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Two-Loop Sudakov Form Factor in a Theory with Mass Gap

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The two-loop Sudakov form factor is computed in a $U(1)$ model with a massive gauge boson and a $U(1) \times U(1)$ model with mass gap. We analyze the result in the context of hard and infrared evolution equations and establish a matching procedure which relates the theories with and without mass gap setting the stage for the complete calculation of the dominant two-loop corrections to electroweak processes at high energy.

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Since the pioneering works by Sudakov [1] and Jackiw [2] the high energy asymptotics of the electromagnetic form factor has been the subject of numerous investigations. The problem is relevant for a wide class of phenomenological applications from Drell-Yan processes to deep inelastic scattering. Recently a new wave of interest to the Sudakov asymptotic regime has been risen in connection with higher-order corrections to electroweak processes at high energies [3–11]. Experimental and theoretical studies of electroweak interactions have traditionally explored the range from very low energies, e.g. through parity violation in atoms, up to energies comparable to the masses of the W- and Z-bosons, e.g. at the LEP or the Tevatron. The advent of multi-TeV colliders like the LHC during the present decade or a future linear electron-positron collider will give access to a completely new energy domain. Once the characteristic energies $\sqrt{s}$ are far larger than the masses of the W- and Z-bosons, $M_{W,Z}$, exclusive reactions like electron-positron (or quark-antiquark) annihilation into a pair of fermions or gauge bosons will receive virtual corrections enhanced by powers of the large *electroweak* logarithm $\ln\left(s/M_{W,Z}^2\right)$. The leading double-logarithmic corrections may well amount to ten or even twenty percent in one-loop approximation and reach a few percent in two-loop approximation. Moreover, in the TeV region, the subleading logarithms turn out to be equally important [6, 10] and a percent accuracy of the theoretical estimates for the cross sections necessary for the search of new physics beyond the standard model can be guaranteed only by including *all* the logarithmic two-loop corrections.

The calculation of the two-loop electroweak corrections even in the high energy limit is an extremely challenging theoretical problem. It is complicated in particular by the presence of the mass gap and mixing in the gauge sector. However, the logarithmic corrections are quite insensitive to fine details of the spontaneous symmetry breaking. The calculation of the leading logarithmic (LL) electroweak corrections can be performed using the fields of the unbroken symmetry phase and how the infrared singular virtual photon contribution can be separated within the infrared evolution equation approach [5]. This scheme has been extended to the next-to-leading (NLL) and next-to-next-to-leading logarithmic (N^{2}LL) approximation in Refs. [6, 10].

In the study of Sudakov corrections the analysis of the form factor plays a special role since it is the simplest quantity which includes the complete information about the universal *collinear* logarithms [12] directly applicable to a process with an arbitrary number of fermions. In this Letter we formulate a general matching procedure which relates the logarithmic corrections in the theories with and without mass gap by combining the hard and infrared evolution equation approach with the explicit two-loop results for the form factor in an Abelian gauge model. This reduces the calculation of the dominant two-loop corrections to electroweak processes at high energy to a single-mass problem without mixing.

The structure of the Letter is as follows. First, we present the explicit two-loop results for the form factor in a $U(1)$ model with a massive gauge boson. Then we introduce the evolution equations, compute the two-loop corrections to the form factor in a $U(1) \times U(1)$ model with mass gap, and establish the matching procedure. Finally we outline how the approach can be applied to the calculation of the two-loop electroweak corrections to neutral current four-fermion processes.

The vector form factor $\mathcal{F}$ determines the fermion scattering amplitude in an external Abelian field. It is a function of the Euclidean momentum transfer $Q^2 = -(p_1 - p_2)^2$ where $p_{1,2}$ is the incoming/outgoing fermion momentum and we consider on-shell massless fermions, $p_1^2 = p_2^2 = 0$. Let us write the perturbative expansion for the form factor as $\mathcal{F}_\alpha(M,Q) = \sum_n \left(\frac{\alpha}{4\pi}\right)^n f^{(n)} \mathcal{F}_B$ where $\mathcal{F}_B$ corresponds to the Born approximation, $f^{(0)} = 1$.

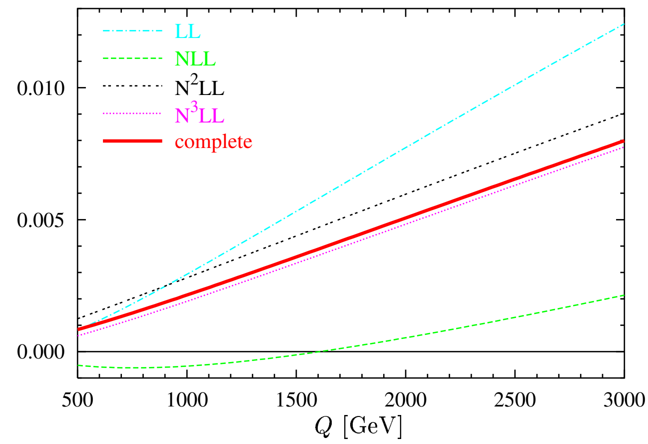

FIG. 1: The two-loop correction to the form factor $\mathcal{F}_\alpha(M,Q)$ in LL (including $\alpha^2\mathcal{L}^4$), NLL (including $\alpha^2\mathcal{L}^3$), N^{2}LL (including $\alpha^2\mathcal{L}^2$), N^{3}LL (including $\alpha^2\mathcal{L}^1$) approximations and the complete two-loop correction as functions of the momentum transfer for $M = 80$ GeV, $\alpha/(4\pi) = 3\cdot 10^{-3}$.

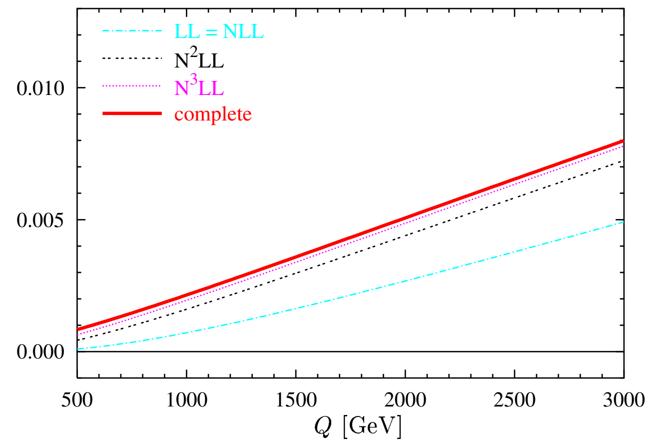

FIG. 2: The same as Fig. 1 after changing the argument of the logarithm.

For the $U(1)$ model with a gauge boson of mass M in the Sudakov limit $M/Q \to 0$ the one-loop correction is well known

$$f^{(1)} = -\mathcal{L}^2 + 3\mathcal{L} - \frac{7}{2} - \frac{2}{3}\pi^2\,, \qquad (1)$$

where $\mathcal{L} = \ln\left(Q^2/M^2\right)$ and all the power-suppressed terms are neglected. For the two-loop term we find

$$\begin{aligned} f^{(2)} &= \frac{1}{2}\mathcal{L}^4 - 3\mathcal{L}^3 + \left(8+\frac{2}{3}\pi^2\right)\mathcal{L}^2 - \left(9+4\pi^2\right. \\ &\left. -24\zeta(3)\right)\mathcal{L} + \frac{25}{2} + \frac{52}{3}\pi^2 + 80\zeta(3) - \frac{52}{15}\pi^4 \\ &-\frac{32}{3}\pi^2\ln^2 2 + \frac{32}{3}\ln^4 2 + 256\,\mathrm{Li}_4\left(\frac{1}{2}\right)\,, \qquad (2)\end{aligned}$$

where $\zeta(3) = 1.202057\ldots$ and $\mathrm{Li}_4\left(\frac{1}{2}\right) = 0.517479\ldots$ are the values of the Riemann's ζ-function and the polylogarithm, respectively. In Eq. (2) we do not include the contribution due to the virtual fermion loop computed in [11]. For the calculation of the leading power behavior of the two-loop on-shell vertex diagrams with two massive propagators in the Sudakov limit we used the expansion by regions approach [13] (for the application to the Sudakov form factor see also [6]). The method is based on the factorization of the contributions of the dynamical modes characteristic for the Sudakov limit [14] in dimensional regularization. Our result for the contribution of the hard modes agrees with the dimensionally regularized massless result of Ref. [15]. Details of our calculation will be published elsewhere.

In Fig. 1 the numerical results for the two-loop correction to the form factor in the different logarithmic approximations are plotted as functions of the momentum for the values of M and α typical for electroweak interactions. The two-loop logarithmic terms have a sign-alternating structure resulting in significant cancellations. In the region of a few TeV the form factor does not reach the double-logarithmic asymptotics. The quartic, cubic and quadratic logarithms are comparable in magnitude and dominate the two-loop corrections. Then the logarithmic expansion starts to converge and, after including the linear-logarithmic contribution, provides a very accurate approximation of the total two-loop correction. Such a behavior is typical for the Sudakov limit and holds for the non-Abelian corrections as well [10, 11]. Note that by rescaling $M \to e^{3/4}M$ in the argument of the logarithm the NLL contribution can be made to disappear. That improves significantly the convergence of the logarithmic expansion and prevents the strong cancellation between the logarithmic terms (see Fig. 2). Still, the N^{3}LL contribution is a must for the quantitative approximation.

The asymptotic dependence of the form factor on Q is governed by the linear *hard* evolution equation [16]. As a consequence, the logarithmic corrections exponentiate. For the purely Abelian contribution the exponent has a particularly simple form

$$\begin{aligned}\mathcal{F}_\alpha(M,Q) &= \exp\left\{\frac{\alpha}{4\pi}\left[-\mathcal{L}^2 + \left(3 - \frac{\alpha}{4\pi}\left(-\frac{3}{2}+2\pi^2\right.\right.\right.\right. \\ &\left.\left.\left.\left. -24\zeta(3)\right) + \mathcal{O}(\alpha^2)\right)\mathcal{L}\right]\right\}\mathcal{F}_\alpha(M,M)\,. \qquad (3)\end{aligned}$$

The double-logarithmic term in the exponent is protected against the Abelian multiloop corrections by the properties of the light-cone Wilson loop [17]. Our two-loop result determines the next-to-next-to-next-to-next-to-leading logarithmic (N^{4}LL) approximation of the form factor which includes the $\alpha^n\mathcal{L}^m$ corrections with $m = 2n-4,\ldots,2n$ to all orders in α.

Let us now turn to the second example with two Abelian gauge bosons of the masses λ and M, $\lambda \ll M$, and couplings α' and α, respectively. We can introduce the *infrared* evolution equation which governs the dependence of the form factor $\mathcal{F}(\lambda, M, Q)$ on λ [5]. The virtual corrections become divergent in the limit $\lambda \to 0$. According to the Kinoshita-Lee-Nauenberg theorem [18], these divergences are cancelled against the ones of the corrections due to the emission of real light gauge bosons of vanishing energy and/or collinear to one of the on-shell fermion lines. The singular behavior of the form factor must be the same in the full $U_{\alpha'}(1) \times U_\alpha(1)$ theory and the effective $U_{\alpha'}(1)$ model with only the light gauge boson. Thus for $\lambda \ll M \ll Q$ the solution of the infrared evolution equation is given by the exponent of Eq. (3) with M, α replaced by λ, α', and the form factor can be written in a factorized form

$$\mathcal{F}(\lambda, M, Q) = \tilde{F}(M, Q)\mathcal{F}_{\alpha'}(\lambda, Q) + \mathcal{O}(\lambda/M)\,, \qquad (4)$$

where the function $\tilde{F}(M, Q)$ depends on α and α' and incorporates all the logarithms of the form $\ln\left(Q^2/M^2\right)$. It can be obtained directly by calculating the ratio

$$\tilde{F}(M, Q) = \left[\frac{\mathcal{F}(\lambda, M, Q)}{\mathcal{F}_{\alpha'}(\lambda, Q)}\right]_{\lambda\to 0}\,. \qquad (5)$$

Since the function $\tilde{F}(M, Q)$ does not depend on the infrared regularization, we compute the ratio in Eq. (5) with $\lambda = 0$ using dimensional regularization for the infrared divergences. The method of calculation of the two-loop diagrams with both massive and massless gauge bosons is similar to the purely massive case. We obtain the two-parameter perturbative expansion $\tilde{F}(M, Q) = \sum_{n,m} \frac{\alpha'^n \alpha^m}{(4\pi)^{n+m}} \tilde{f}^{(n,m)}$, where $\tilde{f}^{(0,0)} = 1$, $\tilde{f}^{(n,0)} = 0$, $\tilde{f}^{(0,m)} = f^{(m)}$, and the two-loop interference term reads

$$\begin{aligned}\tilde{f}^{(1,1)} &= \left(3 - 4\pi^2 + 48\zeta(3)\right)\mathcal{L} - 2 + \frac{20}{3}\pi^2 \\ &\quad -84\zeta(3) + \frac{7}{45}\pi^4\,. \end{aligned}\qquad (6)$$

The numerical structure of the corrections to $\tilde{F}(M, Q)$ is very similar to the one of $\mathcal{F}_\alpha(M, Q)$ (see Fig. 1).

In the equal mass case, $\lambda = M$, we have an additional reparameterization symmetry, and the form factor is determined by Eq. (3) with the effective coupling $\bar{\alpha} = \alpha' + \alpha$ so that $\mathcal{F}(M, M, Q) = \mathcal{F}_{\bar{\alpha}}(M, Q)$. We can now write down the matching relation

$$\mathcal{F}(M, M, Q) = C(M, Q)\tilde{F}(M, Q)\mathcal{F}_{\alpha'}(M, Q)\,, \qquad (7)$$

where the matching coefficient $C(M, Q)$ represents the effect of the power-suppressed terms neglected in Eq. (4). By combining the explicit results for $\mathcal{F}_{\alpha'}(M, Q)$ and $\tilde{F}(M, Q)$ we find the two-loop matching coefficient

$$\begin{aligned}C(M, Q) &= 1 + \frac{\alpha'\alpha}{(4\pi)^2}\left[\frac{59}{4} + \frac{70}{3}\pi^2 + 244\zeta(3) - \frac{113}{15}\pi^4 \right. \\ &\quad \left. -\frac{64}{3}\pi^2 \ln^2 2 + \frac{64}{3}\ln^4 2 + 512\,\mathrm{Li}_4\left(\frac{1}{2}\right)\right]. \end{aligned}\qquad (8)$$

Eq. (8) does not contain logarithmic terms, and up to the N^{3}LL accuracy the product $\tilde{F}(M, Q)\mathcal{F}_{\alpha'}(\lambda, Q)$ continuously approaches $\mathcal{F}(M, M, Q)$ as λ goes to M. Therefore, to get *all* the logarithms of the heavy gauge boson mass in two-loop approximation for the theory with mass gap, it is sufficient to divide the form factor $\mathcal{F}_{\bar{\alpha}}(M, Q)$ of the symmetric phase by the form factor $\mathcal{F}_{\alpha'}(\lambda, Q)$ of the effective $U_{\alpha'}(1)$ theory taken at the symmetric point $\lambda = M$. Thus we have reduced the calculation in the theory with mass gap to the one in the symmetric theory with a single mass parameter. Note that the absence of the linear-logarithmic term in Eq. (8) is an exceptional feature of the Abelian corrections. The general analysis of the evolution equation [10] shows the terms neglected in Eq. (4) to contribute starting from the N^{3}LL approximation. This implies the absence of the second and higher powers of the logarithm in the matching coefficient of Eq. (7), irrespectively of the gauge group and the mass generation mechanism. Moreover, in the approximately equal mass case, $(M - \lambda)/M \equiv \delta \ll 1$, one can compute the form factor as an expansion around the equal mass result. Up to N^{2}LL accuracy only the leading term of Eq. (4) contributes and the expansion takes the form

$$\begin{aligned}\mathcal{F}(\lambda, M, Q)|_{\lambda\to M} &= \left[1 - \delta\frac{\alpha'}{\pi}\left(\mathcal{L} - \frac{3}{2}\right) + \mathcal{O}(\delta^2)\right] \\ &\quad \times\mathcal{F}_{\bar{\alpha}}(M, Q) + \mathcal{O}(\delta\alpha'\alpha\mathcal{L})\,. \end{aligned}\qquad (9)$$

Let us show how the above procedure applies to the calculation of two-loop electroweak corrections. To be specific, we consider a four-fermion neutral current process, which is of primary phenomenological importance, with light fermions. The four-fermion amplitude can be decomposed into (the square of) the form factor and a *reduced* amplitude [6, 10]. The latter carries all the Lorentz and isospin indices and does not contain collinear logarithms in perturbative expansion. The logarithmic corrections to the reduced amplitude are obtained by solving a renormalization group like equation [19]. The corresponding two-loop anomalous dimensions can be extracted from the existing massless QCD calculations [20] (see [10, 21]). Thus, the problem of the calculation of the two-loop electroweak logarithms in the four-fermion processes reduces to the analysis of the form factor.

In Ref. [10] by analyzing the hard evolution equation it has been found that the two-loop electroweak corrections up to the next-to-next-to-leading (quadratic) logarithms are not sensitive to the structure of the theory at the electroweak symmetry breaking scale. The prediction of Ref. [6, 10] for the two-loop logarithmic corrections fully agrees with the available explicit results for the light fermion contribution [11] and the Abelian contribution obtained in this Letter. The only trace of the Higgs mechanism of the gauge boson mass generation in N^{2}LL approximation is the $Z - W$ boson mass splitting which can be systematically taken into account within an expansion around the equal mass approximation similar to Eq. (9). Thus, the calculation of the two-loop electroweak corrections up to the quadratic logarithms can be performed in two steps outlined above: (i) the corrections are evaluated using the fields of unbroken symmetry phase with all the gauge bosons of the same mass $M \approx M_{Z,W}$ introduced by hand; (ii) the QED contribution with an auxiliary photon mass M is factorized as in Eq. (7) leaving the pure electroweak logarithms. The separated virtual QED corrections accompanied by the real photon radiation in the limit of vanishing photon mass result in the universal infrared safe factor independent of $M_{Z,W}$.

By contrast, the N^{3}LL approximation is sensitive to fine details of the gauge boson mass generation and the coefficient of the linear two-loop electroweak logarithm depends *e.g.* on the Higgs boson mass. For the full calculation of this coefficient one has to use the true mass eigenstates of the standard model. Our result, Eqs. (2, 6), is an example of such a calculation when applied to the two-loop diagrams with photon and Z boson exchanges. We can, however, make a reasonable approximation which dramatically simplifies the analysis. Namely, consider a simplified model with a Higgs boson of zero hypercharge. Then the mixing is absent and the hypercharge gauge boson remains massless. The interference diagrams including the heavy $SU_L(2)$ and the light hypercharge $U(1)$ gauge bosons are identical with the ones of the purely Abelian model discussed in this Letter, where the above two-step procedure can be applied to get all the two-loop logarithms including the linear term. In the standard model the mixing of the gauge bosons results in a linear-logarithmic contribution, which is not accounted for within this procedure. It is, however, suppressed by a small factor $\sin^2\theta_W \approx 0.2$, with θ_W being the Weinberg angle. Therefore, the above simplified model gives an estimate of the coefficient in front of the linear electroweak logarithm with 20% accuracy. From the numerical result of Fig. 1, which represents the typical structure of the two-loop corrections, we see that a 20% error in this coefficient leads to an uncertainty comparable to the nonlogarithmic contribution and is practically negligible. Thus we are able to get an accurate estimate of the two-loop correction, which is sufficient for practical applications to the future collider physics, by performing the calculations in the model without mass gap and mixing of the gauge bosons. The last ingredient necessary to complete the calculation of the dominant two-loop electroweak corrections is the generalization of Eq. (2) up to the linear-logarithmic term to the pure $SU_L(2)$ gauge model with the Higgs mechanism of mass generation. Note that up to the quadratic logarithm the two-loop corrections are predicted by the evolution equation [10].

To conclude, we have obtained the complete results for the two-loop corrections to the vector form factor in the Sudakov limit in Abelian theories with one massive gauge boson or with two gauge bosons of essentially different masses. The results are in full agreement with the predictions of the evolution equation approach. We have formulated a systematic procedure of factorizing the infrared singular virtual corrections and reducing the calculation in the theory with mass gap to the single-mass problem. The analysis can directly be generalized to the standard model with spontaneous breaking of the $SU_L(2) \times U(1)$ theory to low-energy QED. This solves the principal problems of the calculation of the dominant two-loop electroweak corrections to the neutral current four-fermion processes which are mandatory for the high-precision physics at the LHC and the next generation of linear colliders.

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