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Precision Measurements and CKM Unitarity

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Determinations of $|V_{ud}|$ and $|V_{us}|$ along with their implications for the unitarity test $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$ are discussed. The leading two loop radiative corrections to neutron β-decay are evaluated and used to derive a refined relationship $|V_{ud}|^2 \tau_n(1 + 3g_A^2) = 4908(4)$sec. Employing $|V_{ud}| = 0.9740(5)$ from superallowed nuclear decays and the measured neutron lifetime, $\tau_n = 885.7(7)$sec leads to the precise prediction $g_A = 1.2703(8)$ which is compared with current direct experimental values. Various extractions of $|V_{us}|$ are described and updated. The long accepted Particle Data Group value of $|V_{us}|$ from fitted K_{e3} decay rates suggests a deviation from CKM unitarity but it is contradicted by more recent experimental results which confirm unitarity with good precision. An outlook for possible future advances is given.

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I. INTRODUCTION

The study of nuclear beta decays played an important historical role in unveiling universal properties of weak charged current interactions and in helping to establish the $SU(2)_L \times U(1)_Y$ Standard Model of electroweak unification. In the limit of neglecting electroweak loop corrections, a special subset of such decays, the superallowed $0^+ \to 0^+$ Fermi transitions, depend only on the vector current which is conserved and, therefore, not renormalized by strong interactions at $q^2 \simeq 0$. Hence, they are ideal for extracting $|V_{ud}|$, a cornerstone of the CKM (Cabibbo-Kobayashi-Maskawa [1, 2]) three generation quark mixing matrix,

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{1}$$

Indeed, those decays currently provide the very precise determination [3]

$$|V_{ud}| = 0.9740(5) \qquad (0^+ \to 0^+ \ \beta\text{-decays}), \tag{2}$$

which we will discuss later in this paper. Combining the value in eq. (2) with knowledge about $|V_{us}|$ from kaon and Hyperon decays along with the fact that $|V_{ub}|^2 \simeq 2.1(3) \cdot 10^{-5}$ [4] is negligibly small, allows one to confront the CKM unitarity relation,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \tag{3}$$

at a high precision level.

That prediction has been tested to about $\pm 0.15\%$, an impressive accomplishment. At that level, it has confirmed the presence of very large, $\sim 4\%$, Standard Model loop corrections in the extraction of $|V_{ud}|^2$ from the data [5, 6]. However, for many years, a small deviation, $\sim 2\sigma$, from exact unitarity has been persistently observed. That issue has been further clouded by various distinct $|V_{us}|$ determinations which seem to be inconsistent with one another. These problems are addressed later in some detail, when new K_{e3} results fully consistent with unitarity are also discussed. Of course, if a real deviation from unitarity expectations is seen, it signals the presence of as yet unaccounted for new physics beyond the Standard Model, an exciting prospect. Alternatively, if unitarity is respected, constraints on new physics can be implied. However, $|V_{ud}|$ and $|V_{us}|$ must be thoroughly scrutinized both theoretically and experimentally before conclusions are drawn.

Neutron beta decay, $n \to pe^-\overline{\nu}_e$, has not until recently been prominent in efforts to determine $|V_{ud}|$ and test unitarity. It depends on both vector and axial-vector charged current interactions. The latter are renormalized by strong interactions at $q^2 \simeq 0$. The size of that effect is parameterized by $g_A \equiv G_A/G_V$, a fundamental quantity in its own right. Indeed, the value of g_A, which has grown over time from about 1.23 to 1.27, is important for predicting the expected solar neutrino flux [7], light element abundances from primordial nucleosynthesis [8], the spin content of nucleons [9, 10] and for testing the Goldberger-Treiman relation [11]. A byproduct of the analysis in this paper will be to provide a very precise determination of g_A that can be compared with more direct neutron decay asymmetry measurements of that important parameter or to refine the above mentioned applications.

As new more intense neutron facilities turn on, experimental measurements of both the neutron lifetime τ_n and g_A from the electron asymmetry in polarized neutron beta decay are expected to become much more precise [12]. Indeed, combining determinations of those two quantities, can yield $|V_{ud}|$ with an anticipated uncertainty competitive with the error in eq. (2), i.e. dominated by theory. In preparation for those improvements, we present in this paper a relationship among $|V_{ud}|$, τ_n and g_A which includes one and some dominant two loop quantum corrections. It can be used to determine g_A from τ_n and the $|V_{ud}|$ input from superallowed nuclear β-decays or as an independent measure of $|V_{ud}|$ using τ_n and neutron decay asymmetry determinations of g_A.

Our plan is as follows: in section II, we update the radiative corrections to neutron decay by incorporating the $\mathcal{O}(\alpha^2)$ effects due to leading logs, (some) small next-to-leading logs and Coulombic effects. The last of these has been considered previously, but with the wrong sign. We take this opportunity to correct that longstanding error. In section III, we review and update (slightly) the extraction of $|V_{ud}|$ from superallowed β-decays. Using that value of $|V_{ud}|$ along with the neutron lifetime τ_n we derive in section IV a very precise prediction for g_A and compare it with direct asymmetry measurements of that parameter. Then, in section V, we review and update various determinations of $|V_{us}|$ and point out inconsistencies among them. The main problem stems from old K_{e3} decay rates obtained from fitted PDG studies and is suggestive of errors in some longstanding (accepted) kaon decay properties. Indeed, recent K_{e3} results from Brookhaven and Fermilab experiments confirm significant errors in the old charged and neutral kaon decay branching ratios and lead to values of $|V_{us}|$ fully consistent with unitarity. Implications of a unitarity violation or confirmation in eq. (3) are briefly discussed and an outlook for future advances is given in section VI.

II. RADIATIVE CORRECTIONS TO NEUTRON DECAY

Our analysis of the radiative corrections to neutron beta decay builds on the results of earlier studies, particularly the classic work by Wilkinson [13]. They included $\mathcal{O}(\alpha)$ radiative corrections as well as effects due to the final state electromagnetic ep interaction embodied in the Fermi function. A number of other small corrections from proton recoil, finite nucleon size etc. have also been examined [14–16].

In the Standard Model, one renormalizes the beta decay amplitude using [17–22]

$$G_\mu = 1.16637(1) \cdot 10^{-5}\ \mathrm{GeV}^{-2}, \tag{4}$$

the Fermi constant as obtained from the muon total decay rate (inverse lifetime). In that way, ultraviolet divergences as well as radiative corrections common to both decay amplitudes are absorbed into G_μ. The remaining loop differences and bremsstrahlung effects can then be factorized into an overall 1+RC correction to the neutron lifetime, τ_n

$$\frac{1}{\tau_n} = \frac{G_\mu^2 \, |V_{ud}|^2}{2\pi^3} m_e^5 \left(1 + 3g_A^2\right)(1 + \mathrm{RC})\, f, \tag{5}$$

where f is a phase space factor,

$$f = 1.6887, \tag{6}$$

which includes a relatively large Fermi function contribution [13] ($\sim 5.6\%$) as well as smaller nucleon mass, size and recoil corrections. It has been somewhat updated in eq. (6) to incorporate slight nucleon mass shifts.

We note that the electroweak radiative corrections, denoted by 1+RC, have been factorized in the same way for both the vector and axial-vector contributions. (Interference and induced coupling corrections are negligibly small in the case of the lifetime [13, 14].) That factorization effectively defines g_A via the relative normalization of the axial-vector current as measured by the lifetime (it incorporates QED as well as strong interaction effects in its definition).

Employing such a definition for g_A means that there will be some $\mathcal{O}(\alpha)$ corrections to the g_A^{asy} measured in neutron decay asymmetries that must be applied before contact with eq. (5) can be made at a level of high precision. (Those corrections will be discussed in section IV.) In this connection, it is worthwhile to note that the normalization of g_A in eq. (5) is consistent with the so-called $1/k$ method, in which the radiative corrections to various observables in β-decay, such as the lifetime, the electron spectrum, and the longitudinal electron polarization are expressed in terms of effective couplings G_V and G_A [23–26]. The same approach has been employed to calculate the corrections to the electron asymmetry [27–31]. The factorization of the short distance contributions of the radiative corrections, implicit in eq. (5), conforms also with an asymptotic theorem concerning their behavior in arbitrary semileptonic decays mediated by $W^{\pm}$ [32].

To order α, the radiative corrections in eq. (5) are given by [5, 6, 33, 34]

$$\mathrm{RC} = \frac{\alpha}{2\pi}\left\{\overline{g}(E_m) + 4\ln\frac{m_Z}{m_p} + \ln\frac{m_p}{m_A} + 2C + A_g\right\}, \tag{7}$$

where $\overline{g}(E_m)$ represents long distance loop corrections and bremsstrahlung effects averaged over the β-decay spectrum [23] ($E_m = 1.292579$ MeV, the end-point electron energy in neutron decay),

$$\frac{\alpha}{2\pi}\overline{g}(E_m) = 0.015056. \tag{8}$$

The other parameters in that expression have the values

$$\begin{aligned} m_Z &= 91.1875 \text{ GeV}, \\ m_p &= 0.9383 \text{ GeV}, \\ m_A &\simeq 1.2 \text{ GeV}, \\ C &\simeq 0.891, \\ A_g &\simeq -0.34. \end{aligned} \tag{9}$$

Here, following [3, 26, 35], we have approximately identified m_A with the mass of the $a_1(1260)$ axial vector meson. The value of C is an update of the calculation discussed in Ref. [33] (using $g_A = 1.27$ for self consistency). This leads to a first order result

$$\mathrm{RC}\,(\mathcal{O}(\alpha)) = 0.03770, \tag{10}$$

which is rather sizeable. (Together with the Fermi function, such (primarily) QED corrections increase the neutron decay rate by 9.37%!)

Given the magnitude of the order α corrections in eq. (10) and our desire for high precision, it becomes imperative to include the leading $\mathcal{O}(\alpha^2)$ contributions and estimate the theoretical uncertainty in the radiative corrections. Regarding the latter issue, the overall uncertainty is usually obtained by allowing m_A in eq. (7) to vary by a factor of 2 up or down. That reflects the fact that the last 3 terms in eq. (7) result from axial-vector current loop effects and their calculation is not perfectly matched in going from long to short-distance contributions. The scale, m_A, uncertainty reflects the matching error in a rough, but numerically realistic, way. With that methodology, the theoretical uncertainty is estimated to be

$$\mathrm{RC}\text{ (uncertainty)} = \pm 0.0008, \tag{11}$$

an error common to all neutron and nuclear β-decay studies. Reducing that dominant theory matching error would be very useful, but it is extremely challenging and beyond the scope of this paper.

Our focus in this section is to include the so called leading log corrections of the form $\alpha^n \ln^n \frac{m_Z}{m_p}$ and $\alpha^n \ln^n \frac{m_p}{2E_m}$, $n = 2, 3, \ldots$ in eq. (7) along with some of the other potentially most important $\mathcal{O}\left(\alpha^2 \ln \frac{m_Z}{m_p}\right)$ and $\mathcal{O}(\alpha^2)$ effects. Regarding the last of those, there is a relatively important Coulomb correction not included in the product of the Fermi function and 1+RC. It is called $\frac{\alpha}{2\pi}\delta$ in the literature and approximated by (see Appendix)

$$\frac{\alpha}{2\pi}\delta \simeq -\alpha^2 \ln\frac{m_p}{m_e} + \ldots = -0.00043. \tag{12}$$

Although considered previously, for some reason it was given the wrong sign. Correcting the value +0.0004 used in the past to eq. (12) corresponds to a -0.00083 shift which is quite significant at the level of our analysis.

In the Appendix, we give formulas that sum the leading log contributions in eq. (7) via the method in ref. [33]. They lead to the replacements

$$\begin{aligned} 1+\frac{2\alpha}{\pi}\ln\frac{m_Z}{m_p} &\rightarrow S(m_p,m_Z)=1.02248,\\ 1+\frac{3\alpha}{2\pi}\ln\frac{m_p}{2E_m} &\rightarrow L(2E_m,m_p)=1.02094. \end{aligned} \tag{13}$$

where the large $\frac{3\alpha}{2\pi}\ln\frac{m_p}{2E_m}$ contribution is hidden in the $\frac{\alpha}{2\pi}\overline{g}(E_m)$ function of eq. (7) [23].

Our final new input is to estimate next-to-leading log (NLL) corrections of the form $\alpha^2\ln\frac{m_Z}{m_p}$ and $\alpha^2\ln\frac{m_p}{m_f}$ coming from fermion vacuum polarization insertions in loops with photon propagators. Because they all enter with the same sign and there are quite a few leptons and quarks that contribute, one might expect those fermion loops to dominate the NLL contribution. However, as illustrated in the Appendix, they turn out to be quite small in the $\overline{\rm MS}$ formalism we employ. We estimate

$$\mathrm{NLL}=-0.0001. \tag{14}$$

Other $\mathcal{O}(\alpha^2)$ contributions are not expected to be significant, but a complete calculation to $\mathcal{O}(\alpha^2)$ would be very difficult and beyond the scope of this paper. Also, such a refined calculation is not obviously warranted until the one loop matching uncertainty in eq. (11) is significantly reduced.

Our next step is to organize the radiative corrections into a factorized form that does not induce spurious contributions when multiplied out. To accomplish that end, we take an effective field theory approach, dividing contributions into very long distance (the Fermi function), long distance, intermediate distance, and short distance factors. Of course, we must make certain that the matching is done correctly, for example by introducing terms such as $\frac{\alpha}{2\pi}\delta$ (see eq. (12)) when appropriate. Overall, we employ the following factorization beyond the Fermi function

$$\begin{aligned} 1+\mathrm{RC} &= \left[1+\frac{\alpha}{2\pi}\left(\overline{g}(E_m)-3\ln\frac{m_p}{2E_m}\right)\right]\\ &\cdot\left[L(2E_m,m_p)+\frac{\alpha}{\pi}C+\frac{\alpha}{2\pi}\delta\right]\\ &\cdot\left[S(m_p,m_Z)+\frac{\alpha(m_p)}{2\pi}\left(\ln\frac{m_p}{m_A}+A_g\right)+\mathrm{NNL}\right], \end{aligned} \tag{15}$$

where $\alpha(m_p)\simeq 1/134$. Employing the values we have given above for the quantities in eq. (15), one finds

$$1+\mathrm{RC} = 1.0390(8), \tag{16}$$

where the uncertainty in eq. (11) has been employed. Comparing eqs. (16) and (10), one sees that the $\mathcal{O}(\alpha^2)$ and summation effects have increased the RC by 0.0013, not a very large shift. Nevertheless, they must be included in precision studies.

Employing eq. (16) in eq. (5), we derive the relationship

$$|V_{ud}|^2\left(1+3g_A^2\right)\tau_n=4908\pm 4\ \text{sec}. \tag{17}$$

That master formula can be used to extract $|V_{ud}|$ via

$$|V_{ud}|=\left(\frac{4908(4)\ \text{sec}}{\tau_n\left(1+3g_A^2\right)}\right)^{1/2}, \tag{18}$$

as τ_n and g_A become more precise or to obtain g_A from τ_n and $|V_{ud}|$. Currently, one finds from the experimental averages

$$\tau_n^{\text{ave}} = 885.7(7)\ \text{sec} \tag{19}$$
$$g_A^{\text{ave}} = 1.2720(18), \tag{20}$$

the CKM parameter

$$|V_{ud}|=0.9729(4)(11)(4), \tag{21}$$

where the errors correspond to τ_n, g_A and RC respectively. In section III, we compare that value with the more precise $|V_{ud}|$ obtained from superallowed beta decays. Of course, the full utility of eq. (18) will be much better realized as τ_n and g_A measurements improve.

The master formula in eq. (17) can also be used to extract a very precise value of g_A via

$$g_A = \left[\frac{1636(1)\ \text{sec}}{\tau_n\, |V_{ud}|^2} - \frac{1}{3} \right]^{1/2} . \tag{22}$$

Note that the theory uncertainty (± 1 sec) in eq. (22) is due to the m_A scale uncertainty in the radiative corrections and it cancels with the same error in determinations of $|V_{ud}|^2$ from nuclear decays. The formula in eq. (22) will be utilized in section IV.

III. SUPERALLOWED β-DECAYS AND $|V_{ud}|$

Superallowed $0^+ \to 0^+$ Fermi transitions have been the focus of many studies, both experimentally and theoretically. Here, we start with the recent results of Towner and Hardy [36]. They have thoroughly scrutinized the nine very well measured superallowed β-decays, using RC last updated in ref. [37, 38], and taking great care to correct for various nuclear Coulombic and structure dependent effects. In the end, they arrive at $\mathcal{F}t$ values that are nucleus independent and can be used to extract $|V_{ud}|$, modulo uncertainties in the radiative corrections. The Z independence of their results is an important consistency check, since the daughter nuclei have Z values ranging from 5 to 26 with correspondingly different Coulomb and structure corrections.

In Table I, we give the (slightly) updated values of $|V_{ud}|$ obtained from the nine best measured superallowed β-decays, incorporating the isospin symmetry-breaking and structure dependent corrections δ_C and δ_{NS} used by Towner and Hardy, in conjunction with our new factorization scheme for the radiative corrections in eq. (15) (changing E_m and δ in that expression as appropriate for each nucleus). Specifically, we include the corrections δ_{NS} in the second factor of eq. (15) and append an additional factor $(1 - \delta_C)$.

TABLE I: The values of $|V_{ud}|$ obtained from superallowed beta-decays. The error given does not include nuclear and theory uncertainties common to the analysis. It is taken from ref. [36] and used to obtain the weighted average.

Nucleus	$\|V_{ud}\|$
^{10}C	0.97388(76)
^{14}O	0.97445(41)
^{26}Al	0.97416(35)
^{34}Cl	0.97431(40)
^{38}K	0.97424(43)
^{42}Sc	0.97351(38)
^{46}V	0.97372(43)
^{50}Mn	0.97396(44)
^{54}Co	0.97409(43)

The values of $|V_{ud}|$ derived from those distinct measurements are very consistent with one another and range from about 0.9735 to 0.9745. The weighted average is centered at $|V_{ud}| = 0.974047$; so, we round down to

$$|V_{ud}| = 0.9740(1)(3)(4), \tag{23}$$

where the uncertainties are experimental, nuclear theory and RC (see eq. (11)). We have checked that the combined small differences in the RC from our new factorization in eq. (15), improvements in the higher order leading logs, central value of m_A, small NLL (-0.0001) correction etc. tend to cancel. For that reason, the result in eq. (23) is essentially the same as the one obtained earlier by Towner and Hardy (cf. eq. (2)). We note that the sign of the $\frac{\alpha}{2\pi}\delta$ corrections for the superallowed decays (which are all e^+ emitters) is correct in the literature and numerically the change in sign of that correction for neutron decay was our biggest modification of previous results. So, $|V_{ud}|$ in eq. (23) remains the current best value. Indeed, comparison with $|V_{ud}|$ extracted from neutron decay τ_n and g_A values in eq. (21) shows that they share the same RC uncertainty, but the error on g_A must be improved by about a factor of 4 before the neutron decay becomes competitive.

One way of reducing the RC uncertainty in $|V_{ud}|$, perhaps by as much as a factor of 2 would be to use the $\pi^+ \to \pi^0 e^+ \nu_e$ decay rate for which the loop induced axial-vector contributions are better controlled and nuclear theory uncertainties are circumvented [39]. However, the small branching ratio $\sim 10^{-8}$ makes that method statistically challenging. Nevertheless, an ongoing PSI experiment finds

$$|V_{ud}| = 0.9737(39) \cdot \left[\frac{\mathrm{BR}\,(\pi^+ \to e^+ \nu_e(\gamma))}{1.2352 \cdot 10^{-4}}\right]^{1/2}, \tag{24}$$

where its dependence on the $\pi^+ \to e^+ \nu_e(\gamma)$ branching ratio (used for normalization) is exhibited. We assume the SM theory value of $1.2352 \cdot 10^{-4}$ is correct and hence obtain $|V_{ud}| = 0.9737(39)$, in excellent agreement with nuclear and neutron results but with a larger error. Alternatively, others have chosen to use the PDG recommended branching ratio of 1.230(4) which leads to $|V_{ud}| = 0.9716(39)$ which is also in agreement within errors even though the central value may appear low [40].

Note, if we average the $|V_{ud}|$ determinations above, we still find

$$|V_{ud}| = 0.9740(5), \qquad \text{(Average)} \tag{25}$$

because the superallowed β-decays dominate and they have an average central value slightly larger than 0.9740.

IV. g_A: THEORY VS. EXPERIMENT

Employing the neutron lifetime measurement in eq. (19) and the value of $|V_{ud}|$ in eq. (23), we find via eq. (22) the Standard Model prediction

$$g_A = 1.2703(6)(5), \tag{26}$$

where the errors are in τ_n and $|V_{ud}|$ (nuclear uncertainty). That precise prediction is larger than the PDG recommended value of 1.2670(30) [41], but smaller than the single best asymmetry value [42] of

$$g_A = 1.2739(19), \tag{27}$$

(after small QED corrections are applied).

For now, the g_A in eq. (26) should be considered the standard, to be used in solar neutrino flux calculations etc. However, to take proper advantage of its precision, one should employ the neutron decay definition we have assumed and correct the weak interaction process under consideration for its own electroweak radiative corrections. As an example, consider the relationship between the g_A defined via the neutron lifetime whose value is given in eq. (26) and one defined by the lowest order polarized neutron decay asymmetry

$$A = \frac{2\hat{g}(1-\hat{g})}{1+3\hat{g}^2}, \qquad \hat{g} \equiv g_A^{\mathrm{asy}}. \tag{28}$$

If we wish to replace $\hat{g}$ by g_A in that expression, then there will be additional (energy dependent) radiative corrections to the asymmetry. Those corrections were computed long ago, originally by Shann and later confirmed by many others. They are quite small, leading to about a 0.1% shift in the asymmetry [27]. That effect is corrected for in the most recent [42] experimental measurement of the asymmetry, although it is well below current experimental uncertainties. It would be useful to compute the QED corrections to other processes where the very precise value of g_A in eq. (26) might prove useful, for example primordial nucleosynthesis or the solar neutrino flux.

V. $|V_{us}|$ DETERMINATIONS AND CKM UNITARITY

The 2002 PDG recommended value for $|V_{us}|$ [41],

$$|V_{us}|_{\mathrm{PDG}} = 0.2196(26) \qquad \text{PDG2002}, \tag{29}$$

has remained fixed (modulo small variations in its uncertainties) for many years. It is based on fitted branching ratios from rather old data for K_{e3} decays, $K^0 \to \pi^- e^+ \nu_e$ and $K^+ \to \pi^0 e^+ \nu_e$, combined with the kaon lifetimes, τ_{K_L} and τ_{K^+}. The resulting K_{e3} decay rates are proportional to $|V_{us}|^2$ and can be used for its extraction. In fact, they are analogous to superallowed nuclear beta decays (or $\pi^+ \to \pi^0 e^+ \nu_e$) in that only the weak vector current contributes at the tree level. Since that current is conserved in the SU(3) flavor limit, strong interaction corrections are of second order in SU(3) breaking. Those effects, characterized by the departure of the form factor $f_+(0)$ from 1 along with isospin breaking effects were considered in the classic study by Leutwyler and Roos [43] that forms the basis for the extracted value of $|V_{us}|$ in eq. (29).

In addition to SU(3) breaking effects, there is a fairly significant first order $m_d - m_u$ correction due to π^0–η mixing ($\sim 4\%$) that must be separately applied to the charged K_{e3}^+ decay rate and an extra Coulombic $\pi^- e^+$ final state QED interaction for K_{e3}^0. The fact that even with those different isospin violating corrections, both the neutral and charged K_{e3} decay rates gave consistent values for $|V_{us}|$ (eq. (29) is their average) has often been used to argue for the validity of eq. (29) as compared with for example Hyperon beta decays which have tended to give somewhat larger values for $|V_{us}|$ but are not as theoretically clean.

We note that combining eq. (29) with the value of $|V_{ud}|$ in eq. (23) and using $|V_{ub}|^2 \simeq 2.1 \times 10^{-5}$ leads to

$$|V_{ud}|^2_{0^+\to 0^+} + |V_{us}|^2_{\mathrm{PDG}} + |V_{ub}|^2 = 0.9969 \pm 0.0015. \tag{30}$$

That roughly 2 sigma deviation from unitarity has been a persistent problem for many years. It has been at times interpreted as a hint of new physics or as an indication that something is wrong with the data and/or theory calculations used to extract $|V_{ud}|$ and $|V_{us}|$. Since perfect unitarity would require a rather large shift (+ 3.2%) in $|V_{us}|$ but a relatively smaller shift (+ 0.16%) in $|V_{ud}|$, the latter has been generally thought to be the root of the problem. As a result, considerable experimental and theoretical scrutiny has been applied to $|V_{ud}|$. Nevertheless, as emphasized in the first part of this paper, its value has remained rather stable.

To illustrate the above approach and some of its underlying uncertainties, we describe the general K_{e3} decay rate formula [43],

$$\Gamma(K \to \pi e \nu(\gamma)) = \frac{G_\mu^2 m_K^5}{192\pi^3} S_{\mathrm{EW}}\,(1+\delta_K^e)\, C^2\, |V_{us}|^2\, f_+^2(0) I_K^e, \tag{31}$$

where $C^2 = 1$ for K_L or K_S decays (to both $\pi^\pm e^\mp$) and $C^2 = 1/2$ for $K^\pm$. $S_{\mathrm{EW}} = 1.022$ is the universal short-distance radiative correction in eq. (15) [32] while δ_K^e are model dependent long-distance QED corrections recently estimated to be (for radiative inclusive studies) [44–46]

$$\begin{aligned}\delta^e_{K^0} &= +1.3 \pm 0.3\%,\\ \delta^e_{K^+} &= -0.1 \pm 0.7\%.\end{aligned} \tag{32}$$

The form factor $f_+(0)$ incorporates SU(3) breaking. Leutwyler and Roos found

$$f_+(0) = f_+^{K^0\pi^-} = 0.961 \pm 0.008, \tag{33}$$

a value recently confirmed by lattice calculation [47] and chiral perturbation theory [46, 48] (up to unresolved order p^6 ambiguities). In the case of charged kaons, $m_d - m_u$ mass splittings give rise to $\pi - \eta$ mixing such that

$$f_+^{K^+\pi^0} \simeq 1.022 f_+^{K^0\pi^-} = 0.9821 \pm 0.008 \pm 0.002. \tag{34}$$

Finally, the phase space factor is determined for a linear form factor to be (for a slope factor of 0.028)

$$\begin{aligned}I^e_{K^0} &= 0.1550,\\ I^e_{K^+} &= 0.1594,\end{aligned} \tag{35}$$

while for a quadratic form factor suggested by KTeV data [49],

$$I^e_{K^0} = 0.1535. \tag{36}$$

The difference, $\sim 1\%$, is somewhat accounted for by assigning an extra $\pm 0.7\%$ form factor decay rate uncertainty. However, we note that the central value of $|V_{us}|$ will depend on which parametrization is used. The value of $|V_{us}|$ in eq. (29) would become 0.2207 if a quadratic parametrization were employed.

Recently, a number of developments have cast doubt on the reliability of eq. (29). First, a new measurement of the K_{e3}^+ branching ratio by the E865 Collaboration at Brookhaven National Laboratory finds it to be about 5.3% larger than the fitted PDG value used to obtain $|V_{us}|$. Their finding conflicts with the earlier K_{e3} decay rates, even after all isospin violating differences are taken into account and on its own leads to [50]

$$|V_{us}| = 0.2236(23)/f_+^{K^+\pi^0}(0), \qquad \text{E865} \tag{37}$$

where $f_+^{K^+\pi^0}(0) = 1+\mathrm{SU}(3)$ breaking effects, including $m_d - m_u$ corrections (but unlike E865, we have not absorbed QED corrections into its definition). For the value $f_+^{K^+\pi^0}(0) = 0.9842(84)$ effectively used by the E865 collaboration after removing the QED correction, one finds $|V_{us}| = 0.2272(30)$ and correspondingly almost perfect unitarity,

$$|V_{ud}|^2_{0^+\to 0^+} + |V_{us}|^2_{\mathrm{E865}} + |V_{ub}|^2 = 1.0003 \pm 0.0017. \tag{38}$$

However, that is not the end of the story. In an even more recent development, the KTeV Collaboration (E832) [51] at Fermilab has reported a thorough analysis of all primary K_L decay modes, including K_{e3}. They find significant disagreement with the PDG fit values for several of the main K_L branching ratios, including K_{e3}, exactly the types of shifts required to bring the K_L system into accord with unitarity and the K^+ results of E865.

The KTeV Collaboration finds (using the PDG K_L lifetime)

$$\Gamma(K_L \to \pi e \nu) = 0.520(4) \times 10^{-14}\ \mathrm{MeV}. \tag{39}$$

That radiatively inclusive K_{e3} partial decay rate is about 5% larger than the 2002 PDG fit value. After accounting for measured form factor effects, a new calculation of QED radiative corrections [44–46] and SU(3) breaking (via Leutwyler and Roos [43]), they find

$$|V_{us}| = 0.2253(23) \qquad \text{KTeV } K_{e3}, \tag{40}$$

where (because of the very high statistics) the error is essentially dominated by SU(3) breaking, form factor shape and K_L lifetime uncertainties. Unlike K_{e3}^+, the K_L extraction is not directly sensitive to the up-down mass difference. For $K_{\mu 3}$, they obtained a similar result $|V_{us}| = 0.2250(23)$, which provides strong confirmation. Taken together with the value of $|V_{ud}|$ in eq. (23), one finds from eq. (40)

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9994(14) \qquad \text{KTeV } K_{e3}\ , \tag{41}$$

in very good agreement with unitarity, a remarkable turn of events. So, E865 and KTeV are in apparent agreement regarding $|V_{us}|$ and unitarity. Of course, they are dependent on the K^+ and K_L lifetimes which, if the history of the kaon branching ratios is any indication, could change upon closer scrutiny.

It is useful to average the K_{e3} and $K_{\mu 3}$ results from E865 and KTeV in order to get a single determination of $|V_{us}|$ and unitarity constraint that can be used to limit new physics appendages to the standard model. To carry out such an average requires a consistent treatment of features common to the analyses of both measurements. In that category we put SU(3) breaking effects and the form factor shape (linear vs. quadratic). We adopt the KTeV quadratic form factor along with its $\pm 0.7\%$ uncertainty and assume the Leutwyler-Roos estimate of SU(3) breaking and $m_d - m_u$ effects. Together they shift the E865 value of $|V_{us}|$ up by about 0.7%. So, we wind up with the following two quantities to be averaged:

$$\begin{aligned} |V_{us}| &= 0.2288(26)(20) \qquad \text{Shifted E865 } K_{e3}^+, \\ |V_{us}| &= 0.2252(13)(20) \qquad \text{KTeV } K_{e3}^0 \text{ and } K_{\mu 3}^0 \text{ average,} \end{aligned} \tag{42}$$

where we employ the KTeV K_{e3} and $K_{\mu 3}$ averages under the assumption of muon-electron universality. The common error from the form factor and SU(3) breaking, (± 20), has been factorized. The remaining uncertainty in the K^+ case comes from adding in quadrature the experimental uncertainties along with errors due to QED, the K^+ lifetime, $m_d - m_u$ effects, and normalization uncertainties associated with correlations among K^+ branching ratios. We estimate the last of these to be $\pm 0.5\%$. For the K_L, the first error comes from experimental uncertainties, the K_L lifetime error (which is appreciable), and QED.

Carrying out a weighted average, using the first set of errors to do the weighting, we find

$$|V_{us}| = 0.2259(12)(20) \qquad \text{E865-KTeV Average,} \tag{43}$$

or, combining errors in quadrature,

$$|V_{us}| = 0.2259(23) \qquad \text{E865-KTeV Average.} \tag{44}$$

Together with the $|V_{ud}|$ value in eq. (2), that gives

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9997(10)(10), \tag{45}$$

where the first and second errors arise from $|V_{ud}|$ and $|V_{us}|$, respectively. Combining the errors in quadrature, we obtain

$$|V_{ud}|^2 + |V_{us}|\,|^2 + |V_{ub}|^2 = 0.9997(14), \tag{46}$$

which confirms unitarity superbly. Agreement with the standard model expectation can be used to constrain various new physics effects. For example, if the muon had some additional, exotic decay mode such as $\mu \to e+$ the wrong neutrinos, it would affect unitarity through the G_μ used in our normalization [52]. The good agreement in eq. (46) limits the branching ratios for those types of hypothetical decays to be $< 0.2\%$, which is similar to the best direct constraints, but more general.

At this point we note that the central values of $|V_{us}|$ in eq. (42) differ by about 1.2σ. It could be a simple effect due to experimental systematics (statistical errors are very small) or might indicate a problem in the Kaon lifetimes, $m_d - m_u$, QED long distance radiative corrections or K^+ correlated branching ratios. Improved measurements in the charged and neutral kaon properties may prove useful in clarifying the difference. However, the total uncertainty in $|V_{us}|$ of about $\pm 1\%$ is dominated by SU(3) breaking and the form factor shape and magnitude. It will be difficult to reduce those errors much further. Unitarity prevails, but unless some new procedure for calculating SU(3) breaking effects with much higher precision is found, it seems unlikely that the error in eq. (44) can be reduced much further, i.e. a $\pm 1\%$ uncertainty in $|V_{us}|$ is near the end of the road for K_{e3}. Similar remarks apply to $|V_{ud}|$ where theory uncertainty dominates. Fortunately, they seem to be ending with a triumph for unitarity and a strong confirmation of the standard model.

Other new analyses of K_{e3} by the KLOE Collaboration at Frascati and NA48 at CERN are being completed and should report results shortly. It will be interesting to see if they confirm E865 and KTeV or reopen the K_{e3} problem.

In another relatively recent development, Cabibbo, Swallow and Winston (CSW) [53, 54] have revisited the extraction of $|V_{us}|$ from Hyperon beta decays. That procedure is sometimes criticized as unreliable because the decay rates are renormalized by first order SU(3) breaking effects in the axial-vector contributions. However, rather than just employing total decay rates, CSW also included decay asymmetry measurements that effectively measure the first order SU(3) breaking effects, analogous to the use of g_A^{Asy} in neutron beta decay to determine $|V_{ud}|$. They found

$$|V_{us}| = 0.2250(27) \qquad \text{Hyperon Decays,} \tag{47}$$

where the error quoted is purely experimental and SU(3) breaking has been neglected. Nevertheless, if taken at face value it gives

$$|V_{ud}|^2_{0^+\to 0^+} + |V_{us}|^2_{\text{Hyperon}} + |V_{ub}|^2 = 0.9993(16), \tag{48}$$

i.e. good agreement with unitarity. Because SU(3) breaking effects and various other theory uncertainties have not been considered, we do not include eq. (47) in our averaging.

Finally, a (preliminary) recent lattice calculation [55] of the pseudoscalar decay constants

$$f_K/f_\pi = 1.211(6)_{\text{stat}} \begin{pmatrix} +15 \\ -13 \end{pmatrix}_{\text{syst}}, \tag{49}$$

can be combined [56] with the experimental quantity

$$\frac{\Gamma\left(K^+ \to \mu^+\nu_\mu(\gamma)\right)}{\Gamma\left(\pi^+ \to \mu^+\nu_\mu(\gamma)\right)} = 1.334(4), \tag{50}$$

to yield

$$\frac{|V_{us}|}{|V_{ud}|} = 0.2277(31), \tag{51}$$

where the uncertainty is lattice dominated. If unitarity is assumed, that ratio implies

$$\begin{aligned} |V_{us}| &= 0.2220(29), \\ |V_{ud}| &= 0.9750(6), \end{aligned} \tag{52}$$

while if $|V_{ud}| = 0.9740(5)$ is employed

$$|V_{us}| = 0.2218(30) \qquad \text{Preliminary Lattice.} \tag{53}$$

It corresponds to a modest 1.2 sigma deviation from unitarity. The beauty of the lattice approach is that it is still in its infancy. With new larger computers and better treatment of chiral symmetry and isospin breaking, the uncertainty in eq. (53) may be reduced by a factor of 4 or more, making the lattice approach to f_K/f_π ultimately the best way to determine $|V_{us}|$. Should that happen, it will be somewhat ironic that the axial current determination via K_{l2} and π_{l2}

decay constants turns out to be theoretically more pristine that the vector current approach using K_{e3}. If something similar were possible for $|V_{ud}|$, the unitarity test (constraint) in eq. (46) might be significantly improved.

The $|V_{us}|$ central values discussed above vary from about 0.22–0.23 depending on the data used and SU(3) breaking corrections applied. That range is to be compared with the value suggested by $0^+ \to 0^+$ beta decays and perfect unitarity

$$|V_{us}| = 0.2265(22) \qquad \text{Unitarity} + 0^+ \to 0^+ \text{ Nuclear.} \tag{54}$$

A comparison of the different determinations of $|V_{us}|$ is illustrated in Fig. 1.

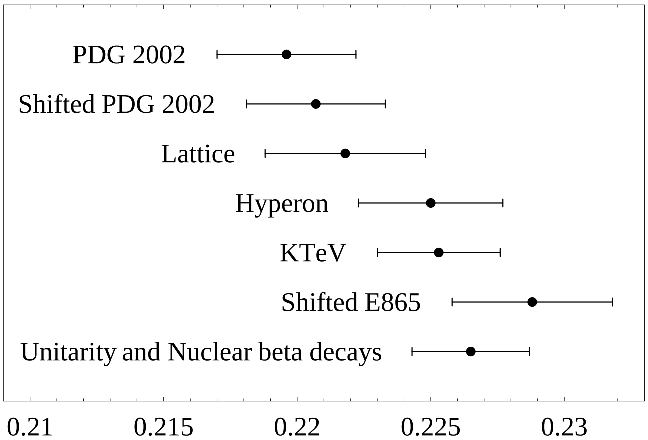

FIG. 1: Determinations of $|V_{us}|$ from various sources. The Hyperon value does not include theory errors. Shifted values correspond mainly to the change of linear to quadratic form factor parametrization.

VI. CONCLUSION AND OUTLOOK

We have presented an update of the electroweak radiative corrections to neutron β-decay. It currently provides the most precise determination of g_A, a quantity that finds many applications in nuclear, particle, and astro-physics. As g_A and τ_n experimental measurements improve, our results can be used to obtain a $|V_{ud}|$ that is competitive with nuclear β-decay determinations (which yield $|V_{ud}| = 0.9740(5)$), but without the nuclear structure dependent uncertainties. In the end, we will still be limited by a ± 0.0004 theory error that comes from uncertainties in the axial-vector induced loop corrections. Reducing the latter error further will require a new theoretical approach or the measurement of $\Gamma(\pi^+ \to \pi^0 e^+ \nu_e)$ which has smaller uncertainties. The 10^{-8} branching ratio makes the latter strategy very difficult statistically.

More problematic than $|V_{ud}|$ in testing CKM unitarity has been inconsistencies in the various $|V_{us}|$ determinations. Of particular concern has been the low value $\lesssim 0.22$ obtained from PDG fits to old K_{e3} data which for a long time has suggested a small but persistent departure from unitarity. However, recently measured K_{e3} branching ratios for both the charged and neutral kaon exhibit large deviations (about 5%) from the PDG fit values, increasing $|V_{us}|$ to a level consistent with unitarity. In fact, the average value they provide $|V_{us}|$=0.2259(23) together with $|V_{ud}|$=0.9740(5) from superallowed beta decays concur with unitarity expectations up to ± 0.0014 (see eq. (46)). That good agreement confirms the predicted large +4% [5] radiative correction of the Standard Model at the 28σ level! A triumph for quantum loop effects. However, before one can be sure that the unitarity problem has been resolved, the recent results should be confirmed by anticipated new measurements of the K_{e3} branching ratios from the KLOE experiment at Frascati and the NA48 experiment at CERN. In addition, given the recent changes in kaon branching fractions, it would be nice to have new confirmation measurements of both the K^+ and K_L lifetimes.

The newest approach to $|V_{us}|$ appears to be very promising [56]. It combines a lattice calculation of f_K/f_π with the experimental measurement of $\frac{\Gamma(K^+ \to \mu^+ \nu_\mu(\gamma))}{\Gamma(\pi^+ \to \mu^+ \nu_\mu(\gamma))}$. The latter ratio is already very well measured, but in view of the changes in K^+ decay rates, the numerator should be experimentally revisited. Also, the electroweak radiative corrections that largely cancel in the ratio should be reexamined.

The lattice calculation of f_K/f_π has some very nice features that suggest its already small uncertainty may be reduced much further. They include a cancellation of the statistical uncertainties which are highly correlated and the scale uncertainty. The challenge is to refine extrapolations to the chiral and continuum limits with full dynamical fermion simulations. That should become possible with the advent of very large special purpose computing facilities.

The current confirmation of unitarity embodied in eq. (46) gives us further confidence in the Standard Model and can be used to constrain "New Physics" appendages to it. Roughly speaking, it rules out additional tree or loop level contributions to the $|V_{ud}|^2 + |V_{us}|^2$ combination at about the $\pm 0.2\%$ level. Effects of about that magnitude could in principle come about from supersymmetry loops [57–59], an extra Z' boson [60], heavy quark or lepton mixing, exotic muon decays (since we normalize in terms of G_μ), compositeness, extra dimensions [61] etc. However, there are no really compelling reasons to expect such a large deviation. For that reason, confirmation of unitarity via experimental or theoretical changes has been anticipated. Its verification is, nevertheless, important. Also, finding new ways to improve the unitarity constraint, e.g. lattice calculations, is difficult, but should be strongly encouraged. Such challenges push our computational and experimental skills to new limits, stimulate ingenuity, and perhaps at some point new physics will be uncovered.

APPENDIX

In this appendix, we describe the input that went into:

1. the leading log summations $L(2E_m, m_p)$ and $S(m_p, m_Z)$;
2. the partial next to leading log calculation due to photonic vacuum polarization insertions and
3. the residual $\frac{\alpha}{2\pi}\delta$ correction that results from a proper matching of the Fermi function and long distance $\mathcal{O}(\alpha)$ corrections.

1. Leading Log Summations

We follow the approach of ref. [33] where a renormalization group summation for the leading short-distance logs was given and a value for $S(m_p, m_Z)$ derived. Here we extend that method down to the intermediate region $2E_m - m_p$. That simply requires a change in the anomalous dimension from $2\alpha/\pi$ to $3\alpha/2\pi$ and an evaluation of the $\overline{\text{MS}}$ (modified minimal subtraction) coupling $\alpha(\mu)$ at low scales.

The leading log summation in an $\overline{\text{MS}}$ approach is simply given by

$$\begin{aligned}
L(2E_m, m_p) &= \left(\frac{\alpha(m_u)}{\alpha(2E_m)}\right)^{9/4} \left(\frac{\alpha(m_d)}{\alpha(m_u)}\right)^{27/28} \left(\frac{\alpha(m_\mu)}{\alpha(m_d)}\right)^{27/32} \left(\frac{\alpha(m_s)}{\alpha(m_\mu)}\right)^{27/44} \left(\frac{\alpha(m_p)}{\alpha(m_s)}\right)^{9/16}, && (55)\\
S(m_p, m_Z) &= \left(\frac{\alpha(m_c)}{\alpha(m_p)}\right)^{3/4} \left(\frac{\alpha(m_\tau)}{\alpha(m_c)}\right)^{9/16} \left(\frac{\alpha(m_b)}{\alpha(m_\tau)}\right)^{9/19} \left(\frac{\alpha(m_W)}{\alpha(m_b)}\right)^{9/20} \left(\frac{\alpha(m_Z)}{\alpha(m_W)}\right)^{36/17}, && (56)
\end{aligned}$$

where the $\alpha(\mu)$ are values of the $\overline{\text{MS}}$ (modified minimal subtraction) QED coupling at a scale μ. This coupling is given to leading log order by

$$\begin{aligned}
\alpha^{-1}(\mu) &= \alpha^{-1}(m_e) - \frac{2}{3\pi}\sum_f Q_f^2 \Theta(\mu - m_f)\ln\frac{\mu}{m_f} + \frac{7}{2\pi}\Theta(\mu - m_W)\ln\frac{\mu}{m_W}, && (57)\\
\alpha^{-1}(m_e) &= 137.036 + \frac{1}{6\pi} = 137.089. && (58)
\end{aligned}$$

In that expression, the sum is over all quarks and lepton flavors f (with a color factor of 3 for quarks). We thereby find

$$\begin{aligned}
\alpha^{-1}(2E_m = 2.585\ \text{MeV}) &= 136.745,\\
\alpha^{-1}(m_u = 62\ \text{MeV}) &= 136.0708,\\
\alpha^{-1}(m_d = 83\ \text{MeV}) &= 135.9263,\\
\alpha^{-1}(m_\mu) &= 135.7896,\\
\alpha^{-1}(m_s = 215\ \text{MeV}) &= 135.2368,\\
\alpha^{-1}(m_p) &= 133.9861,\\
\alpha^{-1}(m_c = 1.35\ \text{GeV}) &= 133.67728,
\end{aligned}$$

$$\begin{aligned}
\alpha^{-1}(m_\tau) &= 133.3662, \\
\alpha^{-1}(m_b = 4.5\ \mathrm{GeV}) &= 132.1174, \\
\alpha^{-1}(m_W = 80.4\ \mathrm{GeV}) &= 128.0389, \\
\alpha^{-1}(m_Z = 91.1875\ \mathrm{GeV}) &= 128.001.
\end{aligned} \tag{59}$$

Relatively low effective quark masses [62] have been used in those results in order to incorporate QCD contributions at low energies. In that way $\alpha^{-1}(m_Z) = 128.0$ is obtained while a more detailed higher order QED and QCD analysis including $e^+e^- \to$ hadrons via a dispersion relation gives $\alpha^{-1}(m_Z) = 127.934$.

Employing the above values in eq. (55) and eq. (56), we find

$$\begin{aligned}
L(2E_m, m_p) &= 1.02094, \\
S(m_p, m_Z) &= 1.02248.
\end{aligned} \tag{60}$$

Those results are not very sensitive to the quark masses employed. For example, changing the m_u value by a factor of 2 leads to a shift in the RC by less than 3×10^{-5} which is well below the overall $\pm 8 \times 10^{-4}$ uncertainty assumed in the RC.

A useful relation, valid for the nine transitions in Table I,is

$$L(2E_m, m_p) = L(m_e, m_p) \left(\frac{\alpha(m_e)}{\alpha(2E_m)} \right)^{9/4}, \tag{61}$$

which leads to

$$L(2E_m, m_p) = 1.026725 \left(1 - \frac{2\alpha(m_e)}{3\pi} \ln \frac{2E_m}{m_e} \right)^{9/4}. \tag{62}$$

2. Next-to-leading logarithmic corrections due to photonic vacuum polarization.

Next-to-leading logarithmic (NLL) corrections to weak decays have been studied in detail for QCD (see, for example, [63, 64]). Here we adapt those results to QED. We consider a subset of contributions, containing fermion loop insertions in the photon propagator, as depicted in Fig. 2. That particular subset is somewhat enhanced by the large number of contributing fermions.

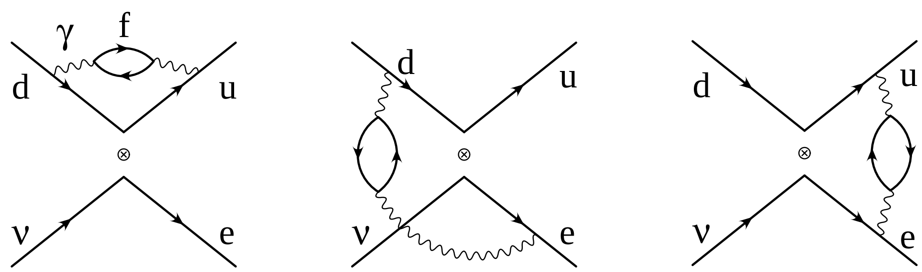

FIG. 2: Vacuum polarization diagrams contributing logarithmic corrections to the beta decay rate. QED loop corrections to external charged particle legs are included but not illustrated.

The effect of short distance corrections can be parameterized as follows. We keep only leading-logarithmic terms, and those NLL ones that arise from fermion loops. Then, the decay width Γ can be expressed in terms of the lowest-order width Γ^0 times a correction factor,

$$\begin{aligned}
\Gamma &= \Gamma^0 \left[1 - \gamma^{(0)} L a - L a^2 \left(\gamma^{(1)} + 2\beta_0 B \right) + \frac{1}{2} L^2 a^2 \gamma^{(0)} \left(\beta_0 + \gamma^{(0)} \right) \right], \\
L &\equiv \ln \frac{M_W^2}{\mu^2}, \qquad a \equiv \frac{\alpha(\mu)}{4\pi},
\end{aligned} \tag{63}$$

and we will be interested in the value of this correction at $\mu \simeq m_p$. A similar analysis can be extended down to $2E_m$.

The leading order anomalous dimension of the four-fermion operator $\bar{u}\gamma^\mu(1-\gamma_5)d\otimes\bar{e}\gamma_\mu(1-\gamma_5)\nu$ is given by diagrams similar to those in Fig. 2, but without fermion loops. It can be expressed in terms of fermion charges,

$$\gamma^{(0)} = (Q_u - Q_d)^2 + Q_e(Q_e + 8Q_u - 2Q_d) = -4. \tag{64}$$

The running of the coupling constant is described by a sum over all contributing fermions with charges Q_f and number of color varieties n_f (equal 3 for quarks and 1 for others),

$$\beta_0 = -\frac{4}{3}\widetilde{n}, \qquad \widetilde{n} \equiv \sum_f n_f Q_f^2. \tag{65}$$

At the NLL order, we also need the two-loop anomalous dimension $\gamma^{(1)}$ and the matching coefficient B. The former is determined by calculating $1/\epsilon$ poles of the diagrams in Fig. 2. The latter is given by the finite part of analogous diagrams without fermion loop insertions. Their individual values depend on the scheme of the calculation (for example, treatment of γ_5), and only their combination occurring in eq. (63) is scheme-independent.

We list here the values of both quantities obtained in the naive dimensional regularization (NDR), with an anti-commuting γ_5, and in the 't Hooft-Veltman scheme (HV),

$$\begin{aligned} \gamma^{(1)}_{\rm HV} &= -\tfrac{20}{3}\widetilde{n}, \qquad & \gamma^{(1)}_{\rm NDR} &= -\tfrac{44}{9}\widetilde{n}, \\ B_{\rm HV} &= -\tfrac{23}{6}, \qquad & B_{\rm NDR} &= -\tfrac{19}{6}. \end{aligned} \tag{66}$$

In both schemes we get

$$2\beta_0 B + \gamma^{(1)} = \frac{32}{9}\widetilde{n}. \tag{67}$$

With these results, the logarithmic corrections of eq. (63) become

$$\Gamma \;=\; \Gamma^0 \left\{ 1 + \frac{2\alpha(\mu)}{\pi}\ln\frac{M_W}{\mu} + \left(\frac{\alpha}{\pi}\right)^2 \ln\frac{M_W}{\mu}\left[\left(\frac{2\widetilde{n}}{3}+2\right)\ln\frac{M_W}{\mu} - \frac{4\widetilde{n}}{9}\right]\right\}. \tag{68}$$

All but the last terms in this correction factor are included and summed up to all orders in the factor S in eq. (56). The last term is the fermionic NLL correction. Because it is very small, we estimate it as though all fermions u, d, s, c, b and e, μ, τ contributed over the whole range from M_W to m_p. In this approximation, we have

$$\widetilde{n} \;=\; \frac{20}{3}, \tag{69}$$

and the numerical value of the NLL fermionic correction is

$$-\frac{4\widetilde{n}}{9}\left(\frac{\alpha}{\pi}\right)^2 \ln\frac{M_W}{m_p} = -0.00007. \tag{70}$$

Small additional NNL contributions from the E_m to m_p region shift that value to about -0.0001.

3. Evaluation of the $\mathcal{O}(Z\alpha^2)$ corrections

In this Appendix we summarize the evaluation of the $\mathcal{O}(Z\alpha^2)$ corrections in the case of neutron β-decay ($Z = 1$). These are defined as the residual corrections of this order not contained in the product of the Fermi function and the $\mathcal{O}(\alpha)$ corrections. For a point nucleus, analytic results for the correction to the charged lepton spectrum has been obtained in the extreme relativistic approximation [65]. The sign is opposite for electron and positron emitters (as can be readily understood by a glance at the figures in Ref. [65]) and, for the neutron decay, we have

$$\Delta P = -\alpha^2\left[\ln\left(\frac{m_p}{m_e}\right) + \frac{43}{18} - \frac{5}{3}\ln\left(\frac{2E}{m_e}\right)\right], \tag{71}$$

(see Eq. (5) of Ref. [65]). It has also been verified that, to good approximation, the leading contributions to Eq. (71) extrapolate smoothly to their non-relativistic limit, which has also been obtained analytically. Next, we evaluate numerically the spectral average of $\ln(2W)$ ($W \equiv E/m$) using the phase space factor $\sqrt{W^2-1}\; W\,(W_{\max}-W)^2$, leading to

$$\langle \ln(2W) \rangle = 1.1390. \tag{72}$$

Combining Eqs. (71) and (72), we obtain

$$\langle \Delta P \rangle = -8.006\alpha^2 = -4.26 \times 10^{-4}. \tag{73}$$

In order to take into account the finite proton size, we employ Eqs. (7-9) of Ref. [37], with the sign again reversed, since we are dealing with an electron emitter. These formulae correspond to a uniformly charged sphere of radius $R = \sqrt{\frac{5}{3}}a$, where $a \equiv \sqrt{6}/\Lambda$ is the rms charge radius of the proton. Using $a = 0.90$ fm [13], we have $\Lambda/m_p = 0.572$. Insertion of this value in Eqs. (7-9) of Ref. [37], leads to a finite proton size effect -5.4×10^{-6}. Combining this result with Eq. (73), we obtain our final answer for this class of corrections in the case of neutron decay:

$$\frac{\alpha}{2\pi}\delta = -4.3 \times 10^{-4}. \tag{74}$$

We have verified that terms of $\mathcal{O}\Big((\Lambda/m_p)^3\Big)$, not contained in Eqs. (7-9) of Ref. [37], as well as estimates of the corrections $\mathcal{O}\big(Z^2\alpha^3\big)$, give negligible contributions in the case of neutron decay.

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