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# QCD Corrections to Hadronic Z and $\tau$ Decays*

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**Abstract.** We present a brief (mainly bibliographical) report on recently performed calculations of terms of order $\mathcal{O}(\alpha_s^4 n_f^2)$ and $\mathcal{O}(\alpha_s^4 n_f^2 m_q^2)$ for hadronic Z and $\tau$ decay rates. A few details about the analytical evaluation of the masters integrals appearing in the course of calculations are presented.

**PACS.** 12.38.Bx Perturbative calculations – 12.38.-t Quantum chromodynamics

## 1 Introduction

Important physical observables like the cross-section of $e^+e^-$ annihilation into hadrons and the decay rate of the $Z$ boson are related to (the absorptive parts of) the vector and axial-vector current correlators (for a detailed review see, e.g. [1]). From the theoretical viewpoint the two-point correlators are ideally suited for evaluations in the framework of perturbative QCD (pQCD) [2]. Indeed, due to the simple kinematics (only one external momentum), even multiloop calculations can be analytically performed.

In many important cases (with the Z- and $\tau$-decay rates as prominent examples) the external momentum is much larger than the masses of the relevant quarks. It is then justified to neglect these masses in a first approximation which significantly simplifies all the calculations. Within this massless approximation of QCD, the absorptive parts of vector and scalar correlators are analytically known to $\alpha_s^3$ [3–6]. The residual quarks mass effects can be taken into account via an expansion in quark masses. This has been done for the quadratic and quartic terms to the same $\alpha_s^3$ order [7–9].

During the past years, in particular through the analysis of $Z$ decays at LEP and of $\tau$ decays, an enormous reduction of the experimental uncertainty has been achieved. Inclusion of the $\mathcal{O}(\alpha_s^3)$ corrections is mandatory already now. Quark mass effects must be included for $Z$-decays. The remaining theoretical uncertainty from uncalculated higher orders is at present comparable to the experimental one [1]. Thus, the full calculation of the next contributions, those of $\mathcal{O}(\alpha_s^4)$, to the two-point quark current correlators is an important next step in testing the Standard Model and crucial for precise determination of the QCD coupling constant.

In this paper we discuss some selected theoretical aspects of recent calculations of terms of order $\mathcal{O}(\alpha_s^4 n_f^2)$ and $\mathcal{O}(\alpha_s^4 n_f^2 m_q^2)$ contributing to the two-point correlator of the vector (and axial) quark currents. More detailed discussion as well as phenomenological applications in the context of hadronic Z and $\tau$ decays can be found in the original publications [10–12].

## 2 Correlator and diagrams

A well-known definition of a two-point quark current correlator reads

$$\begin{aligned} &\Pi^{V/A}_{\mu\nu,ij}(q,m_i,m_j,m,\mu,\alpha_s) = \\ &i\int dx e^{iqx}\langle T[\,j^{V/A}_{\mu,ij}(x)(j^{V/A}_{\nu,ij})^\dagger(0)\,]\rangle \end{aligned} \tag{1}$$

with $m^2 = \sum_f m_f^2$ and $j^{V/A}_{\mu,ij} = \bar{q}_i\gamma_\mu(\gamma_5)q_j$. The two (not necessarily different) quark fields with masses $m_i$ and $m_j$ are denoted by $q_i$ and $q_j$ respectively.

The number of diagrams contributing to (1) grows fast with the order of perturbation theory. While only three diagrams appear at $\mathcal{O}(\alpha_s)$, this number becomes 37 and 738 at $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$ respectively[1]. Finally, one arrives at 19832 five-loop diagrams at $\mathcal{O}(\alpha_s^4)$. Even more important is the fact that the calculational complexity of a single diagram also grows tremendously with every additional loop.

The calculational effort for a full evaluation of $R(s)$ in $\mathcal{O}(\alpha_s^4)$ is enormous and with present techniques exceeds the available computer resources by an order of magnitude. It is for this reason that our calculations were limited to a gauge invariant subset, namely the terms of order $\alpha_s^4 n_f^2$, where $n_f$ denotes the number of fermion flavours.

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[1] The specific numbers are cited as produced by the diagram generator QGRAF [13] for the case of the non-diagonal quark current ($i \neq j$ in (1)).

Some typical representatives of the corresponding set of diagrams are depicted in fig. 1. Note that the terms of

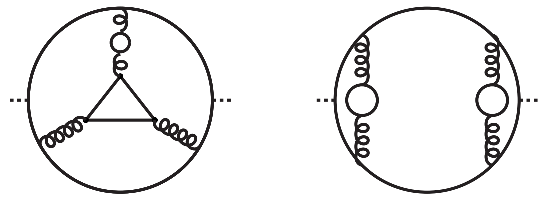

**Fig. 1.** Some representative of four-loop diagrams contributing to (1) in the $\mathcal{O}(\alpha_s^4 n_f^2)$ order.

order $\alpha_s^4 n_f^3$ are rather simple. They had been obtained earlier by summing the renormalon chains [14].

## 3 Reduction to masters

As is well-known [15] the calculation of absorptive part of the correlator (1) in the massless limit[2] at the $L+1$ loop level is reducible to the evaluation of some properly constructed set of massless $L$-loop propagator-type diagrams (the corresponding Feynman integrals will be referred to as "p-integrals"). The completely automatized reduction procedure of [5] is based the method of Infrared Rearrangement [16] enforced by a special technique of dealing with infrared divergences – the $R^*$-operation [17].

Thus, to compute the *five* loop $\mathcal{O}(\alpha_s^4)$ contribution in the absorptive part of (1) one should be able to evaluate generic *four* loop p-integrals. Unfortunately, this problem is far more complicated than the one at three loops. The latter was *in principle* done long ago by a manual consideration of all possible cases within the integration-by-parts technique (IBP) [18]. Nevertheless, it took almost ten years before the corresponding algorithm was reliably implemented in FORM [19,20]. A straightforward extension of the same approach to four loops is barely possible at all.

Fortunately, another method has been developed in [21–24].

According the IBP paradigm every integral is to be reduced to a sum of irreducible ("master") integrals, with coefficient functions being rational functions of the space-time dimension $D$. However, in contrast to the standard approach, where these coefficients are calculated by a recursive procedure, they are in the current approach obtained from an auxiliary integral representation [21] in the form of an expansion in $1/D$. Calculating sufficiently many terms in this expansion, the original $D$-dependence can be reconstructed.

The calculations were done in the following way. First, the set of irreducible integrals involved in the problem

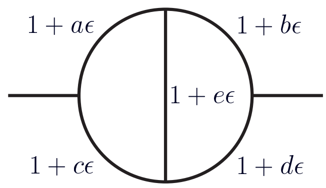


**Fig. 2.** A generic massless two-loop diagram whose expansion in $\epsilon$ was computed to order $\epsilon^7$ in [26]

was constructed, using the criterion irreducibility of Feynman integrals [24]. Second, the coefficients multiplying these master integrals were calculated in the $1/D \to 0$ expansion. This part was performed using the parallel version of FORM [25] running on an 8-alpha-processor-SMP-machine with disk space of 350 GB. The calculations in the massless limit took approximately 500 hours in total. (Approximately the same time went into the calculation of the $\mathcal{O}(\alpha_s^4 n_f^2 m_q^2)$ contribution.) Third, the exact answer was reconstructed from results of these expansions. Extensive tests were performed.

## 4 Master integrals and their evaluation

The master integrals appearing in the course of our calculations are shown in Table 1. These can be separated in three group: simple (m01, m11, m12, m13, m14, m23, m24, m25, m31), semi-simple (m22, m26, m27, m21, m32, m33) and the difficult ones (m34, m35, m41 and m52).

Simple integrals can be immediately performed in terms of $\Gamma$ functions by a repeated application of the textbook one-loop integration formula. The members of semi-simple group prove to be easily reducible to a generic two-loop integral shown on fig. 2 The latter was computed up to pretty high order in the parameter $\epsilon = 2 - D/2$ in the eighties [26,27]. Luckily enough, remaining four difficult integrals were all analytically evaluated[3] long ago in the course of the computation of the five-loop $\beta$-function in the $\phi^4$-theory [28,29] .

## 5 Summary

The calculations performed in [10–12] demonstrate that the approach based on the $1/D$-expansion is suited to obtain genuine QCD results in five loop approximation. Further analysis shows that the other diagrams appearing in order $\alpha_s^4$ can be solved in the same way, given sufficient computer resources. Work in this direction is in progress.

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[2] The statement also holds for the quadratic in quark masses contribution.

[3] We mean the pole and constant parts of the corresponding $\epsilon$ expansions.

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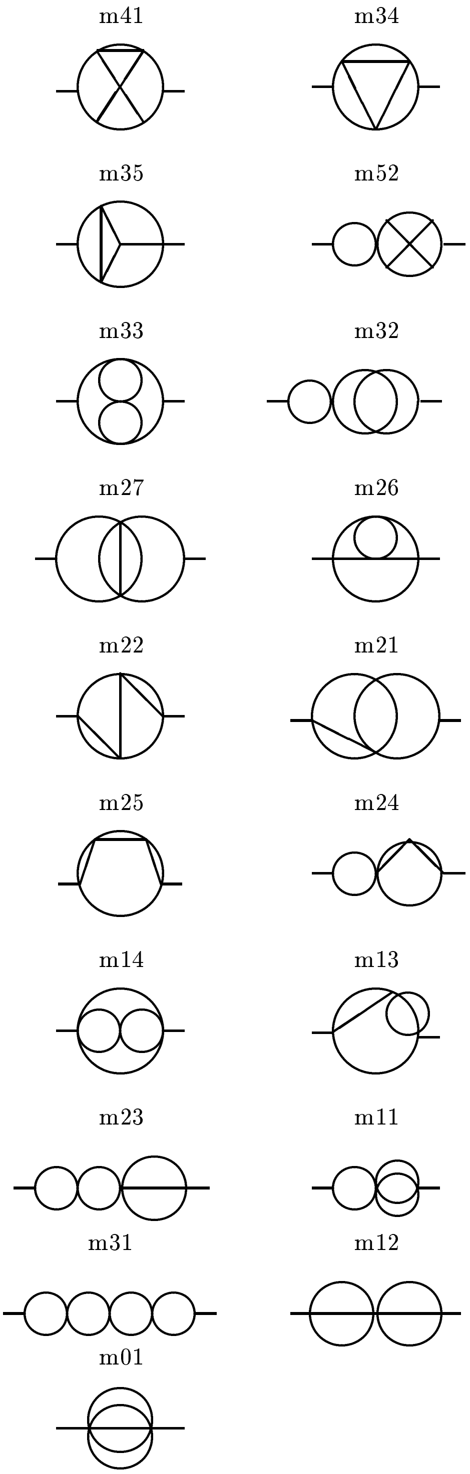
**Table 1.** Master four-loop propagator-type integrals appearing in the course of calculations of the correlator (1) in orders $\mathcal{O}(\alpha_s^4 n_f^2)$ and $\mathcal{O}(\alpha_s^4 n_f^2 m_q^2)$.

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