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Complete Two Loop Bosonic Contributions to the Muon Lifetime in the Standard Model

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The last missing correction to the muon lifetime in the Standard Model at $\mathcal{O}(\alpha^2)$ coming from gauge and Higgs boson loops is presented. The associated contribution to the parameter Δr in the on-shell scheme ranges from 6×10^{-5} to -4×10^{-5} for Higgs boson masses from 100 GeV to 1 TeV. This result translates into a shift of the W boson mass which does not exceed ± 1 MeV in the same range and amounts in particular to approximately -0.8 MeV for a 115 GeV Higgs boson.

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The muon decay lifetime (τ_μ) has long been used as an input parameter for high precision predictions of the Standard Model (SM). It allows for an indirect determination of the mass of the W boson (M_W), which suffers currently from a large experimental error of 39 MeV [1], one order of magnitude worse than that of the Z boson mass (M_Z). A reduction of this error by the Large Hadron Collider (LHC) to 15 MeV [2] and by a future linear collider to 6 MeV [3] would provide a stringent test of the SM by confronting the theoretical prediction with the experimental value.

The extraction of M_W with an accuracy matching that of next experiments, *i.e.* at the level of a few MeV necessitates radiative corrections beyond one loop order. Large two loop contributions from fermionic loops have been calculated in [4]. The current prediction is affected by two types of uncertainties. First, apart from the still unknown Higgs boson mass, two input parameters introduce large errors. The current knowledge of the top quark mass results in an error of about 30 MeV [5], which should be reduced by LHC to 10 MeV and by a linear collider even down to 1.2 MeV. The inaccuracy of the knowledge of the running of the fine structure constant up to the M_Z scale, $\Delta\alpha(M_Z)$, introduces a further 6.5 MeV error. Second, several higher order corrections are unknown, of which only one at the $\mathcal{O}(\alpha^2)$ order. This lacking contribution comes from diagrams with no closed fermion loops. It has been previously estimated to be of the order of the square of the one loop bosonic correction [4]. This, however, is unjustified since both have a different dependence on the Higgs boson mass (logarithmic vs. quadratic). In this Letter, the question of the exact size of this contribution is finally settled.

The muon decay is naturally described in the language of effective field theory. The process' dynamics are driven by the leptons, the five light quarks, and the two massless gauge bosons. Even then the momentum scale set by the muon mass leads to a strong decoupling of the τ lepton and of the heavier of the five quarks [6]. The heavy particles W, Z, the Higgs boson and the top quark generate point interactions. A general effective lagrangian assumes the form

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QED} + \mathcal{L}^{(5)}_{\rm QCD} + \sum_{n,i} \frac{\mathcal{C}^i_n}{(M_W^2)^n} \mathcal{O}^i_n, \qquad (1)$$

where $\mathcal{L}_{QED}$ and $\mathcal{L}^{(5)}_{QCD}$ are the bare QED and the five flavour QCD lagrangians, $\mathcal{O}^i_n$ are composite operators of the light fields and $\mathcal{C}^i_n$ are the respective dimensionless matching coefficients, obtained by comparing the Green functions of the full theory with those of the effective one. The mass of the W boson has been chosen as the heavy scale.

Due to the left-handed nature of the charged current in the SM, the only operator of dimension six relevant to muon decay is the four-fermion interaction of the Fermi model

$$\mathcal{O}_F = \bar{e}\gamma^\alpha(1-\gamma_5)\mu \otimes \bar{\nu_\mu}\gamma_\alpha(1-\gamma_5)\nu_e, \qquad (2)$$

with μ, e, ν_μ, ν_e denoting the spinors of the muon, electron and their neutrinos. Here the so called charge-conserving form has been used (Fierz transformed SM amplitude). The respective matching coefficient is traditionally parametrised as

$$\mathcal{C}_F = M_W^2 \frac{G_F}{\sqrt{2}} = \frac{\pi\alpha}{2 s_W^2}(1+\Delta r), \qquad (3)$$

where G_F is known as the Fermi constant, whereas Δr represents the higher order corrections and vanishes at the Born level. The one loop corrections in the effective theory given by Eqs. 1, 2 and 3 have been calculated in [7] and [8], while the two loop in [9].

The framework for construction of effective lagrangians together with a proof has been given in [10]. It is based on expansions of individual Feynman diagrams. The

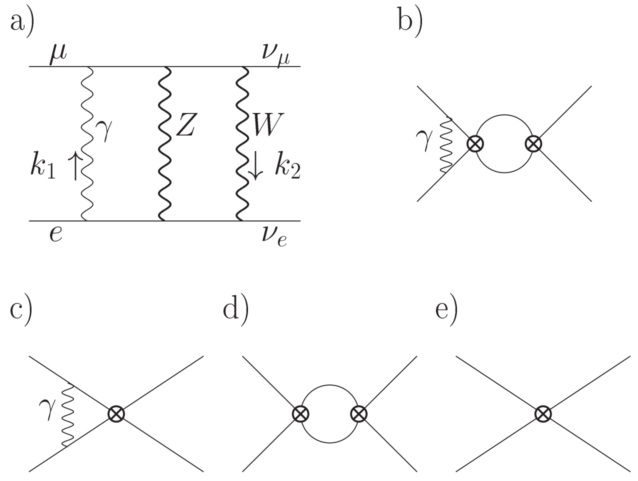

FIG. 1: A typical muon decay diagram a) and the contributions to its large mass expansion according to the momenta b) k_1-soft, k_2-soft; c) soft-hard; d) hard-soft; e) hard-hard.

method is presented on the example of a specific graph contributing to bosonic corrections to muon decay at two loop order in Fig. 1. The terms in the large mass expansion can be classified according to the scale of the loop momenta. When both momenta are "soft" ($\ll M_W$), as in b), the propagators of the W and Z bosons are expanded leading to a correction of order α/M_W^4 in the effective theory. For one momentum "soft" and one "hard" ($\sim M_W$), as in c) and d), corrections of either order, α/M_W^2 or $1/M_W^4$ in the effective theory, are generated. The contribution to the matching coefficient comes only from the region where both momenta are hard, as in e). In this case all of the light particle masses and momenta should be put to zero. By these arguments it can be shown that Δr can be obtained by simply taking the sum of all the diagrams and putting all external momenta and light masses to zero. The procedure should generate no spurious infrared divergences, while the physical divergences connected with the photon should be contained in the corrections of the effective theory. As known the Fermi theory corrections are finite, therefore the Δr correction obtained as above should also be finite.

Previous calculations of Δr have been based on a different method of factorisation originally devised in [11]. This procedure consists of subtracting from the infrared divergent SM diagrams the respective Fermi theory diagrams in Pauli-Villars regularisation. The difference is well defined in the limit of zero light masses and external momenta. It turns out, however, that the QED Ward identity, which is responsible for the finiteness of the corrections in the Fermi theory, implies in this case the vanishing of the sum of the subtracted diagrams. This proves that both procedures are equivalent.

The evaluation of two loop corrections to a four-fermion process requires the full second order renormalisation of the SM lagrangian in all but the Higgs sector, where first order suffices. The comparison with experiment imposes the use of on-shell parameters for the final result. Throughout this work the on-shell scheme was used, with a procedure similar to the one described in [5]. The only substantial difference concerns the treatment of tadpoles.

It is known that gauge invariance of mass counter-terms requires inclusion of tadpoles [12, 13] (at the two loop level this has been explicitely shown in [14]). In this case, however, one cannot use one-particle-irreducible (1PI) Green functions. In order to have gauge invariant counter-terms and 1PI Green functions only, a special procedure was designed. An additional renormalisation constant for the bare vacuum expectation value v_0, denoted Z_v, has been introduced and explicitely split from the bare masses

$$v_0 \longrightarrow v_0 Z_v^{1/2}, \tag{4}$$

$$M^0_{W,Z} \longrightarrow M^0_{W,Z} Z_v^{1/2}. \tag{5}$$

The term linear in the Higgs field H in the lagrangian

$$T^0 H^0 = \frac{M_W^0 s_W^0}{e_0} (M_H^0)^2 Z_v^{1/2} (Z_v - 1) H^0, \tag{6}$$

is then used to determine Z_v, through the requirement that tadpoles are canceled. It can be proved [12, 15] that the bare masses are gauge invariant in this case (an equivalent procedure which makes use of the effective potential has been used in [16]).

The calculation of the two loop bosonic contributions to muon decay was performed by means of a completely automated system. The diagram generation stage was done by the C++ library **DiaGen** [17]. The tensor reduction of two loop propagator diagrams was accomplished with the algorithm described in [18], whereas vacuum diagrams were treated with integration by parts identities [19]. For algebraic manipulations, the program **FORM** [20] was used. The two loop two-point integrals were numerically evaluated with single integral representations of the package **S2LSE** [21]. The latter was modified for quadruple precision, which was needed due to large cancellations (independent terms grow as M_H^8, while the result behaves as M_H^2).

The size of the software required several tests. The following algebraic checks were performed

- ultraviolet and infrared finiteness, by cancellation of poles in dimensional regularisation,
- gauge invariance, by independence of the three gauge parameters of the general R_ξ gauge for the SM,
- Slavnov-Taylor identities for two-point functions, as given in [18], both for on-shell integrals and by expansion in the external momentum to second order.

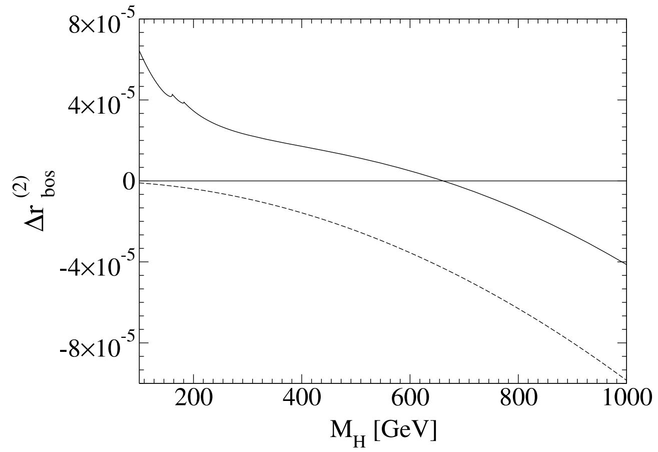

FIG. 2: Δr as function of the Higgs boson mass (solid line) and the leading term in the large Higgs mass expansion (dashed line).

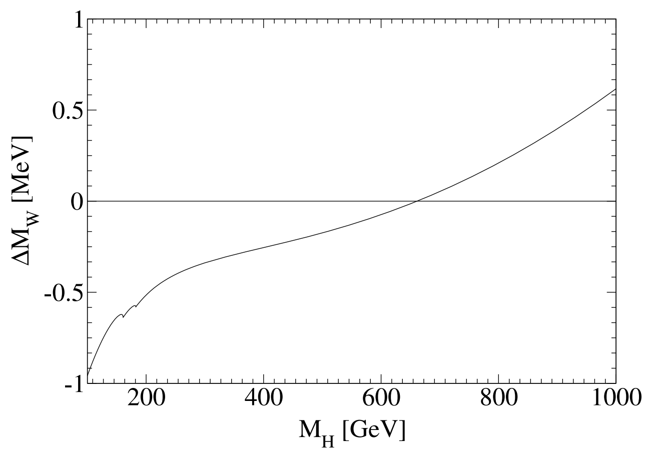

FIG. 3: W boson mass shift generated by two loop bosonic corrections.

Several numerical tests were also done

- all of the master integrals were evaluated independently by means of deep mass difference and large mass expansions,
- each of the two-point on-shell diagrams was calculated separately with the help of small-momentum and different large-mass expansions,
- the result of [14] for the W and Z mass counterterms was reproduced to precision dictated by the order of the expansions contained therein.

A detailed description of the methods used is relegated to a subsequent publication [15].

The variation of the bosonic contribution with M_H subtracted at $M_H = 100$ GeV has also been compared with the one presented in [5]. A discrepancy which grows up to $\sim 16\%$ for $M_H = 1$ TeV, has been found.

All the numerical values have been obtained for the following coupling and mass parameters [1]

$$\begin{aligned} &\alpha^{-1} = 137.03599976(50), \\ &G_F = 1.16639(1) \times 10^{-5}\ \text{GeV}^{-2}, \\ &M_W = 80.423(39)\ \text{GeV}, \\ &M_Z = 91.1876(21)\ \text{GeV}. \end{aligned} \tag{7}$$

The two loop bosonic correction to Δr, denoted $\Delta r^{(2)}_{bos}$, is shown in Fig. 2 as function of the Higgs boson mass M_H. Its behaviour is very smooth apart from the range between 150 GeV and 190 GeV, where two peaks are located. Their positions at ~ 161 GeV and ~ 182 GeV correspond to the W and Z pair production thresholds introduced through the on-shell definition of the Higgs boson mass.

It is interesting to study the correction for large values of M_H. In fact, two different leading M_H^2 terms have been given in [22] and [23], both based on the same results of [24]. In this last work a renormalisation scheme has been used in which M_H^2 terms occur only in two-point functions. The leading behaviour can then be obtained from the formula

$$\Delta r^{(2)}_{\text{Higgs}} = 2\frac{\delta e}{e} + \frac{\Delta M_W^2}{M_W^2} + \frac{c_W^2}{s_W^2}\left(\frac{\Delta M_Z^2}{M_Z^2} - \frac{\Delta M_W^2}{M_W^2} - \Delta\rho\right), \tag{8}$$

where δe is the charge renormalisation and $\Delta M_{W,Z}$ represent the shift of the W and Z self energies from zero external momentum up to the on-shell value. $\Delta\rho$ has been introduced in [25]. With the help of [24], the result reads

$$\begin{aligned} \Delta r^{(2)}_{\text{Higgs}} &= \left(\frac{\alpha}{4\pi s_W^2}\right)^2 \frac{M_H^2}{8M_W^2} \\ &\times \left(9\sqrt{3}\ \text{Cl}_2(\frac{\pi}{3}) + \frac{49}{72} - \frac{11\pi\sqrt{3}}{4} - \frac{25\pi^2}{108}\right), \end{aligned} \tag{9}$$

where Cl_2 is the Clausen function and $\text{Cl}_2(\pi/3) = 1.0149416$. Eq. 9 is in agreement with [22], whereas in [23] the charge renormalisation has not been included. The leading Higgs mass dependence is depicted in Fig. 2. The difference between the full result and Eq. 9 for values below 1 TeV shows the importance of subleading corrections. It has been checked that for $M_H > 5$ TeV both functions agree very well.

With the help of Δr the W boson mass can be estimated by inverting Eq. 3

$$M_W = M_Z\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2}(1+\Delta r)}}. \tag{10}$$

This equation should be solved recursively, since Δr also depends on M_W. In the case of small corrections one can, however, obtain the additional mass shift by means of an expansion. For parameters' values given in Eq. 7, a second order Taylor series yields

$$\Delta M_W = -(1.491 + 1.779\, \overline{\Delta r}) \times 10^4 \Delta r^{(2)}_{bos} \text{ [MeV]}, \quad (11)$$

where $\overline{\Delta r}$ represents the dominating correction, which is composed of one loop electroweak, two loop fermionic and $\mathcal{O}(\alpha\alpha_S)$ QCD contributions. Since $\overline{\Delta r}$ does not exceed 5% [5], the third order in the Taylor series would contribute less than a percent which is comparable to the inaccuracy induced by the dependence of $\Delta r^{(2)}_{bos}$ on the W mass.

The W mass shift, Eq. 11, with $\overline{\Delta r}$ neglected, is plotted in Fig. 3. The dropped term implies an error of less than 6%, which is, however, completely negligible in view of the size of the total effect, which does never exceed ± 1 MeV.

In conclusion, the complete $\mathcal{O}(\alpha^2)$ bosonic contributions to muon decay have been calculated in the on-shell scheme, and shown to lead to a small W boson mass shift below 1 MeV. A framework for evaluation of Δr which does not explicitely refer to the QED corrections in the Fermi theory has been presented and its equivalence to the method based on subtraction of Pauli-Villars regulated diagrams demonstrated. A procedure of obtaining gauge invariant mass counter-terms without explicitely including tadpole graphs has been developed.

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