Multiloop Tadpoles

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We present a semi-numerical approach which allows the calculation of four-loop tadpole diagrams. The method is introduced by presenting the results for a scalar four-loop diagram. Finally, the application of the method to the calculation of a physical quantity like the four-loop vacuum polarization in QCD is discussed.

1. Introduction

The great success of modern high precision phenomenology would not have been possible without the development of a variety of calculational methods for the evaluation of Feynman diagrams. A basic ingredient which enters at many quite different stages are the vacuum diagrams, also called tadpole diagrams. They appear as factorizible parts of complicated Feynman diagrams or directly build up the diagrammatic representation of a quantity of physical interest like, for example, the low momentum expansion of the vacuum polarization. There are also expansion techniques [1] which transform complicated diagrams into a combination of simpler massless diagrams and tadpole diagrams. Since the latter manifest themselves in a variety of applications, the detailed control over this important building block is mandatory. Furthermore, given the increasing resolution of modern experiments, one is forced to also push ahead the efforts on the theory side, calculating higher orders in perturbation theory to be able to keep the theoretical uncertainties below the experimental precision.

The results of the evaluation of a complete set of master diagrams for tadpoles up to the three-loop order, including an algorithm to reduce an arbitrary vacuum diagram to this set of master integrals, are available [2]. This algorithm is implemented in the computer algebra package `MATAD` [3] written in `FORM` [4]. The obvious next step is the treatment of four-loop tadpole diagrams. In many applications the method of Laporta [5] is at present a highly useful approach for the treatment of multiloop diagrams. In contrast, we followed a different strategy which was already successfully applied in [6]. In the next sections we will use the calculation of a scalar tadpole as an example. Then we will indicate how to apply the same technique to the vacuum polarization in QCD.

2. Method

The idea of this approach is the reduction of the number of analytical integrations by one, having then to deal with three-loop diagrams. The remaining one-dimensional integration is performed numerically. The integrand is obtained by cutting one line of the four-loop tadpole, thus obtaining a three-loop self-energy diagram. Since there is no other four-momentum involved, the self-energy diagram only depends on the momentum p^2 flowing through the cut line. Thus one is able to immediately perform the angular part of the remaining loop integration and is left with a one-dimensional integration over the modulus of the momentum p. Following this procedure, the calculation of the four-loop tadpole is transformed to the problem of evaluating a three-loop self-energy.

These three-loop diagrams can be treated with the help of asymptotic expansions. Specifically we obtain the self-energy in the low- and high-energy limit. These two limits can be combined in a Pade-approximation, a rational function with the same low- and high-energy expansion as the exact self-energy. It had been shown in [6] and [7]

$\times 10^{-5}$		high energy					
		3	4	5	6	7	8
low energy	3	-191.42436	-6.01348	0.44044	0.09165	0.00151	-0.01738
	4	-5.15001	1.94591	-0.10397	-0.00344	0.00052	0.00029
	5	-16.06059	0.74717	0.01367	-0.00101	0.00388	-0.00025

Table 1
Difference between the known result 1.348948021709708... by Laporta and our method taking cut 1. The dependence on the number of low- and high-energy input terms is given. Note that each value has to be multiplied by a factor of 10^{-5}.

$\times 10^{-5}$		high energy					
		3	4	5	6	7	8
low energy	3	-579.78184	8.92004	-0.08593	0.01259	0.00456	0.00010
	4	-186.75432	1.21461	-0.04923	-0.00242	-0.00008	-0.00004
	5	-41.00154	0.53958	-0.10894	0.00332	0.00009	-0.00002

Table 2
Difference between the known result 1.348948021709708... by Laporta and our method taking cut 2. The dependence on the number of low- and high-energy input terms is given. Note that each value has to be multiplied by a factor of 10^{-5}.

that this way of reconstructing the general dependence on the external momentum leads to a very good agreement with the full analytical result.

3. A Scalar Four-Loop Tadpole

In the following we will consider the diagram shown in figure 1, a scalar tadpole diagram with

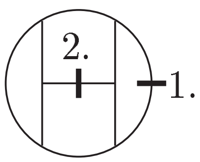

Figure 1. Scalar four-loop diagram with all lines massive. The bold lines indicated by the numbers represent two possible cuts of the diagram, leading to different three-loop self-energy diagrams.

all lines representing a massive propagator. Two independent cuts are taken at the bold lines indicated by number 1. and 2. In this way we obtain two different three-loop self-energy diagrams which have to be evaluated according to section 2. The calculation of a physical quantity involves a large number of diagrams and the set-up thus makes use of several computer algebra packages. The program `EXP` [8] extracts the subdiagrams needed for the asymptotic expansions. After the application of the expansion procedures, the resulting integrals are tadpoles with propagators of one mass scale in combination with massless propagator type integrals. Their computation can be performed using the program packages `MATAD` and `MINCER` [9]. We obtain the low- and high-energy expansion of the diagrams and construct the Pade approximation. After performing the final numerical integration, we find the results summarized in tables 1 and 2. The results are compared to the value 1.348948021709708... given by Laporta in [10]. The numbers (multiplied by 10^{-5}) in the tables 1 and 2 give the difference between the result by Laporta and our method in its dependence on the number of low- and high-energy input terms. One observes that these values converge quite fast towards the precisely known result. Already three terms in both the low- and the high-energy expansion lead to an agreement better than 1%. This also shows that the final result does not depend on the line where the cut is performed, although the speed of convergence differs for the two cases. Given these results, we conclude that this semi-numerical method works very well and can be applied to the calculation of a generic set of tadpole diagrams.

4. Vacuum Polarization

At present our major goal is the calculation of the low energy expansion of the four-loop vacuum polarization in the framework of QCD. The Taylor expansion of the relevant photon self-energy leads to a series in the external momentum q^2, times a set of four-loop tadpole diagrams. On the four-loop level this amounts to about 700 diagrams. Facing this number of diagrams, one has to use automatized methods. We use the program `QGRAF` [11] to generate the diagrams. In figure 2 the implementation is indicated. The cut is always located next to the same external vertex, thus generating vertex type diagrams shown on the right side of figure 2. By using `EXP` to treat the output of `QGRAF`, it is straight forward to implement the expansion with respect to the external momentum q and the low- and high energy expansions with respect to the loop momentum p.

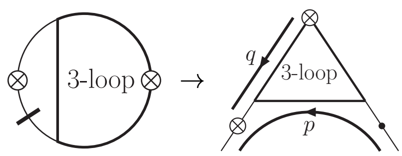

Figure 2. General idea how diagrams are generated

The following steps can, in principle, be performed in the same way as described in the previous section for the scalar example. In contrast to the scalar diagram in figure 1, where we are dealing with finite contributions at all stages, the sum of the whole set of four-loop QCD diagrams obtained as described still exhibits some divergencies. These divergencies cancel after the final integration and the full renormalization. The same approach as presented in section 3 would lead to a numerical cancellation of these divergencies. But as the sum of all QCD diagrams Π_n on the right hand side in figure 2 has the same divergence structure as the quark self-energy in QCD, one could remove these by multiplying with the quark field renormalization Z_2 in the appropriate way:

$$\overline{\Pi}_3 = \left[\frac{1}{Z_2}\left(\quad + \quad + \cdots + \quad + \cdots + \quad + \cdots \right)\right]\Big|_{\alpha_s^3}. \tag{1}$$

The low- and high-energy expansion of the α_s^3 contribution $\overline{\Pi}_3$ is now finite and we are able to apply the procedure described in section 3 to get a finite numerical four-loop result. It is left to correct for the multiplication with Z_2 in equation (1). This is achieved by adding the analytically known low-energy expansion up to the three-loop or the contributions up to order α_s^2 respectively with the appropriate combination of Z_2 shown in equation

$$\Pi_3 = Z_2\left(\int dp\, \overline{\Pi}_3(p^2) + [\frac{1}{Z_2}(\quad + \quad + \cdots + \quad + \ldots)]|_{\leq\alpha_s^2}\right). \tag{2}$$

To obtain a more compact expression this can be rewritten as follows:

$$\Pi_3 = \int dp\, \overline{\Pi}_3(p^2) + \left[\left(1 - \frac{1}{Z_2}\right)\left(\quad + \quad + \cdots + \quad + \ldots\right)\right]\Big|_{\alpha_s^3}. \tag{3}$$

5. Status

At present we have recalculated the first physically interesting moment of the vacuum polarization up to the three-loop level. By taking five terms of the low- and high-energy expansions respectively, we reproduce the known results [7] to an accuracy at the per-mill level. The evaluation of the four-loop diagrams is in progress. Given the same number of terms in the expansions, we expect a four-loop result of an accuracy in the order of 1%.

6. Summary

We have described a semi-numerical method for the calculation of four-loop tadpole diagrams. It has been shown that this framework leads to good agreement with already available results. The calculation of the four-loop vacuum polarization is under way.

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