FPs η_i that give rise to singularities can be factorized. The UV singularities associated with the polynomial $\mathcal{U}_i$ are decomposed as follows:

1. If $\mathcal{U}_i(\vec{0}) = 0$, then we proceed with steps 2–4. Otherwise no further decomposition is performed.

2. We choose a set $\mathcal{R} = \{\eta_{j_1}, \dots, \eta_{j_r}\}$ with a minimal number r of FPs such that

$$\mathcal{U}_i(\vec{\eta}) = 0 \quad \text{if} \quad \eta_{j_1} = \ldots = \eta_{j_r} = 0. \qquad (29)$$

   Here, in order to simplify the notation, we assume that $j_k = k$ such that the polynomial $\mathcal{U}_i$ can be written as

$$\mathcal{U}_i(\vec{\eta}) = \sum_{k=1}^{r} \eta_k \hat{\mathcal{U}}_{ik}(\vec{\eta}). \qquad (30)$$

3. The integration range is decomposed into r subsectors $\mathcal{S}_1, \dots, \mathcal{S}_r$ with $\mathcal{S}_j = \{\vec{\eta}\,|\,\eta_k \leq \eta_j \text{ for } 1 \leq k \leq r\}$, so that

$$G_i = \sum_{j=1}^{r} G_{ij} \quad \text{with} \quad G_{ij} = \int_0^1 \mathrm{d}\eta_j \int_0^{\eta_j} \prod_{\substack{k=1\\k\neq j}}^{r} \mathrm{d}\eta_k \int_0^1 \prod_{l=r+1}^{I-1} \mathrm{d}\eta_l\, \hat{g}_i(\vec{\eta}). \qquad (31)$$

   Each sector $\mathcal{S}_j$ is remapped to the unit cube using the variable transformation

$$\eta_k = \begin{cases} \xi_k & \text{if } r+1 \leq k \leq I-1 \\ \xi_j & \text{if } k = j \\ \xi_j\xi_k & \text{otherwise} \end{cases} \qquad \text{with} \quad \mathrm{d}^{I-1}\vec{\eta} = \mathrm{d}^{I-1}\vec{\xi}\, \xi_j^{r-1}. \qquad (32)$$

As a consequence, in the sector $\mathcal{S}_j$ the variable ξ_j can be factorized by rewriting the polynomial $\mathcal{U}_i$ as

$$\mathcal{U}_i(\vec{\eta}) = \xi_j \mathcal{U}_{ij}(\vec{\xi}) \quad \text{with} \quad \mathcal{U}_{ij}(\vec{\xi}) = \hat{\mathcal{U}}_{ij}(\vec{\eta}) + \sum_{\substack{k=1 \\ k\neq j}}^{r} \xi_k \hat{\mathcal{U}}_{ik}(\vec{\eta}). \tag{33}$$

The resulting sector integrals read

$$G_{ij} = \int_0^1 \mathrm{d}^{I-1}\vec{\xi}\, \xi_j^{r-1-f+(L+1)\varepsilon} \hat{g}_{ij}(\vec{\xi}), \tag{34}$$

where

$$\hat{g}_{ij}(\vec{\xi}) = \Gamma(e+L\varepsilon) \frac{\mathcal{N}_{ij}(\vec{\xi})}{\left[\mathcal{F}_{ij}(\vec{\xi})\right]^{e+L\varepsilon} \left[\mathcal{U}_{ij}(\vec{\xi})\right]^{f-(L+1)\varepsilon}} \tag{35}$$

with[6] $\mathcal{F}_{ij}(\vec{\xi}) = \mathcal{F}_i(\vec{\eta})$ and $\mathcal{N}_{ij}(\vec{\xi}) = \mathcal{N}_i(\vec{\eta})$.

4. For each subsector integral G_{ij}, we restart the decomposition from step 1.

The iterative application of steps 1–4 gives rise to a tree-like structure. At each iteration the subsectors are divided into new subsubsectors, which have to be labelled with new indices

$$G_i \to \sum_{j_1} G_{ij_1}, \qquad G_{ij_1} \to \sum_{j_2} G_{ij_1j_2}, \qquad \ldots \qquad G_{ij_1\ldots j_{n-1}} \to \sum_{j_n} G_{ij_1\ldots j_{n-1}j_n}. \tag{36}$$

At each decomposition new FPs are factorized and the resulting sector integrals have the general form

$$G_{ij_1\ldots} = \int_0^1 \mathrm{d}^{I-1}\vec{\xi}\, P\left(\vec{\xi}^{\vec{T}_{ij_1\ldots}+\varepsilon\vec{\tau}_{ij_1\ldots}}\right) \hat{g}_{ij_1\ldots}(\vec{\xi}), \tag{37}$$

where

$$\hat{g}_{ij_1\ldots}(\vec{\xi}) = \Gamma(e+L\varepsilon) \frac{\mathcal{N}_{ij_1\ldots}(\vec{\xi})}{\left[\mathcal{F}_{ij_1\ldots}(\vec{\xi})\right]^{e+L\varepsilon} \left[\mathcal{U}_{ij_1\ldots}(\vec{\xi})\right]^{f-(L+1)\varepsilon}}, \tag{38}$$

and the factorized FPs are written as $P\left(\vec{\xi}^{\vec{T}_{ij_1\ldots}+\varepsilon\vec{\tau}_{ij_1\ldots}}\right)$ using the shorthands (87)–(88). The iteration of steps 1–4 stops when only sector integrals with $\mathcal{U}_{ij_1\ldots}(\vec{0}) \neq 0$ remain. Then, the same iterative decomposition has to be repeated for the mass singularities that are associated with the polynomial $\mathcal{F}_{ij_1\ldots}$ until

$$\mathcal{U}_{ij_1\ldots}(\vec{0}) \neq 0 \quad \text{and} \quad \mathcal{F}_{ij_1\ldots}(\vec{0}) \neq 0 \tag{39}$$

in all subsectors.

The convergence and the efficiency of the sector-decomposition algorithm depend on the choice of the sets of FPs $\mathcal{R} = \{\eta_{j_1}, \ldots, \eta_{j_r}\}$ in step 2, which determines the subsectors

[6] In some cases the FP ξ_j can be factorized also in $\mathcal{F}_{ij}(\vec{\xi})$ or $\mathcal{N}_{ij}(\vec{\xi})$.

$\mathcal{S}_1, \ldots, \mathcal{S}_r$ in step 3. In principle, the choice of minimal sets $\mathcal{R}$, i.e. sets with a minimal number r of FPs, permits to reduce the number of new subsectors that are generated at each iteration and guarantees a certain efficiency. However, there are in general different minimal sets $\mathcal{R}$ that fulfil (29), and it is not clear how to choose between them in order to minimize the total number of subsectors at the end of the iterated decomposition. Moreover, in some cases it is possible that owing to an unfortunate choice of the minimal sets the conditions (39) are never realized, i.e. the convergence of the algorithm, as it is formulated above, is not guaranteed. The approach that we have adopted to avoid this problem consists in a random choice of the minimal sets $\mathcal{R}$. This strategy is in general not optimal but has the advantage to be non-deterministic, in the sense that every time that a certain integral is computed it is decomposed into different sums of subsector integrals, and this allows for checks on the final result.

4.3 Counting of singularities

In the following we focus on the properties of a generic sector integral. To this end we define

$$\tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = \int_0^1 \mathrm{d}^{I-1}\vec{\xi}\, P\left(\vec{\xi}^{\,\vec{T}+\varepsilon\vec{\tau}}\right) \frac{\tilde{g}(\vec{\xi})}{\left[\tilde{\mathcal{F}}(\vec{\xi})\right]^{\tilde{e}+L\varepsilon}} \tag{40}$$

with

$$\tilde{g}(\vec{\xi}) = \Gamma(e+L\varepsilon) \frac{\tilde{\mathcal{N}}(\vec{\xi})}{\left[\tilde{\mathcal{U}}(\vec{\xi})\right]^{f-(L+1)\varepsilon}} \tag{41}$$

and

$$\tilde{\mathcal{U}}(\vec{0}) \neq 0, \qquad \tilde{\mathcal{F}}(\vec{0}) \neq 0. \tag{42}$$

The sector integrals (37) can be written as $G_{ij_1\ldots} = \tilde{G}(\vec{T}_{ij_1\ldots}, e, g_{ij_1\ldots}, \mathcal{F}_{ij_1\ldots})$ with these conventions. The dependence of $\tilde{G}$ on $\vec{T}$, $\tilde{e}$, $\tilde{g}$ and $\tilde{\mathcal{F}}$ has been kept in explicit form for later convenience, whereas the dependence on all other quantities is implicitly understood. Note also that $\tilde{e}$ can be different from e in (40)–(41).

The FPs that give rise to singularities in the sector integrals (40)–(42) can be identified by simple power counting. In principle all FPs in $\mathcal{S} = \{\xi_k | T_k \leq -1\}$ yield singularities unless these are not compensated by the behaviour of $\tilde{\mathcal{N}}(\vec{\xi})$ at $\vec{\xi} \to \vec{0}$. Therefore, at the end of the sector decomposition, we decompose $\tilde{\mathcal{N}}(\vec{\xi})$ into monomials with respect to the FPs in $\mathcal{S}$ and we treat the contribution from each monomial as a different subsector, such that the FPs $\mathcal{S}$ can be factorized and the resulting numerator $\tilde{\mathcal{N}}$ does not depend on them anymore.

Now, those FPs that do and those that do not give rise to singularities can be immediately identified. If we rename them as $\vec{y} = (y_1, \ldots, y_m)$ and $\vec{x} = (x_1, \ldots, x_l)$, respectively, with $l + m = I - 1$, then the contribution from each subsector assumes the general form

$$\tilde{G} \equiv \tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = \int_0^1 \mathrm{d}^l\vec{x}\, P\left(\vec{x}^{\,\vec{A}+\vec{\alpha}\varepsilon}\right) \int_0^1 \mathrm{d}^m\vec{y}\, P\left(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\right) \frac{\tilde{g}(\vec{x};\vec{y})}{\left[\tilde{\mathcal{F}}(\vec{x};\vec{y})\right]^{\tilde{e}+L\varepsilon}} \tag{43}$$

with

$$\tilde{g}(\vec{x};\vec{y}) = \Gamma(e+L\varepsilon)\frac{\tilde{\mathcal{N}}(\vec{x})}{\left[\tilde{\mathcal{U}}(\vec{x};\vec{y})\right]^{f-(L+1)\varepsilon}} \tag{44}$$

and exponents

$$\begin{aligned}\vec{A} &= (A_1,\ldots,A_l), & \vec{\alpha} &= (\alpha_1,\ldots,\alpha_l),\\ \vec{b} &= (b_1,\ldots,b_m), & \vec{\beta} &= (\beta_1,\ldots,\beta_m).\end{aligned} \tag{45}$$

By construction, all exponents (45) are integer numbers with $A_i > -1$ and $b_j \geq 0$. The exponents $\vec{A}$, $\vec{b}$ and $\vec{\alpha}$, $\vec{\beta}$ are related to $\vec{T}$ and $\vec{\tau}$, respectively, in an obvious way. As mentioned before, we have decomposed $\tilde{\mathcal{N}}(\vec{x},\vec{y})$ into monomials with respect to the FPs $\vec{y}$ and we treat the contribution from each monomial as a different subsector. The integrations over the m FPs $\vec{y}$ give rise to singularities up to the order $1/\varepsilon^m$.

4.4 Integration

The integrations of the FPs $y_1,\ldots,y_m$ for a generic integral of the type (43) are performed in Appendix B. This is done through recursive subtractions of the type (21), which permit to extract the leading and next-to-leading singularities, i.e. the poles of order $1/\varepsilon^m$ and $1/\varepsilon^{m-1}$. The result for the integral (43) is obtained by applying (94) to the integrand

$$h_0(\vec{y}) = \int_0^1 \mathrm{d}^l\vec{x}\, h_0(\vec{x};\vec{y}) \quad \text{with} \quad h_0(\vec{x};\vec{y}) = P\Big(\vec{x}^{\,\vec{A}+\vec{\alpha}\varepsilon}\Big)\frac{\tilde{g}(\vec{x};\vec{y})}{\left[\tilde{\mathcal{F}}(\vec{x};\vec{y})\right]^{\tilde{e}+L\varepsilon}}. \tag{46}$$

Expanding $h_0 \equiv h_0(\vec{x};\vec{y})$ in ε as

$$h_0 \equiv \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}h_0^{(j)}, \tag{47}$$

and similarly[7] $\Gamma(e+\varepsilon)$ and $\tilde{\mathcal{N}}$ we obtain

$$\begin{aligned}\tilde{G} \stackrel{\text{NLL}}{=} \left[P\Big(\vec{\beta}\Big)\right]^{-1}\left(\frac{1}{\varepsilon}\right)^m \int_0^1 \mathrm{d}^l\vec{x}\Bigg\{&\mathrm{D}_{\vec{y}}^{\vec{b}}\,h_0^{(0)}(\vec{x};\vec{0})\\ &+\varepsilon\left[\mathrm{D}_{\vec{y}}^{\vec{b}}\,h_0^{(1)}(\vec{x};\vec{0}) + \sum_{i=1}^m \beta_i\Delta_{\vec{y}}^{\vec{b},i}h_0^{(0)}(\vec{x};\vec{0})\right]\Bigg\}\end{aligned} \tag{48}$$

to NLL accuracy with

$$\begin{aligned}h_0^{(0)} &= \Gamma^{(0)}(e)P\Big(\vec{x}^{\,\vec{A}}\Big)\frac{\tilde{\mathcal{N}}^{(0)}}{\tilde{\mathcal{F}}^{\tilde{e}}\tilde{\mathcal{U}}^f},\\ \frac{h_0^{(1)}}{h_0^{(0)}} &= L\frac{\Gamma^{(1)}(e)}{\Gamma^{(0)}(e)} + \ln\left[P\Big(\vec{x}^{\,\vec{\alpha}}\Big)\right] + \frac{\tilde{\mathcal{N}}^{(1)}}{\tilde{\mathcal{N}}^{(0)}} + (L+1)\ln\tilde{\mathcal{U}} - L\ln\tilde{\mathcal{F}}.\end{aligned} \tag{49}$$

[7]If $e \leq 0$, then the leading term of $\Gamma(e+\varepsilon) \equiv \sum_{j=0}^{\infty}\Gamma^{(j)}(e)\varepsilon^j/j!$ contains a $1/\varepsilon$ pole:

$$\Gamma^{(0)}(e) = \frac{(-1)^e}{(-e)!}\frac{1}{\varepsilon} \quad \text{for} \quad e \leq 0.$$

The derivative operators $\mathrm{D}_{\vec{y}}^{\vec{b}}$ and the subtraction operators $\Delta_{\vec{y}}^{\vec{b},i}$ are defined in (95) and (96), respectively. These latter are associated to those next-to-leading contributions that remain after subtraction of the singularity from the y_i integration. For instance, in the special case $\vec{b}=\vec{0}$ where all $\vec{y}$ integrations are logarithmically singular, we simply have

$$\Delta_{\vec{y}}^{\vec{0},i} h_0^{(0)}(\vec{x};\vec{0}) = \int_0^1 \frac{\mathrm{d}y_i}{y_i}\left[h_0^{(0)}(\vec{x};0,\dots,y_i,\dots,0) - h_0^{(0)}(\vec{x};\vec{0})\right]. \tag{50}$$

By construction, the remaining y_i integration in the $\Delta_{\vec{y}}^{\vec{b},i}$ term is finite and can be performed either numerically or analytically. Also the l-dimensional integral over the FPs $\vec{x}$ in (48) is convergent. The coefficients of the leading and next-to leading poles have thus been expressed in terms of convergent integrals with dimension l and $l+1$. For negative kinematical invariants these integrals are well-defined. For positive kinematical invariants the analytic continuation is provided by the infinitesimal imaginary part ϵ contained in $\tilde{\mathcal{F}}$.

5 Sector decomposition of massive diagrams

Let us now consider the sector decomposition of massive FP integrals (13)–(16) assuming a hierarchy of energy and mass scales as in (2). In order to extract the singularities that arise as combinations of $1/\varepsilon$ poles and logarithms of $w = M^2/s$, which appear in the asymptotic limit $w \ll 1$, we proceed as follows.

First, we perform the primary and iterated sector decompositions as described in Sects. 4.1–4.2. As a result, the sector integrals assume the form (40)–(42), where owing to (42) all FPs which lead to $1/\varepsilon$ poles are factorized. Then, in each sector, we decompose the polynomial $\tilde{\mathcal{F}}$ as

$$\tilde{\mathcal{F}}(\vec{\xi}) = f_0(\vec{\xi}) + w f_1(\vec{\xi}) - \mathrm{i}\epsilon, \tag{51}$$

where $f_0(\vec{\xi})$ and $f_1(\vec{\xi})$ are the contributions associated with the high-energy and mass parameters, and depend linearly on the corresponding ratios s_j/s and M_k^2/M^2, respectively. If $f_0(\vec{0})=0$, there can be mass singularities that are regulated by the mass term $wf_1(\vec{\xi})$ giving rise to logarithms of w in the asymptotic limit. In this case, we perform an additional iterated decomposition analogous to that of Sect. 4.2 until all FPs that lead to $f_0(\vec{0})=0$ are factorized. At the end the original integral consists of a sum of sector integrals of the type

$$\tilde{G}(\vec{T},\tilde{e},\tilde{g},\tilde{\mathcal{F}}) = \int_0^1 \mathrm{d}^{I-1}\vec{\xi}\, P\left(\vec{\xi}^{\,\vec{T}+\vec{\tau}\varepsilon}\right) \frac{\tilde{g}(\vec{\xi})}{\left[\tilde{\mathcal{F}}(\vec{\xi})\right]^{\tilde{e}+L\varepsilon}} \tag{52}$$

where

$$\tilde{g}(\vec{\xi}) = \Gamma(e+L\varepsilon)\frac{\tilde{\mathcal{N}}(\vec{\xi})}{\left[\tilde{\mathcal{U}}(\vec{\xi})\right]^{f-(L+1)\varepsilon}}, \tag{53}$$

and

$$\tilde{\mathcal{F}}(\vec{\xi}) = P\left(\vec{\xi}^{\,\vec{t}}\right)\tilde{f}_0(\vec{\xi}) + w\tilde{f}_1(\vec{\xi}) - \mathrm{i}\epsilon. \tag{54}$$

After the last sector decomposition we have

$$\tilde{\mathcal{U}}(\vec{0}) \neq 0, \qquad \tilde{\mathcal{F}}(\vec{0}) \neq 0, \qquad \text{and} \quad \tilde{f}_0(\vec{0}) \neq 0 \tag{55}$$

in all subsectors. Without loss of generality we can assume that[8]

$$\vec{t} = (t_1, \dots t_p, 0, \dots, 0) \qquad \text{with} \quad t_i > 0 \quad \text{for} \quad i = 1, \dots, p, \tag{56}$$

i.e. that the FPs that have been factorized in (54) are the first p FPs, where $0 \leq p \leq I-1$.

Simple power counting indicates that the set of FPs which can give rise to $1/\varepsilon$ poles or mass-singular logarithms is given by $\mathcal{S} = \{\xi_k | T_k \leq -1 \text{ or } T_k - \tilde{e} t_k \leq -1\}$. As discussed in Sect. 4.3, the numerator $\tilde{\mathcal{N}}(\vec{\xi})$ has to be decomposed into monomials with respect to the FPs in $\mathcal{S}$, and each monomial has to be treated as a different subsector where the FPs in $\mathcal{S}$ can be factorized.

5.1 Transformation of non-logarithmic singularities

Let us discuss the behaviour of the integrations over the FPs $\xi_1, \dots, \xi_p$ in the asymptotic limit $w \to 0$. If $T_k > -1$ for $1 \leq k \leq p$, the ξ_k-integrations are convergent in $D = 4$ dimensions. For $w \to 0$ they behave as

$$\int \mathrm{d}\xi_k \frac{\xi_k^{T_k}}{\left[\xi_k^{t_k} + w\right]^{\tilde{e}}} \sim \int \mathrm{d}\xi_k^{t_k} \frac{(\xi_k^{t_k})^{\tilde{e}-d_k-1}}{\left[\xi_k^{t_k} + w\right]^{\tilde{e}}} \sim \begin{cases} 1 & \text{if } d_k < 0 \\ \ln(w) & \text{if } d_k = 0 \\ w^{-d_k} & \text{if } d_k > 0 \end{cases}, \tag{57}$$

where we have introduced the degree of singularity

$$d_k := \tilde{e} - \frac{T_k + 1}{t_k} \tag{58}$$

for $k = 1, \dots, p$. Note that if $d_k > 0$, the ξ_k-integral yields a non-logarithmic mass singularity of order w^{-d_k} for $T_k > -1$. Similarly, for $T_k \leq -1$ one obtains singular contributions of order $w^{-d_k}\varepsilon^{-1}$ in $D = 4 - 2\varepsilon$ dimensions. Independently of the values of T_k, these non-logarithmically divergent integrals can be easily expressed as a linear combination of logarithmically-divergent integrals with $d_k \leq 0$ for $k = 1, \dots, p$. To this end, we define the maximal degree of singularity

$$d := \max_{1 \leq k \leq p} (d_k), \tag{59}$$

and if $d > 0$ we multiply the integrand in (52) by

$$1 = \left[\frac{\tilde{\mathcal{F}} - \tilde{f}_0 P\left(\vec{\xi}^{\vec{t}}\right)}{w\tilde{f}_1 - \mathrm{i}\epsilon}\right]^d = \left(w\tilde{f}_1 - \mathrm{i}\epsilon\right)^{-d} \sum_{r=0}^{d} \binom{d}{r} \tilde{\mathcal{F}}^{d-r} \left[-\tilde{f}_0 P\left(\vec{\xi}^{\vec{t}}\right)\right]^r. \tag{60}$$

This yields

$$\tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = w^{-d} \sum_{r=0}^{d} \binom{d}{r} (-1)^r \tilde{G}(\vec{T}_r, \tilde{e}_r, \tilde{g}_r, \tilde{\mathcal{F}}), \tag{61}$$

[8]This requires a permutation of the FPs that is, in general, different for different subsectors.

where

$$\vec{T}_r = \vec{T} + r\vec{t}, \qquad \tilde{e}_r = \tilde{e} - d + r, \qquad \tilde{g}_r = \tilde{g}(\tilde{f}_0)^r(\tilde{f}_1 - \mathrm{i}\epsilon)^{-d}. \tag{62}$$

As a result of the above transformation, the term w^{-d} involving inverse powers of masses is factorized and the degree of divergence d_k is reduced by d in the resulting integrals $\tilde{G}(\vec{T}_r, \tilde{e}_r, \tilde{g}_r, \tilde{\mathcal{F}})$. This can be easily seen from

$$\left(\vec{d}_r\right)_k := \tilde{e}_r - \frac{\left(\vec{T}_r\right)_k + 1}{t_k} = d_k - d. \tag{63}$$

Thus, the integrations with maximal degree of singularity $d_k = d$ become logarithmically divergent, whereas all other integrations with $d_k < d$ become non-divergent.

Since the integrals resulting from (61)–(62) have the same structure as the original ones, we need to compute only integrals of the type (52)–(56) with

$$d_k \leq d \leq 0 \quad \text{for} \quad k = 1, \ldots, p \tag{64}$$

in the following. Integrals with $T_k \leq -1$ for some $1 \leq k \leq p$, where the corresponding ξ_k-integrations give rise to additional $1/\varepsilon$ poles, do not need to be computed explicitly[9] since, as we show in Appendix D, this kind of integrals can be eliminated by means of integration-by-parts identities. Thus, we can restrict ourselves to integrals with

$$T_k > -1 \quad \text{for} \quad k = 1, \ldots, p. \tag{65}$$

We note that, as a consequence of (64)–(65),

$$d = 0 \quad \Rightarrow \quad \tilde{e} > 0, \tag{66}$$

since $d = 0$ implies that there is a k with $1 \leq k \leq p$ and $d_k = 0$. Thus $\tilde{e} = (T_k + 1)/t_k > 0$ owing to (58) and (65).

5.2 Classification and counting of singularities

In the following, we consider sector integrals of the type (52)–(56) with (64)–(65). In order to extract the various $1/\varepsilon$ poles and $\ln(w)$ contributions, we first classify the FP integrations in (52)–(56) according to their singular behaviour.

- The FPs $\{\xi_k | 1 \leq k \leq p \text{ and } d_k = 0\}$, for which we can assume $t_k > 0$ and $T_k > -1$, are renamed as $\{z_1, \ldots, z_n\}$. As we see in Sect. 5.3, the corresponding n integrations give rise to singular contributions up to the order $\ln^n(w)$.
- The FPs $\{\xi_j | p + 1 \leq j \leq I - 1 \text{ and } T_j \leq -1\}$, for which we can assume $t_j = 0$, are renamed as $\{y_1, \ldots, y_m\}$. The corresponding m integrations give rise to poles up to the order $1/\varepsilon^m$.

[9] In practical calculations, we never encountered contributions of this type at the NLL level. However, in general we cannot exclude their presence, especially in view of an extension of the algorithm to the NNLL level or beyond.

- The remaining FPs, $\{\xi_i | T_i > -1 \text{ and if } 1 \le i \le p, d_i < 0\}$, are called $\{x_1, \ldots, x_l\}$. The l integrations over these FPs are free of singularities.

In general we have $0 \le l, m, n \le I-1$ and $l+m+n = I-1$. As a result of the above classification, the contribution from each subsector assumes the general form

$$\tilde{G} \equiv \tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = \\ = \int_0^1 \mathrm{d}^l \vec{x}\, P\Big(\vec{x}^{\,\vec{A}+\vec{\alpha}\varepsilon}\Big) \int_0^1 \mathrm{d}^m \vec{y}\, P\Big(\vec{y}^{\,\vec{\beta}\varepsilon - \vec{b} - 1}\Big) \int_0^1 \mathrm{d}^n \vec{z}\, P\Big(\vec{z}^{\,\vec{c}\tilde{e} - 1 + \vec{\gamma}\varepsilon}\Big) \frac{\tilde{g}(\vec{x};\vec{y};\vec{z})}{\Big[\tilde{\mathcal{F}}(\vec{x};\vec{y};\vec{z})\Big]^{\tilde{e}+L\varepsilon}} \tag{67}$$

with

$$\tilde{g}(\vec{x};\vec{y};\vec{z}) = \Gamma(e + L\varepsilon) \frac{\tilde{\mathcal{N}}(\vec{x})}{\Big[\tilde{\mathcal{U}}(\vec{x};\vec{y};\vec{z})\Big]^{f-(L+1)\varepsilon}} \tag{68}$$

and

$$\tilde{\mathcal{F}}(\vec{x};\vec{y};\vec{z}) = P\Big(\vec{x}^{\,\vec{a}}\Big)\, P\Big(\vec{z}^{\,\vec{c}}\Big)\, \tilde{f}_0(\vec{x};\vec{y};\vec{z}) + w \tilde{f}_1(\vec{x};\vec{y};\vec{z}) - \mathrm{i}\epsilon, \tag{69}$$

where $\tilde{\mathcal{U}}(\vec{0};\vec{0};\vec{0}) \neq 0$, $\tilde{\mathcal{F}}(\vec{0};\vec{0};\vec{0}) \neq 0$, and $\tilde{f}_0(\vec{0};\vec{0};\vec{0}) \neq 0$. The exponents that are associated to the FPs $\vec{x}$ and $\vec{y}$,

$$\begin{aligned} \vec{A} &= (A_1, \ldots, A_l), \qquad & \vec{a} &= (a_1, \ldots, a_l), \qquad & \vec{\alpha} &= (\alpha_1, \ldots, \alpha_l), \\ \vec{b} &= (b_1, \ldots, b_m), \qquad & \vec{\beta} &= (\beta_1, \ldots, \beta_m), \end{aligned} \tag{70}$$

are integer numbers and satisfy

$$A_i > -1, \qquad A_i > a_i \tilde{e} - 1, \qquad a_i \ge 0, \qquad b_j \ge 0, \tag{71}$$

by construction. The number n of FPs $\vec{z}$ depends on the maximal degree of singularity d. If $d < 0$ then $n = 0$, otherwise $n > 0$ and for the exponents of the FPs $\vec{z}$,

$$\vec{c}\,\tilde{e} = (c_1 \tilde{e}, \ldots, c_n \tilde{e}), \qquad \vec{\gamma} = (\gamma_1, \ldots, \gamma_n), \tag{72}$$

which are integer numbers, we have

$$c_k > 0, \qquad c_k \tilde{e} > 0. \tag{73}$$

Again, the exponents $\vec{A}$, $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are related to $\vec{T}$, $\vec{t}$ and $\vec{\tau}$ in an obvious way, $\mathcal{N}(\vec{x}, \vec{y}, \vec{z})$ has been decomposed into monomials with respect to the FPs $\vec{y}$ and $\vec{z}$ and the contribution from each monomial has been treated as a different subsector. Note that $n > 0$, or equivalently $d = 0$, implies $\tilde{e} > 0$, as already observed in (66).

5.3 Integration

Before we proceed, we rewrite (67)–(69) as

$$\tilde{G} = \int_0^1 \mathrm{d}^l\vec{x} \int_0^1 \mathrm{d}^m\vec{y}\, P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big) \int_0^1 \mathrm{d}^n\vec{z}\, h_0(\vec{x};\vec{y};\vec{z})\, \frac{P\Big(\vec{z}^{\,\vec{c}\tilde{e}-1+\vec{\gamma}\varepsilon}\Big)}{\left[P(\vec{z}^{\,\vec{c}}) + w h_1(\vec{x};\vec{y};\vec{z})\right]^{\tilde{e}+L\varepsilon}} \tag{74}$$

with

$$h_0(\vec{x};\vec{y};\vec{z}) = P\Big(\vec{x}^{\,\vec{A}-\tilde{e}\vec{a}+(\vec{\alpha}-L\vec{a})\varepsilon}\Big)\, \frac{\tilde{g}(\vec{x};\vec{y};\vec{z})}{\Big[\tilde{f}_0(\vec{x};\vec{y};\vec{z}) - \mathrm{i}\epsilon\Big]^{\tilde{e}+L\varepsilon}} \tag{75}$$

and

$$h_1(\vec{x};\vec{y};\vec{z}) = \frac{\tilde{f}_1(\vec{x};\vec{y};\vec{z}) - \mathrm{i}\epsilon}{\Big[\tilde{f}_0(\vec{x};\vec{y};\vec{z}) - \mathrm{i}\epsilon\Big]\, P(\vec{x}^{\,\vec{a}})}. \tag{76}$$

The singular integrations over the FPs $\vec{y}$ and $\vec{z}$ for a generic integral of the type (74) are performed in Appendix C to NLL accuracy. The resulting $1/\varepsilon$ poles and $\ln(w)$ contributions can be read off from (113). Expanding $h_0 \equiv h_0(\vec{x};\vec{y};\vec{z})$ in ε as

$$h_0 \equiv \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} h_0^{(j)} \tag{77}$$

and similarly $\Gamma(e+\varepsilon)$ and $\tilde{\mathcal{N}}$, we obtain

$$\begin{aligned}
\tilde{G} \overset{\mathrm{NLL}}{=} & \Big[P\Big(\vec{\beta}\Big)\, P(\vec{c})\Big]^{-1} (-1)^n \sum_{p\geq 0} \left(\frac{1}{\varepsilon}\right)^{m-p} \int_0^1 \mathrm{d}^l\vec{x} \Bigg\{ F_p(\vec{r})\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0^{(0)}(\vec{x};\vec{0};\vec{0})\mathrm{L}_{n+p}(w) \\
& + \Bigg[\theta(n-1)F_p(\vec{r})\mathrm{D}_{\vec{y}}^{\vec{b}}\Big[h_0^{(0)}(\vec{x};\vec{0};\vec{0})\Big[\ln\Big(h_1(\vec{x};\vec{0};\vec{0})\Big) + C_{\tilde{e}}\Big]\Big] \\
& - \sum_{j=1}^{n} c_j F_p(\vec{r}_{[j]})\Delta_{\vec{z}}^{\vec{0},j}\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0^{(0)}(\vec{x};\vec{0};\vec{0}) \\
& + F_{p-1}(\vec{r})\Bigg[\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0^{(1)}(\vec{x};\vec{0};\vec{0}) + \sum_{i=1}^{m} \beta_i \Delta_{\vec{y}}^{\vec{b},i} h_0^{(0)}(\vec{x};\vec{0};\vec{0})\Bigg]\Bigg]\mathrm{L}_{n+p-1}(w)\Bigg\},
\end{aligned} \tag{78}$$

where L_n is defined in (83) and

$$\begin{aligned}
h_0^{(0)} &= \Gamma^{(0)}(e) P\Big(\vec{x}^{\,\vec{A}-\tilde{e}\vec{a}}\Big)\, \frac{\tilde{\mathcal{N}}^{(0)}}{\Big(\tilde{f}_0 - \mathrm{i}\epsilon\Big)^{\tilde{e}}\, \tilde{\mathcal{U}}^{f}}, \\
\frac{h_0^{(1)}}{h_0^{(0)}} &= L\frac{\Gamma^{(1)}(e)}{\Gamma^{(0)}(e)} + \ln\Big(P\Big(\vec{x}^{\,\vec{\alpha}-L\vec{a}}\Big)\Big) + \frac{\tilde{\mathcal{N}}^{(1)}}{\tilde{\mathcal{N}}^{(0)}} + (L+1)\ln\tilde{\mathcal{U}} - L\ln\Big(\tilde{f}_0 - \mathrm{i}\epsilon\Big).
\end{aligned} \tag{79}$$

The derivative operator $\mathrm{D}_{\vec{y}}^{\vec{b}}$ and the subtraction operators $\Delta_{\vec{y}}^{\vec{b},i}$ and $\Delta_{\vec{z}}^{\vec{0},j}$ are defined in (95) and (96) and (108), respectively. The constant $C_{\tilde{e}}$ is the sum defined in (85). The vector $\vec{r} = (r_1, \ldots, r_n)$ has components

$$r_j = \frac{\gamma_j}{c_j} - L, \tag{80}$$

and $\vec{r}_{[j]}$ and $F_p(\vec{r})$ are defined in (89) and (90), respectively.

As observed at the end of Appendix C, the result (48) for massless integrals, can be obtained from the general result (78) for massive integrals in the special case $n = 0$. In the massive case, for the integrals (67)–(69), where the integrations over the FPs $\vec{z}$ are logarithmically singular, the $m + n$ integrations over the FPs $\vec{y}$ and $\vec{z}$ give rise to singularities up to the order $\varepsilon^{-m+p} \ln^{n+p}(w)$ with $p \geq 0$, as one can see in (78). Instead, for the asymptotic behaviour of non-logarithmically singular integrals we observe the following. For an integral (52)–(56) with $d > 0$ and (65), which is expressed in terms of logarithmically singular integrals of the type (67)–(69) using (61)–(62), each term in the sum (61) yields a contribution of order $\varepsilon^{-m+p} w^{-d} \ln^{n+p}(w)$ with $p \geq 0$. However, from (62), (75) and (78) one can see that these leading contributions are independent of r and thus cancel in the sum (61), i.e. to LL accuracy

$$
\begin{aligned}
& w^{-d} \sum_{r=0}^{d} \binom{d}{r} (-1)^r \tilde{G}(\vec{T}_r, \tilde{e}_r, \tilde{g}_r, \tilde{\mathcal{F}}) \\
& \quad \stackrel{\mathrm{LL}}{=} w^{-d} \tilde{G}(\vec{T}, \tilde{e} - d, \tilde{f}_1^{-d} \tilde{g}, \tilde{\mathcal{F}}) \sum_{r=0}^{d} \binom{d}{r} (-1)^r = 0.
\end{aligned}
\tag{81}
$$

Consequently, only the subleading contributions proportional to $C_{\tilde{e}_r}$ survive, i.e. contributions of order $\varepsilon^{-m+p} w^{-d} \ln^{n+p-1}(w)$ with $p \geq 0$.

The result (78) is expressed as l-dimensional integral over the FPs $\vec{x}$. Analogously to the massless case, the next-to-leading contributions involve one additional integration over the FPs y_i or z_j, which appears in the subtraction operator $\Delta_{\vec{y}}^{\vec{b},i}$ or $\Delta_{\vec{z}}^{\vec{0},j}$, respectively. Thus, the coefficients of the LLs and NLLs have been expressed in terms of convergent integrals with dimension l and $l + 1$. For negative kinematical invariants these integrals are well-defined. For positive kinematical invariants the analytic continuation is provided by the infinitesimal imaginary part $\mathrm{i}\epsilon$ contained in h_0 and h_1.

6 Discussion

The sector-decomposition algorithm that we have presented in Sects. 4 and 5 permits to reduce arbitrary massive (or massless) FP integrals to a sum of sector integrals of the type (67)–(69), where all FPs that can lead to UV or mass singularities are factorized. Each sector integral gives rise to a tower

$$
\sum_{j=0}^{J} \sum_{k=-j}^{\infty} a_{j,k} \, \varepsilon^k \ln^{j+k}(w),
\tag{82}
$$

involving $1/\varepsilon$ poles and logarithms. As discussed in Sect. 5.2, the FPs $y_1, \ldots, y_m$ and $z_1, \ldots, z_n$, which give rise to these singular contributions are easily identified by power counting. The maximal total power J of the resulting poles and logarithms is given by the number of these FPs $\vec{y}$ and $\vec{z}$, i.e. $J = m + n$. This simple counting permits to select the sector integrals that give rise to the LLs and NLLs by requiring that $J = 2L$ and $J \geq 2L - 1$, respectively, at L-loop level. Here we have assumed that the overall factor $\Gamma(e + L\varepsilon)$ in (15) does not yield $1/\varepsilon$ poles since the LLs and NLLs originate from diagrams involving mass singularities, i.e. diagrams with an exponent $e > 0$ for the mass-dependent denominator $\mathcal{F}$ in (15).

The singular contributions have been extracted from the integrals over the FPs $\vec{y}$ and $\vec{z}$ in (67)–(69) using the subtraction technique discussed in Sect. 3.3 and Appendix C [see (109)–(110)]. Each of these J integrations has been split into a singular part that has been solved analytically (giving rise to a pole or a logarithm) and a non-singular part that has not been integrated in (78). The contributions with total power j in the resulting tower (82) originate from the combinations of j' singular parts and $J - j'$ non-singular parts with $j \leq j' \leq J$. Their coefficients are thus expressed as integrals over $(J - j')$ remaining FPs y_i or z_k. Finally, together with these FPs, also the FPs $x_1, \ldots, x_l$, which do not give rise to singularities, remain to be integrated. In the general result (78), the coefficients of the leading ($j = J$) and next-to-leading ($j = J - 1$) contributions for the generic sector integral (67)–(69) are expressed as $(I - 1 - J)$- and $(I - J)$-dimensional integrals,[10] where $I = l + m + n + 1$ is the number of internal lines of the Feynman diagram. These remaining integrations are free of UV and mass singularities by construction.

For the L-loop corrections to processes with E external legs, which involve a maximal number $I = E + 3L - 3$ of internal lines, the coefficients of the LLs ($j = 2L$) and NLLs ($j = 2L-1$) consist of $(E+L-4)$- and $(E+L-3)$-dimensional integrals, respectively. For the most complicated Feynman diagrams that we have computed up to now [18,19], which correspond to 3- and 4-point functions at two loops, the NLLs require 2- and 3-dimensional integrals, respectively, and such integrals turn out to be sufficiently simple to be solved analytically by standard computer algebra programs such as MATHEMATICA[27].

To our knowledge, the algorithm we have presented is the only existing tool that permits to extract NLLs from arbitrary Feynman diagrams in a completely automated way. At present, the accuracy is limited only by the fact that the divergent integrals have been solved in NLL approximation. In order to extend the algorithm beyond the NLL level it is sufficient to solve these divergent integrations, which correspond to a well-defined class of standard integrals resulting from the sector decomposition, with higher accuracy.

The algorithm has been checked against all one-loop tensor integrals given in Ref. [24] and the two-loop master integrals of Ref. [5]. Its first applications were the two-loop calculations of the angular-dependent subset of the electroweak NLLs for arbitrary processes [18] and the complete set of NLLs for the electroweak-singlet massless fermionic form factor [19]. In the future, it can be applied to derive the two-loop electroweak logarithms for more complicated processes in order to check the existing resummation prescriptions, which still rely on arguments based on symmetric gauge theories. Alternatively, it can be used to extend the approach that has been used in Ref. [28] from the one- to the two-loop level, in order to derive the two-loop NLLs for generic processes within the spontaneously broken electroweak theory.

7 Conclusions

At energies far above the electroweak scale, $\sqrt{s} \gg M_{\mathrm{W}}$, the electroweak radiative corrections are dominated by large logarithms of s/M_{W}^2. These logarithms originate from

[10] The additional integration for the coefficients of the NLLs is associated with the subtraction operators $\Delta_{\vec{y}}^{\vec{b},i}$ and $\Delta_{\vec{z}}^{\vec{0},j}$ defined (96).

ultraviolet and mass singularities. In this paper, we have presented an algorithm that permits to extract these singularities and the corresponding large logarithms from arbitrary multi-loop Feynman integrals by using sector decomposition. We have elaborated the algorithm for the case of a single mass scale and a single energy scale but allowing for various different mass and energy parameters. This permits, in particular, to compute higher-order next-to-leading logarithmic electroweak corrections for processes involving various kinematical invariants of the order of hundreds of GeV and masses $M_{\mathrm{W}} \sim M_{\mathrm{Z}} \sim M_{\mathrm{H}} \sim m_{\mathrm{t}}$ of the order of the electroweak scale in the approximation where the masses of the light fermions are neglected.

We have provided explicit formulas for the extraction of the leading and next-to-leading mass singularities. At the next-to-leading level, the algorithm has been successfully checked against one-loop and two-loop results available in the literature, and first applications have already been published. This method will be useful for the calculation of next-to-leading logarithmic electroweak two-loop corrections and for the check of existing resummation prescriptions at this level.

The method can be extended beyond the next-to-leading level. To this end, a well-defined set of mass-singular integrals has to be solved in the required approximation.

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Appendix

A Notation and conventions

For logarithms we introduce the notation

$$\mathrm{L}_n(w) := \frac{\theta(n)}{n!} \ln^n(w). \tag{83}$$

The θ and the $\bar{\delta}$ functions with integer arguments are defined as

$$\theta(n) := \begin{cases} 1 & \text{for} \quad n \geq 0 \\ 0 & \text{for} \quad n < 0 \end{cases}, \qquad \bar{\delta}(n) := \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n \neq 0 \end{cases}. \tag{84}$$

In order to keep our results as compact as possible it is useful to define[11]

$$C_e := \sum_{j=1}^{e-1} \frac{1}{j}. \tag{85}$$

[11]Note that $C_e = \Psi(e) + \gamma_{\mathrm{E}}$, where $\Psi(e) = \Gamma'(e)/\Gamma(e)$.

Here and in the following all sums have to be understood as

$$\sum_{j=j_0}^{j_1} \equiv \theta(j_1 - j_0) \sum_{j=j_0}^{j_1} . \tag{86}$$

Similarly, we use the convention $\prod_{j=j_0}^{j_1} \ldots = 1$ for $j_0 > j_1$.

For vectors $\vec{z} = (z_1, \ldots, z_n)$ we introduce the shorthands

$$\int_0^1 \mathrm{d}^n \vec{z} := \int_0^1 \prod_{j=1}^{n} \mathrm{d}z_j, \qquad P(\vec{z}\,) := \prod_{j=1}^{n} z_j. \tag{87}$$

In the special case $\dim(\vec{z}) = n = 0$ the above expressions have to be understood as $\int_0^1 \mathrm{d}^0 \vec{z} = \int_0^1 \prod_{j=1}^{0} \mathrm{d}z_j = 1$ and $P(\vec{z}\,) = \prod_{j=1}^{0} z_j = 1$. Powers of vectors $\vec{z} = (z_1, \ldots, z_n)$ with vector exponents $\vec{a} = (a_1, \ldots, a_n)$ or scalar exponents b are defined as

$$\vec{z}^{\,\vec{a}+b} \equiv (z_1^{a_1+b}, \ldots, z_n^{a_n+b}). \tag{88}$$

The $(n-1)$-dimensional vector resulting from the omission of the jth component of an n-dimensional vector $\vec{z} = (z_1, \ldots, z_n)$ is denoted as

$$\vec{z}_{[j]} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n). \tag{89}$$

For vectors $\vec{r} = (r_1, \ldots, r_n)$ with real components and dimension $n \geq 0$ we introduce the function

$$F_p(\vec{r}\,) := \theta(p) \sum_{q_1, \ldots, q_n = 0}^{p} \bar{\delta}\Big(p - \sum_{j=1}^{n} q_j\Big) \prod_{j=1}^{n} r_j^{q_j} \tag{90}$$

involving all monomials of degree p. In the special case $n = 0$ we have $F_p(\vec{r}\,) = \bar{\delta}(p)$. Note also that $F_0(\vec{r}\,) = 1$ for $n \geq 0$.

B Massless integrals

In this section we compute FP integrals of the type (43), which result from the sector decomposition of massless loop integrals. In particular, we perform the singular integrations over the FPs $\vec{y}$ to NLL accuracy, i.e. including the leading and next-to-leading $1/\varepsilon$ poles.

Let us first consider one-dimensional integrals of the type

$$I = \int_0^1 \mathrm{d}y\, y^{\beta\varepsilon - b - 1} h_0(y) \tag{91}$$

with β real and $b \geq 0$ integer. We assume that the singular behaviour of the integrand at $y \to 0$ has been isolated in the term $y^{\beta\varepsilon - b - 1}$ such that h_0 is an analytic function free of poles and finite at $y = 0$, i.e. $h_0(0) \neq 0$. The $1/\varepsilon$ pole, which originates from an end-point singularity at $y = 0$, can be isolated by adding and subtracting from $h_0(y)$ the term

$$\hat{h}_0(y) = \sum_{k=0}^{b} y^k \mathrm{D}_y^k\, h_0(0) \quad \text{with} \quad \mathrm{D}_y^k\, h_0(y_0) = \frac{1}{k!} \frac{\partial^k h_0(y)}{\partial y^k}\bigg|_{y=y_0} . \tag{92}$$

To NLL accuracy we obtain

$$
\begin{aligned}
I &= \int_0^1 \mathrm{d}y\, y^{\beta\varepsilon-b-1}\hat{h}_0(y) + \int_0^1 \mathrm{d}y\, y^{\beta\varepsilon-b-1}\Big[h_0(y) - \hat{h}_0(y)\Big] \\
&= \sum_{k=0}^{b} \frac{1}{k-b+\beta\varepsilon}\,\mathrm{D}_y^k\, h_0(0) + \int_0^1 \mathrm{d}y\, y^{\beta\varepsilon-b-1}\Big[h_0(y) - \hat{h}_0(y)\Big] \\
&= \frac{1}{\beta\varepsilon}\,\mathrm{D}_y^b\, h_0(0) + \sum_{k=0}^{b-1} \frac{1}{k-b}\,\mathrm{D}_y^k\, h_0(0) + \int_0^1 \frac{\mathrm{d}y}{y^{b+1}}\Big[h_0(y) - \hat{h}_0(y)\Big] + \mathcal{O}(\varepsilon). \qquad (93)
\end{aligned}
$$

As usual in dimensional regularization, divergent integrals with $b > 0$ are defined via analytic continuation. The generalization to m-dimensional integrals is obtained by recursive application of (93). This yields

$$
\begin{aligned}
&\int_0^1 \mathrm{d}^m\vec{y}\, P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big)\, h_0(\vec{y}) \\
&\quad = \Big[P\Big(\vec{\beta}\Big)\Big]^{-1}\left\{\frac{1}{\varepsilon^m}\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0(\vec{0}) + \frac{1}{\varepsilon^{m-1}}\sum_{i=1}^{m}\beta_i \Delta_{\vec{y}}^{\vec{b},i} h_0(\vec{0}) + \mathcal{O}\left(\frac{1}{\varepsilon^{m-2}}\right)\right\}, \qquad (94)
\end{aligned}
$$

where the derivative operator $\mathrm{D}_{\vec{y}}^{\vec{b}}$ and the subtraction operator $\Delta_{\vec{y}}^{\vec{b},i}$ are defined as

$$
\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0(\vec{y}_0) = \left[\prod_{i=1}^{m}\frac{1}{b_i!}\right]\frac{\partial^{b_1}}{\partial y_1^{b_1}}\cdots\frac{\partial^{b_m}}{\partial y_m^{b_m}} h_0(\vec{y})\bigg|_{\vec{y}=\vec{y}_0}, \qquad (95)
$$

and

$$
\begin{aligned}
\Delta_{\vec{y}}^{\vec{b},i}\, h_0(\vec{y}_0) &= \sum_{k_i=0}^{b_i-1}\frac{1}{k_i-b_i}\Big[\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0(\vec{y}_0)\Big]_{b_i=k_i} \\
&\quad + \int_0^1 \frac{\mathrm{d}y_{\mathrm{i}}}{y_{\mathrm{i}}^{b_i+1}}\bigg\{\Big[\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0(y_{01},\dots,y_{0i-1},y_i,y_{0i+1},\dots,y_{0m})\Big]_{b_i=0} \\
&\qquad - \sum_{k_i=0}^{b_i} y_i^{k_i}\Big[\mathrm{D}_{\vec{y}}^{\vec{b}}\, h_0(\vec{y}_0)\Big]_{b_i=k_i}\bigg\}. \qquad (96)
\end{aligned}
$$

The m integrations over the FPs $\vec{y}$ in (94) yield leading and next-to-leading singularities of order $1/\varepsilon^m$ and $1/\varepsilon^{m-1}$, respectively. Note that the subtracted terms (96), which enter the coefficients of the next-to-leading poles, involve one-dimensional integrals over one of the FPs y_i. These integrals are free of the singularity at $y_i = 0$ since the corresponding terms within the curly brackets in (96) are of order $y_i^{b_i+1}$.

C Massive integrals

In this section we compute FP integrals of the type (74), which result from the sector decomposition of massive loop integrals. In particular, we focus on the singular integrations over the FPs $\vec{y}$ and $\vec{z}$. To be specific, we consider the generic integral

$$\int_0^1 \mathrm{d}^m \vec{y}\, P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big) \int_0^1 \mathrm{d}^n \vec{z}\, h_0(\vec{y};\vec{z})\, \frac{P\Big(\vec{z}^{\,\vec{c}\tilde{e}-1+\vec{\gamma}\varepsilon}\Big)}{\left[P(\vec{z}^{\,\vec{c}}) + w h_1(\vec{y};\vec{z})\right]^{\tilde{e}+L\varepsilon}}. \tag{97}$$

We assume m, n and $\tilde{e}$ integer with $m, n \geq 0$ and $\tilde{e} > 0$ if $n > 0$.[12] The exponents $\vec{\beta} = (\beta_1, \dots, \beta_m)$ and $\vec{b} = (b_1, \dots, b_m)$ are integer numbers. Moreover, we assume $b_i \geq 0$ such that the integrations over the FPs $\vec{y} = (y_1, \dots y_m)$ give rise to a singularity of order $1/\varepsilon^m$. The exponents $\vec{\gamma} = (\gamma_1, \dots, \gamma_n)$ and $\vec{c}\tilde{e} = (c_1\tilde{e}, \dots, c_n\tilde{e})$ are integer numbers with $c_j > 0$. The integrations over the FPs $\vec{z} = (z_1, \dots z_n)$ give rise to mass singularities that are regulated by the mass-to-energy ratio $w = M^2/s$.

All mass and UV singularities appear as end-point singularities at $\vec{y} \to \vec{0}$ or $\vec{z} \to \vec{0}$. As a result of sector decomposition, all FPs that are responsible for such singularities are isolated in the terms $P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big)$ and $P\Big(\vec{z}^{\,\vec{c}}\Big)$. Thus, h_0 and h_1, which are analytic functions corresponding to (75) and (76), are free of end-point singularities and finite at $\vec{y} \to \vec{0}$, $\vec{z} \to \vec{0}$, i.e. $h_0(\vec{0}, \vec{0}) \neq 0$ and $h_1(\vec{0}, \vec{0}) \neq 0$. We also assume that possible poles of h_0 and h_1 that approach the integration contour as $\mathrm{i}\epsilon \to 0$ can be avoided by appropriate contour deformations.

The integration is performed in the asymptotic limit

$$0 < w \ll 1 \tag{98}$$

to NLL accuracy, as explained in Sect. 2. In the following we present some intermediate results which correspond to special cases of (97) and have been used to compute the above integral in the most general case:

- Let us start with the special case $h_0 \equiv h_1 \equiv 1$ and $m = 0$. Here we define

$$B^{\tilde{e}}_{n,\vec{q}}(w) := \int_0^1 \mathrm{d}^n \vec{z}\, \frac{P(\vec{z}^{\,\tilde{e}-1})}{\left[P(\vec{z}) + w\right]^{\tilde{e}}} \prod_{j=1}^{n} \mathrm{L}_{q_j}(z_j), \tag{99}$$

  where we include combinations of logarithms which originate from the ε-expansion of the integrand in (97). For $\vec{q} = (q_1, \dots, q_n)$ with integer $q_j \geq 0$ we obtain

$$B^{\tilde{e}}_{n,\vec{q}}(w) \overset{\mathrm{NLL}}{=} (-1)^n \left[\mathrm{L}_{\tilde{n}}(w) + C_{\tilde{e}} \mathrm{L}_{\tilde{n}-1}(w)\right] \quad \text{with} \quad \tilde{n} = n + \sum_{j=1}^{n} q_j, \tag{100}$$

  where $C_{\tilde{e}}$ is defined in (85). For $n = 0$ the result (100) is trivial, since

$$B^{\tilde{e}}_0(w) = \frac{1}{(1+w)^{\tilde{e}}} = 1 + \mathcal{O}(w). \tag{101}$$

  For $n \geq 1$ and $\tilde{e} = 1$ we have a polylogarithm

$$\begin{aligned} B^1_{n,\vec{q}}(w) &= (-1)^{\tilde{n}-n+1} \mathrm{Li}_{\tilde{n}}\left(-\frac{1}{w}\right) \\ &= (-1)^n \mathrm{L}_{\tilde{n}}(w) + \mathcal{O}\left[\mathrm{L}_{\tilde{n}-2}(w)\right] + \mathcal{O}(w). \end{aligned} \tag{102}$$

[12] As noted at the end of Sect. 5.2, for $n > 0$ we need only the case $\tilde{e} > 0$.

For $n \geq 1$ and $\tilde{e} \geq 2$, (100) can be proven by induction using the recursive relation

$$B^{\tilde{e}}_{n,(q_1,\ldots,q_n)}(w) = B^{\tilde{e}-1}_{n,(q_1,\ldots,q_n)}(w) + \frac{1}{\tilde{e}-1}\Bigg[B^{\tilde{e}-1}_{n,(q_1,\ldots,q_{n-1},q_n-1)}(w) \\ - \bar{\delta}(q_n)B^{\tilde{e}-1}_{n-1,(q_1,\ldots,q_{n-1})}(w)\Bigg], \tag{103}$$

which follows from partial integration in z_n.

- Using (100) one can easily prove that

$$\int_0^1 \mathrm{d}^n\vec{z}\frac{P(\vec{z}^{\,\tilde{e}-1})}{(P(\vec{z}\,)+w)^{\tilde{e}}}\ \mathrm{L}_p\left(P\left(\vec{z}^{\,\vec{r}}\right)\right) \stackrel{\mathrm{NLL}}{=} (-1)^n F_p(\vec{r})\left[\mathrm{L}_{n+p}\left(w\right)+C_{\tilde{e}}\mathrm{L}_{n+p-1}\left(w\right)\right] \tag{104}$$

for any real vector $\vec{r}=(r_1,\ldots,r_n)$ and integer number p, where $F_p(\vec{r})$ is the combinatorial factor defined in (90).

- In the special case $h_0 \equiv h_1 \equiv 1$ and $\vec{b}=\vec{0}$ we obtain

$$\int_0^1 \mathrm{d}^m\vec{y}\,P\left(\vec{y}^{\,\vec{\beta}\varepsilon-1}\right)\int_0^1 \mathrm{d}^n\vec{z}\frac{P\left(\vec{z}^{\,\vec{c}\tilde{e}-1+\vec{\gamma}\varepsilon}\right)}{\left[P(\vec{z}^{\,\vec{c}})+w\right]^{\tilde{e}+L\varepsilon}} \\ \stackrel{\mathrm{NLL}}{=} \left[P\left(\vec{\beta}\right)P(\vec{c}\,)\right]^{-1}(-1)^n\sum_{p\geq 0}F_p(\vec{r}\,)\left(\frac{1}{\varepsilon}\right)^{m-p}\left[\mathrm{L}_{n+p}\left(w\right)+C_{\tilde{e}}\mathrm{L}_{n+p-1}\left(w\right)\right], \tag{105}$$

where the components of the vector $\vec{r}=(r_1,\ldots,r_n)$ read

$$r_j = \frac{\gamma_j}{c_j} - L. \tag{106}$$

The integration of the variables $y_1,\ldots,y_m$ in (105) is trivial and gives rise to a pole of order $1/\varepsilon^m$. The remaining integrand has to be expanded up to the needed order in ε. Then, after neglecting irrelevant terms of order w, it can be integrated with the help of (104).

- In the special case $m=0$, including logarithms as in (99), we obtain

$$\int_0^1 \mathrm{d}^n\vec{z}\,h_0(\vec{z}\,)\,\frac{P(\vec{z}^{\,\tilde{e}-1})}{\left[P(\vec{z}\,)+wh_1(\vec{z}\,)\right]^{\tilde{e}}}\prod_{j=1}^n \mathrm{L}_{q_j}(z_j) \stackrel{\mathrm{NLL}}{=} (-1)^n\Bigg\{h_0(\vec{0})\mathrm{L}_{\tilde{n}}\left(wh_1(\vec{0})\right) \\ + \left[h_0(\vec{0})C_{\tilde{e}} - \sum_{j=1}^n\bar{\delta}(q_j)\Delta^{\vec{0},j}_{\vec{z}}h_0(\vec{0})\right]\mathrm{L}_{\tilde{n}-1}\left(w\right)\Bigg\}, \tag{107}$$

where $\tilde{n}=n+\sum_{j=1}^n q_j$ and analogously to (96) the subtraction operator $\Delta^{\vec{0},j}_{\vec{z}}$ is defined as

$$\Delta^{\vec{0},j}_{\vec{z}}h_0(\vec{z}_0) = \int_0^1\frac{\mathrm{d}z_j}{z_j}\left[h_0(z_{01},\ldots,z_{0j-1},\,z_j,z_{0j+1},\ldots,z_{0n})-h_0(\vec{z}_0)\right]. \tag{108}$$

The above result has been obtained by means of the subtraction technique discussed in Sect. 3.3. Each integration has been split into singular and non-singular parts as

$$\int_0^1 \mathrm{d}z_j I(h_0,h_1;\vec{z}) = \int_0^1 \mathrm{d}z_j I(h_0,h_1;\vec{z})\Big|_{h_i\equiv h_i(z_1,\dots,z_{j-1},0,z_{j+1},\dots,z_n)}$$
$$+\int_0^1 \mathrm{d}z_j \left[I(h_0,h_1;\vec{z}) - I(h_0,h_1;\vec{z})|_{h_i\equiv h_i(z_1,\dots,z_{j-1},0,z_{j+1},\dots,z_n)}\right], \quad (109)$$

where $I(h_0,h_1;\vec{z})$ represents the integrand on the left-hand side of (107). The recursive application of this subtraction to all n integrations gives rise to various terms, which consist of combinations of l singular parts and $n-l$ non-singular parts with $0\leq l\leq n$. The LL and NLL contributions correspond to the terms with $l=n,n-1$, and read

$$\int_0^1 \mathrm{d}^n\vec{z}\, I(h_0,h_1;\vec{z}) \stackrel{\text{NLL}}{=} \int_0^1 \mathrm{d}^n\vec{z}\, I(h_0,h_1;\vec{z})\Big|_{h_i\equiv h_i(\vec{0})}$$
$$+\sum_{j=1}^n \int_0^1 \mathrm{d}^n\vec{z}\left[I(h_0,h_1;\vec{z})|_{h_i\equiv h_i(0,\dots,0,\,z_j,0,\dots,0)} - I(h_0,h_1;\vec{z})|_{h_i\equiv h_i(\vec{0})}\right]. \quad (110)$$

The first term on the right-hand-side of (107) originates from the first term on the right-hand-side of (110) and has to be expanded to NLL accuracy as

$$\mathrm{L}_{\tilde{n}}\left(wh_1(\vec{0})\right) \stackrel{\text{NLL}}{=} \mathrm{L}_{\tilde{n}}\left(w\right) + \ln\left(h_1(\vec{0})\right)\mathrm{L}_{\tilde{n}-1}\left(w\right). \quad (111)$$

The second term on the right-hand-side of (110) gives rise to the Δ contributions in (107). Here the function $\bar{\delta}(q_j)$ indicates that only terms with $q_j=0$ yield a NLL contribution.

- Using (107) it is easy to prove that for any real vector $\vec{r}=(r_1,\dots,r_n)$ and integer number p,

$$\int_0^1 \mathrm{d}^n\vec{z}\, h_0(\vec{z})\,\frac{P(\vec{z}^{\,\tilde{e}-1})}{\left[P(\vec{z}\,)+wh_1(\vec{z}\,)\right]^{\tilde{e}}}\mathrm{L}_p\left(P\left(\vec{z}^{\,\vec{r}}\right)\right) \stackrel{\text{NLL}}{=} (-1)^n\Bigg\{h_0(\vec{0})F_p(\vec{r})\mathrm{L}_{n+p}\left(wh_1(\vec{0})\right)$$
$$+\left[h_0(\vec{0})F_p(\vec{r})C_{\tilde{e}} - \sum_{j=1}^n F_p(\vec{r}_{[j]})\Delta_{\vec{z}}^{\vec{0},j}h_0(\vec{0})\right]\mathrm{L}_{n+p-1}\left(w\right)\Bigg\}, \quad (112)$$

where $\vec{r}_{[j]}$ and F_p are defined in (89) and (90), respectively.

Finally, combining (94) and (112) we arrive at the result

$$\int_0^1 \mathrm{d}^m\vec{y}\, P\left(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\right)\int_0^1 \mathrm{d}^n\vec{z}\, h_0(\vec{y};\vec{z})\,\frac{P\left(\vec{z}^{\,\vec{c}\tilde{e}-1+\vec{\gamma}\varepsilon}\right)}{\left[P(\vec{z}^{\,\vec{c}})+wh_1(\vec{y};\vec{z})\right]^{\tilde{e}+L\varepsilon}} =$$
$$\stackrel{\text{NLL}}{=} \left[P\left(\vec{\beta}\right)P(\vec{c})\right]^{-1}(-1)^n\sum_{p\geq 0}\left(\frac{1}{\varepsilon}\right)^{m-p}\Bigg\{F_p(\vec{r})\mathrm{D}_{\vec{y}}^{\vec{b}}h_0^{(0)}(\vec{0};\vec{0})\mathrm{L}_{n+p}\left(w\right)$$

$$+ \left[\theta(n-1)F_p(\vec{r})\mathrm{D}^{\vec{b}}_{\vec{y}}\left[h_0^{(0)}(\vec{0};\vec{0})\left[\ln\Big(h_1(\vec{0};\vec{0})\Big)+C_{\tilde{e}}\right]\right]\right.$$

$$-\sum_{j=1}^{n} c_j F_p(\vec{r}_{[j]})\Delta^{\vec{0},j}_{\vec{z}}\mathrm{D}^{\vec{b}}_{\vec{y}}h_0^{(0)}(\vec{0};\vec{0})$$

$$\left.+F_{p-1}(\vec{r})\left[\mathrm{D}^{\vec{b}}_{\vec{y}}h_0^{(1)}(\vec{0};\vec{0})+\sum_{i=1}^{m}\beta_i\Delta^{\vec{b},i}_{\vec{y}}h_0^{(0)}(\vec{0};\vec{0})\right]\right]\mathrm{L}_{n+p-1}(w)\Bigg\}, \tag{113}$$

where

$$h_0 \equiv \sum_{j=0}^{\infty}\frac{\varepsilon^j}{j!}h_0^{(j)} \tag{114}$$

is the ε-expansion[13] of the function $h_0 \equiv h_0(\vec{y};\vec{z})$. The vector $\vec{r}=(r_1\ldots,r_n)$, $\vec{r}_{[j]}$, and F_p are defined in (106), (89), and (90), respectively. Moreover, the derivative operator $\mathrm{D}^{\vec{b}}_{\vec{y}}$ and the subtraction operators $\Delta^{\vec{b},i}_{\vec{y}}$ and $\Delta^{\vec{0},j}_{\vec{z}}$ are defined in (95), (96) and (108), respectively. In particular, we have

$$\Delta^{\vec{0},j}_{\vec{z}}\mathrm{D}^{\vec{b}}_{\vec{y}}h_0^{(0)}(\vec{0};\vec{0}) = \int_0^1\frac{\mathrm{d}z_j}{z_j}\left[\mathrm{D}^{\vec{b}}_{\vec{y}}h_0^{(0)}(\vec{0};0,\ldots,z_j,\ldots,0)-\mathrm{D}^{\vec{b}}_{\vec{y}}h_0^{(0)}(\vec{0};\vec{0})\right]. \tag{115}$$

The $m+n$ integrations over the FPs $\vec{y}$ and $\vec{z}$ in (113) yield leading and next-to-leading singularities of order $\varepsilon^{-m+p}\ln^{n+p}(w)$ and $\varepsilon^{-m+p}\ln^{n+p-1}(w)$, respectively. Similarly to the massless case, the subtracted Δ terms which enter the coefficients of the next-to-leading poles, involve convergent one-dimensional integrals over one of the FPs y_i or z_j.

We note that in the trivial case $n=0$ the integral (97) simplifies to

$$\int_0^1\mathrm{d}^m\vec{y}\,P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big)\frac{h_0(\vec{y})}{[1+wh_1(\vec{y})]^{\tilde{e}+L\varepsilon}} = \int_0^1\mathrm{d}^m\vec{y}\,P\Big(\vec{y}^{\,\vec{\beta}\varepsilon-\vec{b}-1}\Big)h_0(\vec{y})+\mathcal{O}(w). \tag{116}$$

Since no singularity is regulated by the mass term w, in the asymptotic limit $w\to 0$ (116) corresponds to the massless integral that is obtained by setting $w=0$ in the integrand. Indeed, in the special case $n=0$, the general result (113) for massive integrals is equivalent to the result (94) for massless integrals, as one can easily verify using $F_p(\vec{r})=\bar{\delta}(p)$ for $\dim(\vec{r})=n=0$.

D Factorization of $1/\varepsilon$ poles from massive integrals

In this section, we consider again the generic sector integrals (52)–(56), which have been computed in Sect. 5 assuming (65), such that all singularities resulting from the integrations over the FPs $\xi_1,\ldots,\xi_p$, i.e. the FPs that have been factorized in (54), are regulated by the mass term w in (54). Here we discuss the more general case where $T_k\le -1$ for some $1\le k\le p$ and the corresponding ξ_k-integrations yield additional $1/\varepsilon$ poles. As we have already anticipated, this kind of integrals can be eliminated by means of (recursive) integration by parts in the variables ξ_k, which permits to increase the exponents $T_k\le -1$ until the $1/\varepsilon$ poles are factorized and only integrals respecting (65) remain.

[13]The leading term $h_0^{(0)}(\vec{y};\vec{z})$ of this ε expansion can be of order $1/\varepsilon^s$.

In the following we consider sector integrals (52)–(56) with (64) assuming that possible non-logarithmic singularities have been eliminated as described in Sect. 5.1. An integration by parts in the variable ξ_k yields

$$\tilde{G}(\vec{T},\tilde{e},\tilde{g},\tilde{\mathcal{F}}) = \frac{1}{T_k+1+\tau_k\varepsilon}\int_0^1 \mathrm{d}^{I-1}\vec{\xi}\, P\Big(\vec{\xi}^{\vec{T}+\vec{\tau}\varepsilon}\Big)\left[\delta(1-\xi_k)-\xi_k\partial_{\xi_k}\right]\left(\frac{\tilde{g}}{\tilde{\mathcal{F}}^{\tilde{e}+L\varepsilon}}\right), \quad (117)$$

where the $\vec{\xi}$-dependence of $\tilde{g}$ and $\tilde{\mathcal{F}}$ is implicitly understood. The $\delta(1-\xi_k)$ contribution on the right-hand side represents the boundary term at $\xi_k = 1$, whereas the boundary term at $\xi_k = 0$ vanishes.[14] The remaining $\xi_k\partial_{\xi_k}$ contribution being proportional to ξ_k has the desired property to increase the power T_k associated with the FP ξ_k. However, as one can see from

$$\xi_k\partial_{\xi_k}\left(\frac{\tilde{g}}{\tilde{\mathcal{F}}^{\tilde{e}+L\varepsilon}}\right) = -\frac{(\tilde{e}+L\varepsilon)\tilde{g}}{\tilde{\mathcal{F}}^{\tilde{e}+1+L\varepsilon}}\left\{P\Big(\vec{\xi}^{\vec{t}}\Big)\left[t_k+\xi_k\partial_{\xi_k}\right]\tilde{f}_0 + w\xi_k\partial_{\xi_k}\tilde{f}_1\right\} + \frac{\xi_k\partial_{\xi_k}\tilde{g}}{\tilde{\mathcal{F}}^{\tilde{e}+L\varepsilon}}, \quad (118)$$

the degrees of singularity $d_j = \tilde{e}-(T_j+1)/t_j$ of the various FP integrations $1 \le j \le p$ can increase for the term proportional to $w\xi_k\partial_{\xi_k}\tilde{f}_1$, where the increase of $\tilde{e}$ is not compensated by the factor $P\Big(\vec{\xi}^{\vec{t}}\Big)$ in the numerator. In order to preserve (64), this must be avoided by rewriting

$$w\xi_k\partial_{\xi_k}\tilde{f}_1 = \frac{\xi_k(\partial_{\xi_k}\tilde{f}_1)}{\tilde{f}_1-\mathrm{i}\epsilon}\left[\tilde{\mathcal{F}} - P\Big(\vec{\xi}^{\vec{t}}\Big)\tilde{f}_0\right], \quad (119)$$

in (118). The integration by parts in the FP ξ_k combined with (119) results in a sum of three contributions,

$$\tilde{G}(\vec{T},\tilde{e},\tilde{g},\tilde{\mathcal{F}}) = \frac{1}{T_k+1+\tau_k\varepsilon}\sum_{r=1}^{3}\tilde{G}(\vec{T}_r,\tilde{e}_r,\tilde{g}_r,\tilde{\mathcal{F}}), \quad (120)$$

that have the same structure as the original integral with

$$\begin{aligned}
\tilde{e}_1 &= \tilde{e}, \qquad \vec{T}_1 = \vec{T}, \qquad \tilde{g}_1 = \delta(1-\xi_k)\,\tilde{g},\\
\tilde{e}_2 &= \tilde{e}+1, \qquad \vec{T}_2 = \vec{T}+\vec{t}, \qquad \tilde{g}_2 = (\tilde{e}+L\varepsilon)\,\tilde{g}\left[t_k - \frac{\xi_k(\partial_{\xi_k}\tilde{f}_1)}{\tilde{f}_1-\mathrm{i}\epsilon}+\xi_k\partial_{\xi_k}\right]\tilde{f}_0,\\
\tilde{e}_3 &= \tilde{e}, \qquad \left(\vec{T}_3\right)_j = T_j+\delta_{jk}, \qquad \tilde{g}_3 = (\tilde{e}+L\varepsilon)\frac{(\partial_{\xi_k}\tilde{f}_1)}{\tilde{f}_1-\mathrm{i}\epsilon}\tilde{g} - \partial_{\xi_k}\tilde{g}. \qquad (121)
\end{aligned}$$

The exponent T_k associated with the FP ξ_k grows in all terms, i.e. $(\vec{T}_r)_k > T_k$, apart from the boundary term $\tilde{G}(\vec{T}_1,\tilde{e}_1,\tilde{g}_1,\tilde{\mathcal{F}})$, where the ξ_k integration is irrelevant. Thus, in order to eliminate all integrals that do not satisfy (65), it is in principle sufficient to iterate the integration by parts to all FPs ξ_k with $T_k \le -1$. However, since our aim is to reduce by one the number of integrations every time that we extract a singularity, care must be taken when $1/\varepsilon$ poles are extracted through the overall factor $1/(T_k+1+\tau\varepsilon)$ in (120), i.e. every time that we perform an integration by parts in a FP ξ_k with $T_k = -1$. In practice,

[14] This holds for arbitrary values of T_k within dimensional regularization, i.e. also for $T_k < -1$ where the boundary term at $\xi_k = 0$ is divergent for $\varepsilon = 0$.

as we show below, it is possible to restrict the use of (120)–(121) to the cases where $\tilde{e} = 0$ and $\partial_{\xi_k}\tilde{g} = 0$ if $T_k = -1$, such that $\tilde{G}(\vec{T}_r, \tilde{e}_r, \tilde{g}_r, \tilde{\mathcal{F}}) = \mathcal{O}(\varepsilon)$ for $r = 2,3$ and the only $1/\varepsilon$ singularity results from the boundary term $\tilde{G}(\vec{T}_1, \tilde{e}_1, \tilde{g}_1, \tilde{\mathcal{F}})$.

Let us now outline our general strategy to eliminate all contributions that do not satisfy (64) and/or (65). Starting from a generic sector integral (52)–(56) we proceed as follows:

1. If non-logarithmic singularities are present, i.e. $d > 0$, we eliminate them by means of (61)–(62), such that the resulting integrals satisfy (64).

2. If there is a FP ξ_k with $1 \le k \le p$ and $T_k < -1$, this FP is integrated by parts using (120)–(121). This step is repeated until all integrals satisfy $T_k \ge -1$ for $1 \le k \le p$. Here no $1/\varepsilon$ pole appears.

3. For the remaining integrals involving a ξ_k with $1 \le k \le p$ and $T_k = -1$ we have $\tilde{e} = d_k \le d \le 0$ as a result of step 1 and owing to (58). If $\tilde{e} < 0$, then
$$\tilde{\mathcal{F}}^{-(\tilde{e}+L\varepsilon)} = \tilde{\mathcal{F}}^{-L\varepsilon} \left\{ \left[P\left(\vec{\xi}^{\,\vec{t}}\right) \tilde{f}_0 \right]^{-\tilde{e}} + \mathcal{O}(w) \right\}, \tag{122}$$
and the contributions of order w can be omitted. Thus, we can rewrite the corresponding integral as
$$\tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = \tilde{G}(\vec{T} - \tilde{e}\vec{t}, 0, \tilde{g}\tilde{f}_0^{-\tilde{e}}, \tilde{\mathcal{F}}), \tag{123}$$
and the integral on the right-hand side, with exponents $\vec{T} \to \vec{T} - \tilde{e}\vec{t}$, already satisfies (65) as a consequence of step 2 ($T_k \ge -1$) and $\tilde{e} < 0$.

4. Finally, we have to consider the integrals with $\tilde{e} = 0$ and a ξ_k with $1 \le k \le p$ and $T_k = -1$. In this case, we split $\tilde{g}$ as
$$\tilde{g}(\vec{\xi}) = \tilde{g}_k(\vec{\xi}) + \Delta\tilde{g}_k(\vec{\xi}) \quad \text{with} \quad \tilde{g}_k(\vec{\xi}) = \tilde{g}(\xi_1, \ldots, \xi_{k-1}, 0, \xi_{k+1}, \ldots, \xi_{I-1}) \tag{124}$$
and treat these two contributions separately, i.e.
$$\tilde{G}(\vec{T}, \tilde{e}, \tilde{g}, \tilde{\mathcal{F}}) = \tilde{G}(\vec{T}, \tilde{e}, \tilde{g}_k, \tilde{\mathcal{F}}) + \tilde{G}(\vec{T}, \tilde{e}, \Delta\tilde{g}_k, \tilde{\mathcal{F}}). \tag{125}$$
On the one hand, for the contribution $\tilde{G}(\vec{T}, \tilde{e}, \tilde{g}_k, \tilde{\mathcal{F}})$ we can safely perform an integration by parts in ξ_k since, as discussed above, owing to $\partial_{\xi_k}\tilde{g}_k(\vec{\xi}) = 0$ and $\tilde{e} = 0$, only the boundary term effectively contributes to the resulting $1/\varepsilon$ singularity. On the other hand, the remaining contribution is free of ξ_k singularities owing to
$$\lim_{\xi_k \to 0} \xi_k^{-1} \Delta\tilde{g}_k(\vec{\xi}) \;=\; \partial_{\xi_k}\tilde{g}(\vec{\xi})\Big|_{\xi_k=0}, \tag{126}$$
and can thus be written as
$$\tilde{G}(\vec{T}, \tilde{e}, \Delta\tilde{g}_k, \tilde{\mathcal{F}}) = \tilde{G}(\vec{T}', \tilde{e}, \xi_k^{-1}\Delta\tilde{g}_k, \tilde{\mathcal{F}}), \tag{127}$$
where $\vec{T}' = (T_1, \ldots, T_{k-1}, 0, T_{k+1}, \ldots, T_{I-1})$. The power T_k of ξ_k has been reduced in both contributions without generating unwanted $1/\varepsilon$ poles. At the end, if other FPs ξ_j with $1 \le j \le p$ and $T_j = -1$ are still present this step has to be iterated until all integrals fulfil (65).

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