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Four-loop renormalization of QCD: full set of renormalization constants and anomalous dimensions

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Abstract

The anomalous dimensions of the gluon and ghost fields as well as those of the ghost-ghost-gluon and quark-quark-gluon vertexes are analytically computed at four loops in pQCD. Taken together with already available anomalous dimensions of the coupling constant, the quark field and the mass the results lead to complete knowledge of *all* renormalization constant entering into the renormalization of the QCD Lagrangian at the four-loop level. As a by-product we get scale and scheme invariant gluon and ghost propagators at NNNLO. Using a theorem due to Dudal, Verschelde and Sorella, we also construct the four-loop anomalous dimension of the "gluon mass operator", A^2, in the Landau gauge.

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1 Introduction

The renormalization group equation is a powerful tool in investigating the properties of the Green functions of a renormalizable field theory. Its crucial ingredients are the anomalous dimensions of quantum fields as well as those of mass and coupling constant(s).

In recent years there has been achieved a significant progress in perturbative calculation of higher orders corrections to renormalization group functions For example, the most physically important RG functions of QCD—the β function and the quark mass and field anomalous dimensions—have been computed at a record-setting four-loup level [1–4].

In the same time the anomalous dimensions of the gluon and ghost fields are available in literature only at three-loop level [4, 5]. It is unfortunate for at least two reasons. First, the gluon and ghost field anomalous dimensions are important in comparison of the non-perturbative results for the momentum dependence of the corresponding propagators with perturbative predictions [6–17]. Second, only the knowledge of anomalous dimensions of *all* fields of the QCD Lagrangian leads, together with the β- function, to complete reconstruction of *all* Renormalization Constants (RCs) entering into the renormalization of the QCD Lagrangian (see below).

In the present paper we fill the gap by analytically computing the anomalous dimensions of the gluon and ghost fields as well as that of the ghost-ghost-gluon vertex at four loops. All calculations have been done in the general covariant gauge.

We apply our results to find the scheme and scale invariant gluon and ghost propagators at Next-Next-Next-Leading Order (NNNLO) as well as the four-loop anomalous dimension of the composite operator A^2.

2 Notations and generalities

The QCD Lagrangian with n_f quark flavors in the covariant gauge reads:

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{f=1}^{n_f}\bar{\psi}^f(\mathrm{i}\not{D} - m_f)\psi^f \tag{1}$$

$$- \frac{1}{2\xi_L}(\partial^\mu A^a_\mu)^2 + \partial^\mu\bar{c}^a(\partial c^a - g f^{abc}c^b A^c_\mu), \tag{2}$$

where

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,(A_\mu \times A_\nu)^a, \quad (A\times B)^a = f^{abc}A^b B^c, \tag{3}$$

$$D_\mu = \partial_\mu - \mathrm{i}g A^a_\mu T^a, \quad \not{D} = \gamma^\mu D_\mu. \tag{4}$$

The quark field ψ_i^f has a mass m_f and transforms as the fundamental representation and the gluon fields A_μ^a as the adjoint representation of the gauge group SU(3). T_{ij}^a and f^{abc} are the generators of the fundamental and adjoint representation of the corresponding Lie algebra. The c^a are the ghost fields and ξ_L is the gauge parameter ($\xi_L = 0$ corresponds to the Landau gauge).

By adding to (1) all counterterms necessary to remove UV divergences from Green functions, one arrives at the bare QCD Lagrangian written in terms of the renormalized fields[2] :

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{4} Z_3 \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2} g\, Z_1^{3g} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) \left(A_\mu \times A_\nu\right)^a \\ &- \frac{1}{4} g^2\, Z_1^{4g} \left(A_\mu \times A_\nu\right)^2 - Z_\xi^{2g} \frac{1}{2\xi_L} \left(\partial_\nu A_\mu\right)^2 + Z_3^c\, \partial_\nu \bar{c} \left(\partial_\nu c\right) \\ &+ g\, Z_1^{ccg}\, \partial^\mu \bar{c} \left(A \times c\right) + Z_2 \sum_{f=1}^{n_f} \bar{\psi}^f \left(\mathrm{i}\partial\!\!\!/ + g Z_1^{\psi\psi g} Z_2^{-1} A\!\!\!/ - Z_m m_f\right) \psi^f .\end{aligned} \tag{5}$$

Here Z_ξ^{2g} is expressed through the RC of the gauge fixing parameter ξ_L as follows

$$\xi_{L,0} = Z_\xi \xi_L, \quad Z_\xi^{2g} = Z_3 / Z_\xi .$$

Z_3, Z_2, Z_3^c are the wave-function RCs appearing in the relations between the renormalized and bare gluon, quark and ghosts fields, viz.

$$A_0^{a\mu} = \sqrt{Z_3}\; A^{a\mu}, \quad \psi_0^f = \sqrt{Z_2}\; \psi_0^f, \quad c_0^a = \sqrt{Z_3^c}\; c^a. \tag{6}$$

The full set of the vertex RCs

$$Z_1^V, \quad V \in \{3\mathrm{g},\ 4\mathrm{g},\ \mathrm{ccg},\ \psi\psi\mathrm{g}\} \tag{7}$$

serve to renormalize 3-gluon, 4-gluon, ghost-ghost-gluon, quark-quark-gluon vertex functions respectively.

The Slavnov-Taylor identities allows one to express all four vertex RCs in terms of an independent one, $Z_g = \frac{g_0 \mu^{-\epsilon}}{g}$, and the above listed wave function RCs. The corresponding relations are:

$$Z_\xi = Z_3, \tag{8}$$

$$Z_g = \sqrt{Z_1^{4g}}\, (Z_3)^{-1}, \tag{9}$$

$$Z_g = Z_1^{3g} (Z_3)^{-3/2}, \tag{10}$$

$$Z_g = Z_1^{ccg} (Z_3)^{-1/2} (Z_3^c)^{-1}, \tag{11}$$

$$Z_g = Z_1^{\psi\psi g} (Z_3)^{-1/2} (Z_2)^{-1}. \tag{12}$$

[2]For simplicity we set the t′ Hooft mass $\mu = 1$ in eq. (5) below.

Within the $\overline{\mathrm{MS}}$ scheme each RC does not depend on dimensional parameters (masses and momenta) and can be represented as follows

$$Z(h) = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}(h)}{\epsilon^n}, \tag{13}$$

where $h = g^2/(16\pi^2)$ and the parameter ϵ is related to the running space time dimension D via $D = 4 - 2\epsilon$. Given a RC $Z(h)$, the corresponding anomalous dimension is defined as

$$\gamma(h) = -\mu^2 \frac{\mathrm{d} \log Z(h)}{\mathrm{d}\mu^2} = h \frac{\partial z^{(1)}(h)}{\partial h} = -\sum_{n=0}^{\infty} (\gamma)_n \, h^{(n+1)}. \tag{14}$$

Customarily, one also defines $Z_h = Z_g^2$ and refers to the corresponding anomalous dimension as the QCD β-function:

$$\beta(h) = 2\gamma_g(h) = 2h \, \frac{\partial z_g^{(1)}(h)}{\partial h} = -\sum_{n=0}^{\infty} \beta_n h^{(n+1)}. \tag{15}$$

Eqs. (9-12) imply that

$$\beta = \gamma_1^{4g} - 2\,\gamma_3, \tag{16}$$
$$\beta = 2\,\gamma_1^{3g} - 3\,\gamma_3, \tag{17}$$
$$\beta = 2\,\gamma_1^{ccg} - 2\,\gamma_3^c - \gamma_3, \tag{18}$$
$$\beta = 2\,\gamma_1^{\psi\psi g} - 2\,\gamma_2 - \gamma_3. \tag{19}$$

As is well-known there is a one-to-one correspondence between an anomalous dimension and the corresponding RC. For instance, Z_h obeys an equation

$$(-\epsilon + \beta(h)) \, h \, \frac{\partial \log Z_h}{\partial h} = -\beta(h) \tag{20}$$

which leads to

$$\log Z_h = \int \frac{\mathrm{d}h}{h} \, \frac{\beta}{\epsilon - \beta}.$$

In general case Z depends on μ through both h and ξ_L and an analog of eq. (20) assumes the form:

$$(-\epsilon + \beta(h)) \, h \, \frac{\partial \log Z}{\partial h} + \gamma_3(h) \, \xi_L \, \frac{\partial \log Z}{\partial \xi_L} = -\gamma(h). \tag{21}$$

Eq. (21) can be easily utilized to reconstruct Z from γ and β. As anomalous dimensions are more compact than corresponding RCs in what follows we will write explicitly only the former.

3 Calculations and Results

Relations (9-12) demonstrate that a minimal set of the RCs necessary to reconstruct all coefficients of the bare QCD Lagrangian (5) consists of Z_m, all three wave-function RCs Z_3, Z_2, Z_3^c, and the coupling constant RC Z_g or, instead, at least one from the collection $Z_1^V, \quad V \in \{3\text{g}, \ 4\text{g}, \ \text{ccg}, \ \psi\psi\text{g}\}$. Taking into account that Z_m, Z_2 and Z_g are known with four loop accuracy from the works [1–4], one is left with just two specific RC to compute, say, Z_3 and Z_3^c.

At present there are basically two different ways to perform RG calculations at the four-loop level. Both approaches make use of the method of Infrared Rearrangement (IRR) Ref. [18] in order to set zero (possibly after a proper Taylor expansion) masses and external momenta. Both eventually employ the traditional integration by parts method to compute the resulting Feynman integrals[3].

The first one, pioneered in the yearly works of Dubna group [24–26], amounts to adding an artificial mass or an external momentum to a properly chosen propagator of a given Feynman diagram before the expansion in masses and true external momenta is made. The artificial external momentum has to be introduced in such a way that all spurious infrared divergences are removed and the obtained Feynman integral is calculable. In practical multiloop calculations the condition of absence of the infrared divergences leads to unnecessary complications and, in some cases, even prevents from reduction to the simplest integrals. The problem was solved with elaborating a special technique of subtraction of IR divergences — the R^*-operation [27–29]. This technique succeeds in expressing the UV counterterm of every (L+1)-loop Feynman integral in terms of divergent and finite parts of some L-loop massless propagators.

In the second approach the infrared rearrangement is performed by introducing a *single* auxiliary mass to *all* propagators in each Feynman diagram at hand [1, 30, 31]. No IR divergences can ever appear due to absence of any massless propagators. Next, after a proper expansion in all the particle masses (except the auxiliary one) and external momenta is performed. The resulting integrals are completely massive tadpoles, i.e. Feynman integrals without external momenta and with only a single mass inserted in all the propagators.

In our calculation of Z_3 and Z_3^c we have used the first, "massless" approach. It proved also to be more convenient to compute the RC Z_1^{ccg} instead of Z_3 and then to find Z_3 from eq. (11).

The four-loop diagrams contributing to the ghost propagator and to the ghost-ghost-gluon vertex to order α_s^4 (altogether about 35000) have been generated with the program QGRAF [32], then globally rearranged to a product of some three-loop p-integrals with a trivial (essentially one-loop) massive Feynman integral

[3] Though the technical implementations could be quite different, cf. works [19–23].

and, finally, computed with the program MINCER [33, 34]. The total amount of CPU time needed to compute RC Z_3^c and Z_1^{ccg} was about a month of work of a standard PC with an Athlon XP 2000+ processor. For testing purposes we have also computed the RC Z_1^{qqg}, which has required an almost double amount of calculational time[4].

Our results for the anomalous dimensions γ_1^{ccg} and γ_3^c read

$$(\gamma_1^{ccg})_0 = \frac{3}{2}\,\xi_L, \tag{22}$$

$$(\gamma_1^{ccg})_1 = \frac{45}{8}\,\xi_L + \frac{9}{8}\,\xi_L^2, \tag{23}$$

$$(\gamma_1^{ccg})_2 = \frac{5427}{64}\,\xi_L + \frac{1053}{64}\,\xi_L^2 + \frac{135}{32}\,\xi_L^3 - \frac{135}{16}\,\xi_L\,n_f, \tag{24}$$

$$\begin{aligned}
(\gamma_1^{ccg})_3 = & \\
& \frac{635749}{384}\,\xi_L + \frac{21519}{32}\,\zeta_3\,\xi_L + \frac{729}{64}\,\zeta_4\,\xi_L - \frac{91125}{128}\,\zeta_5\,\xi_L + \frac{29547}{128}\,\xi_L^2 \\
& + \frac{6993}{64}\,\zeta_3\,\xi_L^2 + \frac{243}{16}\,\zeta_4\,\xi_L^2 - \frac{8505}{128}\,\zeta_5\,\xi_L^2 + \frac{7371}{128}\,\xi_L^3 + \frac{729}{16}\,\zeta_3\,\xi_L^3 \\
& + \frac{243}{64}\,\zeta_4\,\xi_L^3 - \frac{6075}{128}\,\zeta_5\,\xi_L^3 + \frac{1539}{128}\,\xi_L^4 - \frac{459}{64}\,\zeta_3\,\xi_L^4 + \frac{945}{128}\,\zeta_5\,\xi_L^4 \\
+ \quad & n_f\left[-\frac{54623}{288}\,\xi_L - \frac{663}{8}\,\zeta_3\,\xi_L - \frac{99}{4}\,\zeta_4\,\xi_L - \frac{453}{32}\,\xi_L^2 - \frac{45}{8}\,\zeta_3\,\xi_L^2\right] \\
+ \quad & n_f^2\left[-\frac{251}{54}\,\xi_L + 6\,\zeta_3\,\xi_L\right],
\end{aligned} \tag{25}$$

$$(\gamma_3^c)_0 = -\frac{9}{4} + \frac{3}{4}\,\xi_L, \tag{26}$$

$$(\gamma_3^c)_1 = -\frac{285}{16} - \frac{9}{16}\,\xi_L + \frac{5}{4}\,n_f, \tag{27}$$

$$\begin{aligned}
(\gamma_3^c)_2 \quad = \quad & -\frac{15817}{64} - \frac{243}{32}\,\zeta_3 + \frac{459}{32}\,\xi_L - \frac{81}{8}\,\zeta_3\,\xi_L + \frac{81}{32}\,\xi_L^2 \\
& - \frac{81}{32}\,\zeta_3\,\xi_L^2 + \frac{81}{64}\,\xi_L^3 \\
+ \quad & n_f\left[\frac{637}{24} + \frac{33}{2}\,\zeta_3 - \frac{63}{16}\,\xi_L\right] + \frac{35}{36}\,n_f^2,
\end{aligned} \tag{28}$$

[4] These figures should be understood as *effective* ones; that is in reality we have used a sort of trivial parallelization by distributing diagrams between a few PC's.

$$(\gamma_3^c)_3 =$$
$$-\frac{2857419}{512} - \frac{1924407}{512}\,\zeta_3 + \frac{8019}{64}\,\zeta_4 + \frac{40905}{8}\,\zeta_5 + \frac{368231}{1536}\,\xi_L$$
$$- \frac{75573}{256}\,\zeta_3\,\xi_L + \frac{17901}{128}\,\zeta_4\,\xi_L - \frac{12015}{256}\,\zeta_5\,\xi_L + \frac{17613}{512}\,\xi_L^2 - \frac{4131}{128}\,\zeta_3\,\xi_L^2$$
$$+ \frac{81}{2}\,\zeta_4\,\xi_L^2 - \frac{21465}{256}\,\zeta_5\,\xi_L^2 + \frac{4185}{512}\,\xi_L^3 + \frac{9045}{256}\,\zeta_3\,\xi_L^3 + \frac{729}{128}\,\zeta_4\,\xi_L^3$$
$$- \frac{17145}{256}\,\zeta_5\,\xi_L^3 + \frac{81}{32}\,\xi_L^4 - \frac{3213}{512}\,\zeta_3\,\xi_L^4 + \frac{945}{256}\,\zeta_5\,\xi_L^4\Big]$$
$$+ \quad n_f\Big[\frac{1239661}{1152} + \frac{48857}{48}\,\zeta_3 - \frac{8955}{32}\,\zeta_4 - \frac{3355}{4}\,\zeta_5 - \frac{11107}{288}\,\xi_L$$
$$- \frac{39}{8}\,\zeta_3\,\xi_L - \frac{153}{8}\,\zeta_4\,\xi_L - \frac{651}{128}\,\xi_L^2 - \frac{27}{8}\,\zeta_3\,\xi_L^2 - \frac{27}{32}\,\zeta_4\,\xi_L^2\Big]$$
$$+ \quad n_f^2\Big[-\frac{586}{27} - \frac{55}{2}\,\zeta_3 + \frac{33}{2}\,\zeta_4 - \frac{779}{432}\,\xi_L + 3\,\zeta_3\,\xi_L\Big]$$
$$+ \quad n_f^3\Big[\frac{83}{108} - \frac{4}{3}\,\zeta_3\Big]. \tag{29}$$

Finally, a use of eq. (18) immediately leads us to

$$(\gamma_3)_0 = -\frac{13}{2} + \frac{3}{2}\,\xi_L + n_f\frac{2}{3}, \tag{30}$$

$$(\gamma_3)_1 = -\frac{531}{8} + \frac{99}{8}\,\xi_L + \frac{9}{4}\,\xi_L^2 + \frac{61}{6}\,n_f, \tag{31}$$

$$(\gamma_3)_2 =$$
$$-\frac{29895}{32} + \frac{243}{16}\,\zeta_3 + \frac{4509}{32}\,\xi_L + \frac{81}{4}\,\zeta_3\,\xi_L + \frac{891}{32}\,\xi_L^2$$
$$+ \frac{81}{16}\,\zeta_3\,\xi_L^2 + \frac{189}{32}\,\xi_L^3$$
$$+ \quad n_f\Big[\frac{8155}{36} - 33\,\zeta_3 - 9\,\xi_L\Big] - \frac{215}{27}\,n_f^2, \tag{32}$$

$$(\gamma_3)_3 =$$
$$-\frac{10596127}{768} + \frac{1012023}{256}\,\zeta_3 - \frac{8019}{32}\,\zeta_4 - \frac{40905}{4}\,\zeta_5 + \frac{2174765}{768}\,\xi_L$$
$$+ \frac{247725}{128}\,\zeta_3\,\xi_L - \frac{16443}{64}\,\zeta_4\,\xi_L - \frac{170235}{128}\,\zeta_5\,\xi_L + \frac{100575}{256}\,\xi_L^2 + \frac{18117}{64}\,\zeta_3\,\xi_L^2$$
$$- \frac{405}{8}\,\zeta_4\,\xi_L^2 + \frac{4455}{128}\,\zeta_5\,\xi_L^2 + \frac{25299}{256}\,\xi_L^3 + \frac{2619}{128}\,\zeta_3\,\xi_L^3 - \frac{243}{64}\,\zeta_4\,\xi_L^3$$
$$+ \frac{4995}{128}\,\zeta_5\,\xi_L^3 + \frac{1215}{64}\,\xi_L^4 - \frac{459}{256}\,\zeta_3\,\xi_L^4 + \frac{945}{128}\,\zeta_5\,\xi_L^4$$
$$+ \quad n_f\Big[\frac{23350603}{5184} - \frac{387649}{216}\,\zeta_3 + \frac{8955}{16}\,\zeta_4 + \frac{3355}{2}\,\zeta_5 - \frac{10879}{36}\,\xi_L$$

$$
\begin{aligned}
& - 156\,\zeta_3\,\xi_L - \frac{45}{4}\,\zeta_4\,\xi_L - \frac{1161}{64}\,\xi_L^2 - \frac{9}{2}\,\zeta_3\,\xi_L^2 + \frac{27}{16}\,\zeta_4\,\xi_L^2\Big] \\
+ \quad & n_f^2\left[-\frac{43033}{162} - \frac{2017}{81}\,\zeta_3 - 33\,\zeta_4 - \frac{1229}{216}\,\xi_L + 6\,\zeta_3\,\xi_L\right] \\
+ \quad & n_f^3\left[-\frac{4427}{1458} + \frac{8}{3}\,\zeta_3\right].
\end{aligned}
\tag{33}
$$

For an important particular case of the Landau gauge $\xi_L = 0$ we get:

$$
(\gamma_3)_0 = -\frac{13}{2} + n_f\frac{2}{3}, \quad (\gamma_3)_1 = -\frac{531}{8} + n_f\frac{61}{6},
\tag{34}
$$

$$
(\gamma_3)_2 = -\frac{29895}{32} + \frac{243}{16}\,\zeta_3 + n_f\left[\frac{8155}{36} - 33\,\zeta_3\right] - n_f^2\frac{215}{27},
\tag{35}
$$

$$
\begin{aligned}
(\gamma_3)_3 = \quad & - \quad \frac{10596127}{768} + \frac{1012023}{256}\,\zeta_3 - \frac{8019}{32}\,\zeta_4 - \frac{40905}{4}\,\zeta_5 \\
& + \quad n_f\left[\frac{23350603}{5184} - \frac{387649}{216}\,\zeta_3 + \frac{8955}{16}\,\zeta_4 + \frac{3355}{2}\,\zeta_5\right] \\
& + \quad n_f^2\left[-\frac{43033}{162} - \frac{2017}{81}\,\zeta_3 - 33\,\zeta_4\right] \\
& + \quad n_f^3\left[-\frac{4427}{1458} + \frac{8}{3}\,\zeta_3\right].
\end{aligned}
\tag{36}
$$

or, numerically,

$$
\begin{aligned}
\gamma_3 \quad = \quad & h(6.5 - 0.666667 n_f) + h^2(66.375 - 10.1667 n_f) \\
+ \quad & h^3(915.963 - 186.86 n_f + 7.96296 n_f^2) \\
+ \quad & h^4(19920.2 - 4692.27 n_f + 331.285 n_f^2 - 0.169134 n_f^3).
\end{aligned}
\tag{37}
$$

4 Applications

In this section we consider some applications of our results. The case of the massless QCD with the Landau gauge fixing is understood in both subsections.

4.1 Scheme-invariant Gluon and Ghost Propagators in NNNLO

In general case a (multiplicatively renormalizable) Green function G depends on both a renormalization prescription (*scheme*) and the choice of the normalization scale μ. In many cases it is more convenient to deal with the scheme and scale invariant version of G which we will denote as $\hat{G}$. Given the RG equation for G

$$
\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}G(h,\mu) \equiv \left(\mu^2\frac{\partial}{\partial\mu^2} + \beta(h)h\frac{\partial}{\partial h}\right)G = \gamma(h)\,G(h,\mu),
\tag{38}
$$

then a formal solution for $\hat{G}$ reads

$$\hat{G} = G(h,\mu)/f(h), \quad f(h) = \exp\left\{ \int^h \frac{\mathrm{d}x}{x} \, \frac{\gamma(x)}{\beta(x)} \right\}, \tag{39}$$

$$\begin{aligned}
f(h) &= (h)^{\bar{\gamma}_0} \Big\{ 1 + (\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0) h \\
&+ \frac{1}{2}\Big[(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0 \Big] h^2 \\
&+ \Big[\frac{1}{6}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^3 + \frac{1}{2}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0) \\
&+ \frac{1}{3}\Big(\bar{\gamma}_3 - \bar{\beta}_1{}^3\bar{\gamma}_0 + 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_0 - \bar{\beta}_3\bar{\gamma}_0 + \bar{\beta}_1^2\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_2 \Big) \Big] h^3 + \mathcal{O}(h^4) \Big\} .
\end{aligned} \tag{40}$$

Here $\bar{\gamma}_i = \gamma_i/\beta_0$, $\bar{\beta}_i = \beta_i/\beta_0$, (i=1,2,3), the coefficients γ_i, β_i are defined in eqs. (14,15).

Now, after writing the gluon and ghost propagators in the form

$$D^{ab}_{\mu\nu}(q) = \frac{\delta^{ab}}{-q^2}\left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right] D(-q^2), \quad \Delta^{ab}(q) = \frac{\delta^{ab}}{-q^2} \ \Delta(-q^2) \tag{41}$$

and a use of explicit expressions for the very propagators from [35] we arrive at the following NNNLO predictions for the asymptotic behavior of the scheme and scale invariant functions $\hat{D}$ and $\hat{\Delta}$ at large Euclidean $Q^2 = -q^2$ in the $\overline{\mathrm{MS}}$ scheme. First, for the case of pure gluodynamics ($n_f = 0$)

$$\begin{aligned}
h^{\frac{13}{22}} \ \hat{D}^{-1}(-q^2)|_{n_f=0} &= 1 - \frac{25085}{2904}\, h + h^2 \left[-\frac{412485993}{1874048} + \frac{9747}{352}\zeta_3 \right] \\
&+ \ h^3 \left[-\frac{141629801206331}{16326706176} + \frac{80968605}{42592}\zeta_3 + \frac{477315}{704}\zeta_5 \right] \\
&= 1 - 8.63809 h - 186.819\, h^2 - 5686.55\, h^3,
\end{aligned} \tag{42}$$

$$\begin{aligned}
h^{\frac{9}{44}} \ \hat{\Delta}^{-1}(-q^2)|_{n_f=0} &= 1 - \frac{5271}{1936}\, h + h^2 \left[-\frac{615512003}{7496192} + \frac{5697}{704}\zeta_3 \right] \\
&+ \ h^3 \left[-\frac{430343889400537}{130613649408} + \frac{674654895}{1362944}\zeta_3 + \frac{73845}{352}\zeta_5 \right] \\
&= 1 - 2.72262\, h - 72.3825\, h^2 - 2482.24\, h^3.
\end{aligned} \tag{43}$$

In order to illustrate the n_f dependence we give below the results for $n_f = 3$ and $n_f = 6$ (to save space only in the numerical form)

$$h^{\frac{1}{2}} \ \hat{D}^{-1}(-q^2)|_{n_f=3} = 1 - 5.18056\, h - 85.0853\, h^2 - 2178.1\, h^3, \tag{44}$$

$$h^{\frac{1}{4}} \ \hat{\Delta}^{-1}(-q^2)|_{n_f=3} = 1 - 2.78472\, h - 52.591\, h^2 - 1359.11\, h^3, \tag{45}$$

$$h^{\frac{5}{14}}\ \hat{D}^{-1}(-q^2)|_{n_f=6} = 1 - 0.857993\,h + 16.6153\,h^2 + 220.455\,h^3, \tag{46}$$

$$h^{\frac{9}{28}}\ \hat{\Delta}^{-1}(-q^2)|_{n_f=6} = 1 - 3.27934\,h - 31.5908\,h^2 - 372.071\,h^3 g. \tag{47}$$

Note that in eqs. (42-47) the coupling constant h should be understood as $h(\mu^2 = -q^2)$.

4.2 Four-Loop Anomalous Dimension of the Composite Operator A^2

Recently there has been a lot of activity in studying the possibility of a condensate in Yang-Mills-theory of mass dimension two (see, e.g. recent works [36, 37] and references therein). The relevance of the operator $A^2 \equiv A^a_\mu A^a_\mu$ in the Landau gauge in that context has been widely discussed. In this connection a thorough investigation of the renormalization properties of the composite operator $A^2 \equiv A^a_\mu A^a_\mu$ has been carried out in [38, 39]. In particular, the author of [38] has discovered by explicit three loop computation a remarkable relation[5]

$$-2\gamma_{A^2}|_{\xi_L=0} = \beta - \gamma_3 \tag{48}$$

expressing the anomalous dimension γ_{A^2} of the operator A^2 in terms of the β-function and the gluon field anomalous dimension γ_3. An all orders proof of (48) have been later constructed in [40] using algebraic renormalization methods. As both ingredients of (48) are known now at four loops one arrives at

$$\begin{aligned}
\gamma_{A^2}|_{\xi_L=0} &= h\left[\frac{35}{4} - \frac{2}{3}n_f\right] + h^2\left[\frac{1347}{16} - \frac{137}{12}\,n_f\right] \\
&+ \quad h^3\left[\frac{75607}{64} - \frac{243}{32}\,\zeta_3 + n_f\left(-\frac{18221}{72} + \frac{33}{2}\,\zeta_3\right) + \frac{755}{108}\,n_f^2\right] \\
&+ \quad h^4\left[\frac{29764511}{1536} - \frac{99639}{512}\,\zeta_3 + \frac{8019}{64}\,\zeta_4 + \frac{40905}{8}\,\zeta_5\right. \\
&\qquad + n_f\left(-\frac{57858155}{10368} + \frac{335585}{432}\,\zeta_3 - \frac{3355}{4}\,\zeta_5 - \frac{8955}{32}\,\zeta_4\right) \\
&\qquad + n_f^2\left(\frac{46549}{162} + \frac{8489}{162}\,\zeta_3 + \frac{33}{2}\,\zeta_4\right) \\
&\qquad \left. + n_f^3\left(\frac{6613}{2916} - \frac{4}{3}\,\zeta_3\right)\right].
\end{aligned} \tag{49}$$

5 Discussion and Conclusion

Calculation of the two missing RCs Z_3 and Z_2^c completes renormalization of the QCD Lagrangian at four loops. An important issue relevant for any calculation

[5]We have adjusted the coefficients in (48) to our notations.

of such complexity is the correctness of the obtained anomalous dimensions. The following comments are in order.

- The FORM program MINCER used by us to compute three loop massless propagators was developed more than a decade ago and has been since heavily cross-checked in a number of various multiloop calculations.

- At three loop level we have full agreement with the results of [4, 5]. This checks our way to use R^*-operation because the three loop results of [4, 5] have been obtained with direct application of MINCER to three-loop propagators. In the present work the *three loop* contributions come exclusively from *two loop* propagators.

- The leading n_f behaviour of the quark, gluon and ghost anomalous dimensions as well as the anomalous dimensions of the quark-quark-gluon and ghost-ghost-gluon vertices was investigated in all orders of perturbation theory in the work [41]. At the four loop level the predictions of [41] are in full agreement to the corresponding leading n_f pieces of our results[6].

- We have also computed the $\mathcal{O}(\alpha_s^4)$ anomalous dimension of the quark-quark-gluon vertex and found that it satisfies eq. (19) as it should.

- As is known from [42, 43] the ghost-ghost-gluon vertex is unrenormalized in the Landau gauge, that is
$$\gamma_1^{ccg}|_{\xi_L=0} = 0. \tag{50}$$
Eq. (50) is in obvious agreement to eqs. (22 - 25).

- We have *not* performed a *direct* calculation of the gluon wave function RC Z_3 but rather extract the result from the β-function of [1] and the anomalous dimensions γ_3^c and γ_1^{ccg}. Thus, an independent reevaluation of either γ_3 or/and the β-function at four loops is highly desirable.

An interesting feature of the scheme and scale invariant functions $\hat{D}$ and $\hat{\Delta}$ is the full absence of the irrational constant ζ_4 as illustrated in eqs. (42,43). It stems from a noni-trivial mutual cancellation of the terms proportional to ζ_4 which enter into the ingredients—the Green function and its anomalous dimension—of the definition of $\hat{G}$ (see eq. (39)). We have checked that this is true also for the $SU(N)$ color group and a generic value of n_f.

Finally, a comment about the gauge group dependence of our results. Preferring compact and readable formulas to huge and general ones, the author has deliberately formatted all results in this paper for the most practically important

[6] We thank John Gracey for this comment.

case of the $SU(3)$ color group. In reality all calculations have been carried out for a little bit more general case of the SU(N) color group. Full expressions of RCs (and the corresponding anomalous dimensions) describing the renormalization of the Lagrangian (5) in the general covariant gauge and for the SU(N) color group are available (in a computer-readable form) in http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp04/ttp04-08/.

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Note 1

Just before the submission of the manuscript for publication we have been informed that our main assumption — the validity of the result for four-loop QCD beta-function first obtained in [1] — is confirmed by a completely independent calculation [44].

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