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Master Integrals for the 2-loop QCD virtual corrections to the Forward-Backward Asymmetry

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Abstract

We present the Master Integrals needed for the calculation of the two-loop QCD corrections to the forward-backward asymmetry of a quark-antiquark pair produced in electron-positron annihilation events. The abelian diagrams entering in the evaluation of the vector form factors were calculated in a previous paper. We consider here the non-abelian diagrams and the diagrams entering in the computation of the axial form factors, for arbitrary space-like momentum transfer Q^2 and finite heavy quark mass m. Both the UV and IR divergences are regularized in the continuous D-dimensional scheme. The Master Integrals are Laurent-expanded around $D = 4$ and evaluated by the differential equation method; the coefficients of the expansions are expressed as 1-dimensional harmonic polylogarithms of maximum weight 4.

Key words:Feynman diagrams, Multi-loop calculations, Vertex diagrams
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1 Introduction

The measurement of $A_{FB}^{q\bar{q}}$, the forward-backward asymmetry of the production of quark-antiquark pairs in the processes $e^+e^- \to q\bar{q}$, is a stringent test of the Standard Model, as it provides a precise determination of the effective weak mixing angle $\sin^2\theta_W^{eff}$ [1]. In particular, the most precise determination of $\sin^2\theta_W^{eff}$ comes from the measurement of the forward-backward asymmetry in heavy flavour production (c and b quarks) on the Z^0 peak [2]. At the next generation of Linear Colliders it will be also possible the measurement of $A_{FB}^{q\bar{q}}$ for the top quark [3].

The precision reached by the actual and aimed by future measurements requires, from the theoretical counterpart, the control of the second-order perturbative corrections. As concerning QCD, the order $\mathcal{O}(\alpha_S^2)$ corrections for massless quarks were calculated numerically in [4] and analytically in [5]. For the b quarks the order $\mathcal{O}(\alpha_S^2)$ corrections were calculated numerically in [6], retaining terms that do not vanish in the small-mass limit (constants and logarithmically-enhanced terms), but neglecting both terms containing linear mass corrections, like m_b/Q, and terms in which such a ratio is enhanced by a power of the logarithm $\log(Q/m_b)$. In order to take into account also this kind of terms, a full analytic calculation in which the mass of the heavy quark is kept systematically different from zero is required.

In this paper we explore the possibility of such analytic computation, limiting, for the moment, the analysis to the $\mathcal{O}(\alpha_S^2)$ virtual corrections.

The Feynman diagrams necessary for the calculation are shown in Fig. 1. We indicate with a double line the heavy quark and with a simple line the light quark of the diagram. The dashed line, carrying momentum Q, stands for the vector or axial current.

All the scalar integrals entering the calculation of the diagrams (a)–(e) were evaluated analytically in [7]. Among them, those corresponding to diagrams (a), (c), (d), and (e) have been tested numerically by the TOPSIDE collaboration [8]. The vector form factors, corresponding to the coupling of the fermion line with a photon of momentum Q, have been already calculated in [9], in the framework of QED. In this work we present the analytical results for the scalar integrals entering the calculation of the diagrams of Fig. 1 (f)–(l), for arbitrary space-like momentum transfer Q^2 (the continuation to time-like values can be carried out with the usual replacement $Q^2 = -(s + i\epsilon)$). We keep the mass m of the heavier quark in the diagram as finite and we consider the lighter flavours as massless. The explicit values of the form factors will be given elsewhere.

All the amplitudes are regularized within the continuous D-dimensional regularization scheme [10] in which both the IR and UV divergences, parametrized by the same parameter D, show up as poles in $(D-4)$.

We will follow closely the approach already used in [7, 9]. By systematic use of integration by parts identities (IBPs) [11], Lorentz invariance identities (LI) [12], and general symmetry relations, any scalar integral entering the game is expressed as a linear combination of a relatively small number of independent scalar integrals, the so called Master Integrals (MIs). In the present work we are left with 35 MIs. 17 of them were already calculated in [7]. We present here the analytical evaluation of

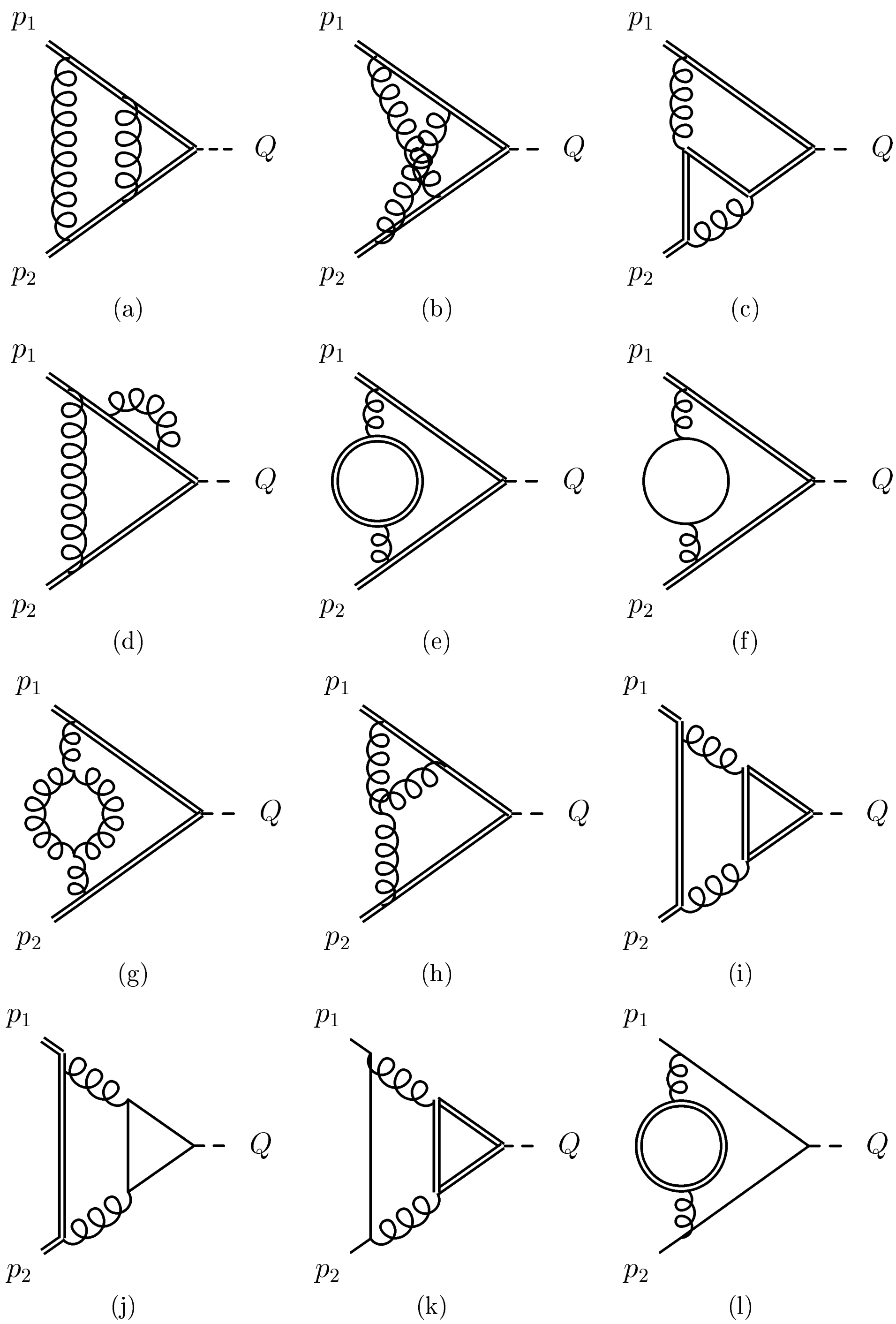


Figure 1: The 2-loop vertex diagrams involved in the calculation of A_{FB} at order $\mathcal{O}(\alpha_S^2)$. The curly lines are massless gluons; the double straight lines, quarks of mass m; the single straight lines, massless quarks. All the external fermion lines are on the mass-shell: $p_1^2 = p_2^2 = -m^2$, the double lines; $p_1^2 = p_2^2 = 0$, the single lines. The dashed line on the r.h.s. carries momentum $Q = p_1 + p_2$, with the metrical convention $Q^2 > 0$ when Q is space-like.

the remaining 18, obtained by means of the differential equations method [13, 14, 15] or, when all the propagators are massless, *via* direct integration with the Feynman parameters. The Master Integrals are Laurent-expanded around $D = 4$ and the coefficients of the Laurent-expansion are then expressed in terms of 1-dimensional harmonic polylogarithms (HPLs) [16, 17]. As an "empirical" rule, we expanded all the MIs up to the order in $(D-4)$ which contains HPLs with maximum weight $w = 4$; that is expected to be sufficient in order to express the 2-loop form factors up to their finite part in $(D-4)$ (but higher order terms, when needed, could be immediately provided by our method).

The paper is structured as follows.

In Section 2 we recall the main steps of the reduction to the MIs and their calculation with the differential equations method. In Section 3 we list the results for the MIs. In Section 4 we give, for completeness, the results of the 6-denominator vertices which are not MIs, and in Section 5 the large momentum expansion of all the 6-denominator diagrams. In Appendix A we give the routing used for the explicit calculations and finally, in Appendix B, the results for the 1-loop subdiagrams, which enter in the calculation.

2 Topologies and Master Integrals

The Feynman diagrams contributing to the order $\mathcal{O}(\alpha_S^2)$ virtual corrections to $A_{FB}^{q\bar{q}}$, are the 2-loop vertices of Fig. 1, describing the annihilation of a quark and an anti-quark, of incoming momenta p_1 and p_2, into a virtual boson of outgoing momentum Q.

The diagrams of Fig. 1 may involve: *1)* a single massive flavour, as in the case of figures (a), (b), (c), (d), (e), (g), (h), and (i); *2)* two flavours, one massive and the other taken as massless, as in figures (f), (j), (k), and (l). The double straight lines stand for the quark (antiquark) of mass m, while the single straight lines for a lighter quark (antiquark), which in our approximation is treated as massless. All the external fermion lines are on the mass-shell: $p_1^2 = p_2^2 = -m^2$ in diagrams (a)–(j) and $p_1^2 = p_2^2 = 0$ in diagrams (k) and (l).

In some details, the forward-backward asymmetry of the t-quark gets contributions only from the diagrams (a)–(j), where all the other lighter flavours running in internal loops are considered as massless, diagrams (f) and (j). In the case of the b-quark, the diagrams (a)–(j) account for the cases in which the t-quark is absent from the internal fermion loops, and b is the massive flavour; in the diagrams (k) and (l), the fermion of the internal loop is a t-quark, and the external b-quark is considered as massless. In the latter approximation, contributions proportional to m_b^2/m_t^2 are neglected.

By using suitably projectors (see for instance Section 2 of [9]), the (Lorentz invariant) form factors of any of the vertex graphs of Fig. 1 can be expressed in terms of several (typically a few hundreds) scalar integrals, whose integrands are combinations of scalar products of the external and loop momenta divided by the denominators appearing in the propagators of the internal lines. Following [7], from

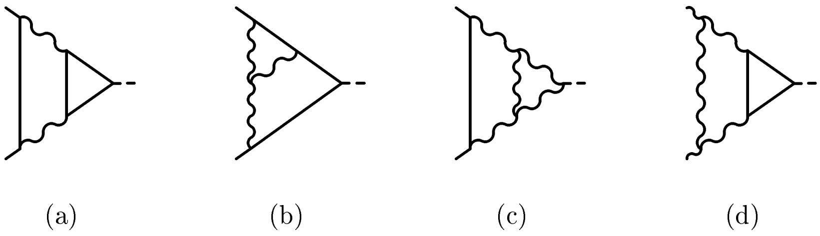


Figure 2: The four independent 6-denominator topologies. Internal straight lines carry mass m, internal wavy lines are massless; the mass shell conditions are $p_1^2 = p_2^2 = -m^2$ for the external straight lines and $p_1^2 = p_2^2 = 0$ for the external wavy lines; the dashed line on the right carries momentum $Q = p_1 + p_2$, with $Q^2 > 0$ when Q is space-like.

now on we will switch our attention from the Feynman diagrams to their topologies – or combinations of different denominators.

2.1 Reduction to MIs

Skipping the diagrams (a)–(e) of Fig. 1, which have already been considered in [7], the topologies corresponding to the diagrams (f)–(l) are the 6-denominator topologies shown in Fig. 2, and the two 5-denominator topologies shown in Fig. 3 (g) and (h).

The following graphical conventions apply: internal straight and wavy lines stand for propagators of mass m and zero respectively; the mass-shell conditions are $p_1^2 = p_2^2 = -m^2$ for external straight lines and $p_1^2 = p_2^2 = 0$ for external wavy lines; the dashed line on the right carries (space-like) momentum $Q = p_1 + p_2$ with $Q^2 > 0$. According to these conventions, wavy lines correspond both to gluons (or photons) and light fermions, so that the topology (g) of Fig. 3, for instance, refers to both Feynman diagrams (f) and (g) of Fig. 1.

The tree of the subtopologies, generated top-down by removing the denominators one by one from the four topologies in Fig. 2 and the two topologies in Fig. 3 (g) and (h), overlaps partially with the set of subtopologies already considered in [7]. The new independent subtopologies are the 15 5-denominator ones of Fig. 3, the 15 4-denominator topologies shown in Fig. 4 and, finally, the 3 3-denominator topologies of Fig. 5. The only non-trivial topology with 2 denominators is the product of two tadpoles, already considered in [7].

As already said, the whole set of the scalar integrals, belonging to all the above topologies, is regularized within the D-continuous dimensional regularization scheme, in which both IR and UV divergences show up as poles in $(D-4)$.

As already discussed at length in [7], thanks to the continuous dimensional regularization one can easily write several identities among the scalar integrals associated to a given topology (and its subtopologies), by using mainly the Integration by Parts

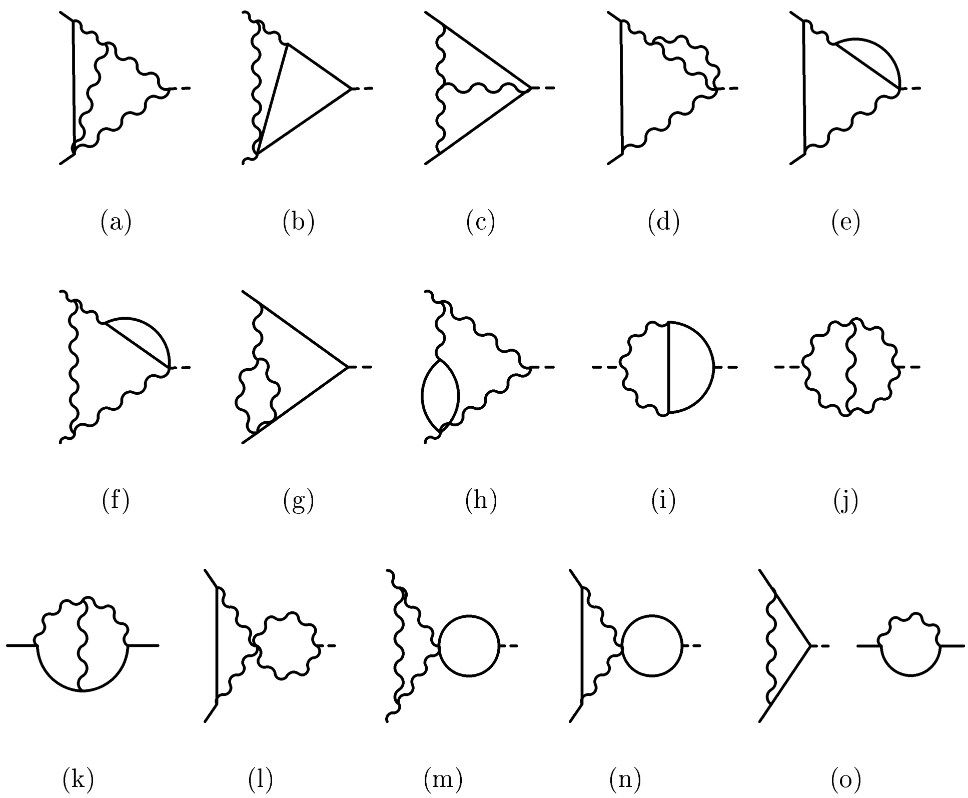


Figure 3: The set of the 15 independent 5-denominator topologies. The graphical conventions are the same as in Fig. 2.

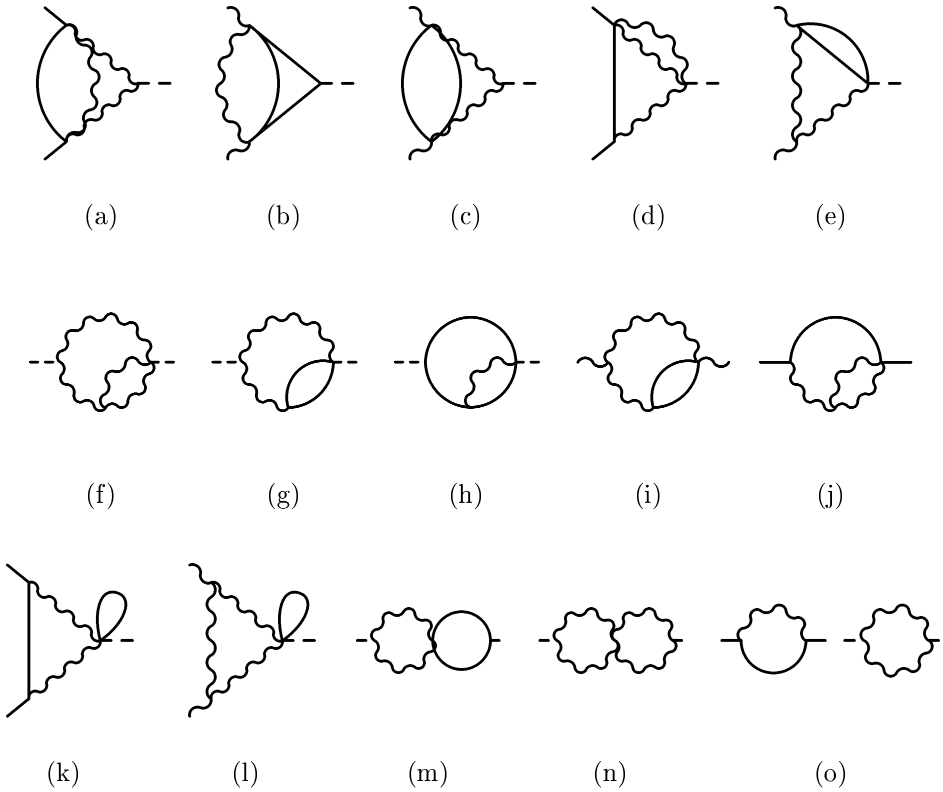


Figure 4: The set of the 15 independent 4-denominator topologies. The graphical conventions are the same as in Fig. 2.

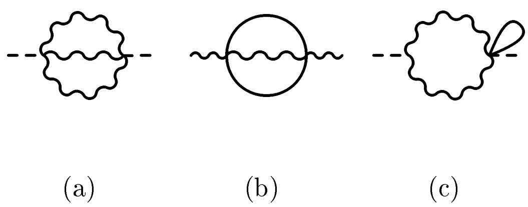


Figure 5: The set of the 2 independent 3-denominator topologies. The graphical conventions are the same as in Fig. 2.

method of [11], but also the Lorentz invariance identities [12] and the symmetry relations, that can occur in particular mass configurations. The identities can then be solved by standard techniques (Gauss substitution rule), whose implementation is, however, algebraically very demanding. Referring again for more details to [7], one can express in this way all the scalar integrals associated to the considered topologies as linear combinations of a small number of integrals, called the Master Integrals (MIs) for that topology.

In our case, we find 18 new MIs, shown as "decorated graphs" in Fig. 6. According to our graphical notations, a simple dot on a propagator line, like in figures (e) and (k), means that, in the integrand of the concerned MI, the corresponding denominator is squared; a dot labeled by the number "3", like in figure (h), means that the corresponding denominator is raised to the third power; an explicitly written scalar product, like in (f) and (i), means that the corresponding integrand has that scalar product in the numerator.

2.2 The differential equations

The calculation of the MIs is performed by means of the differential equations method [13, 14, 15].

In our case, the MIs are functions of the squared momentum transfer Q^2, of the mass m of the heavy quark, and of the dimension parameter D. According to [14, 15], the h MIs of a given topology $M_i(D, m^2, Q^2)$, $\; i = 1, .., h$, satisfy a system of h coupled first-order linear differential equations in Q^2 of the form:

$$\begin{aligned} \frac{d}{dQ^2} M_i(D, m^2, Q^2) &= \sum_{j=1}^{h} A_{ij}(D, m^2, Q^2) M_j(D, m^2, Q^2) \\ &\quad + \sum_{ik} B_k(D, m^2, Q^2) N_k(D, m^2, Q^2) \ , \end{aligned} \qquad (1)$$

where the $N_k(D, m^2, Q^2)$ are MIs of the sub-topologies and the coefficients $A_{ij}(D, m^2, Q^2)$, $B_{ik}(D, m^2, Q^2)$ are ratios of polynomials in D, m^2 and Q^2.

Obviously, the equations for the $M_i(D, m^2, Q^2)$ are not homogeneous, due to the presence of the $N_k(D, m^2, Q^2)$ corresponding to the sub-topologies. It is therefore natural to proceed bottom-up in the solution of the equations for the whole set of MIs, starting from the equations for the MIs of the simplest topologies and using their solutions within the equations for the MIs of the more complicated ones, whose non-homogeneous part can then be considered as known.

We search for a solution of the system (1) in Laurent series of $(D-4)$:

$$M_i(D, m^2, Q^2) = \sum_{j=-2}^{n} (D-4)^j M_i^{(j)}(m^2, Q^2) + \mathcal{O}((D-4)^{(n+1)}) \ , \qquad (2)$$

where n is the required order in $(D-4)$. The solution of the system proceeds order-by-order in $(D-4)$, as explained in [7]; one obtains a set of chained systems, one for each power of $(D-4)$, all with the same homogeneous parts.

In our case, 10 topologies out of 13 have a single MI, which means that they satisfy a single first-order linear differential equation; the topology (c) of Fig. 5 has the two MIs of Fig. 6 (j)–(k), and finally the topologies (a) and (b) of Fig. 5 both have three MIs, shown in Fig. 6 (d)–(f) and (g)–(i).

When the MIs for a given topology are two or more, the actual choice of the scalar integrals to be chosen as MIs, which is in principle arbitrary, can be of great help for simplifying the problem. It turns out, indeed, that with the choice of the three MIs shown in Fig. 6 (d)–(f), the system expanded in $(D-4)$ decouples into a two-by-two system for the two MIs (d), (f) and an equation involving the MI (e) only. The same happens for the three MIs corresponding to Fig. 6 (g)–(i). Likewise, the system for the two MIs (l) and (m) of Fig. 6 decouples, when expanded in $(D-4)$, in two first-order linear differential equations.

Following again [15, 7], the two-by-two first-order systems are transformed in the equivalent single equations of the second order, and all the resulting first and second-order single equations are solved by using Euler's method of the variation of the constants. The method requires the explicit knowledge of the solutions of the associated homogeneous equations; as in previous work, the solutions of all the homogeneous equations were simple algebraic functions, found immediately by inspecting the equations, so that we will not report them here too.

As a last remark, the boundary conditions for the differential equations were found by exploiting the known analytical properties of the MIs under consideration, imposing, typically, the regularity or the finiteness of the solution at the pseudo-thresholds of the MI. This qualitative information was completely sufficient for the quantitative determination of the otherwise arbitrary integration constants, which naturally arise when solving a system of differential equations.

The equations for the MIs of the massless 2-loop sunrise, Fig. 6 (q), and for the MIs of the massless 1-loop bubbles of Fig. 6 (m), (o), (p) and (r) are entirely homogeneous, so that only the scaling dependence of the MIs on Q^2 could be derived by solving the differential equations. In those cases the complete result was easily obtained by direct integration with the Feynman parameters.

3 Results for the MIs

In this Section we give the results of the MIs of Fig. 6 as a Laurent series in $(D-4)$ and we express the coefficients of the series in terms of HPLs [16, 17] of one variable.

In all the cases, except the MIs (j) and (k) of Fig. 6, coming from the reduction tree of the topology (h) of Fig. 3, we express the result in terms of the variable x, defined as:

$$x = \frac{\sqrt{Q^2+4m^2}-\sqrt{Q^2}}{\sqrt{Q^2+4m^2}+\sqrt{Q^2}} \, , \qquad Q^2 = m^2 \frac{(1-x)^2}{x} \, . \tag{3}$$

When Q is space-like and Q^2 varies from 0 to $+\infty$, x varies from 1 to 0. When Q is time-like, the analytic continuation is performed by putting $Q^2 = -(s+i\epsilon)$, with $s>0$. For $0<s<4m^2$, x varies in the upper unit circle; when $4m^2<s<+\infty$,

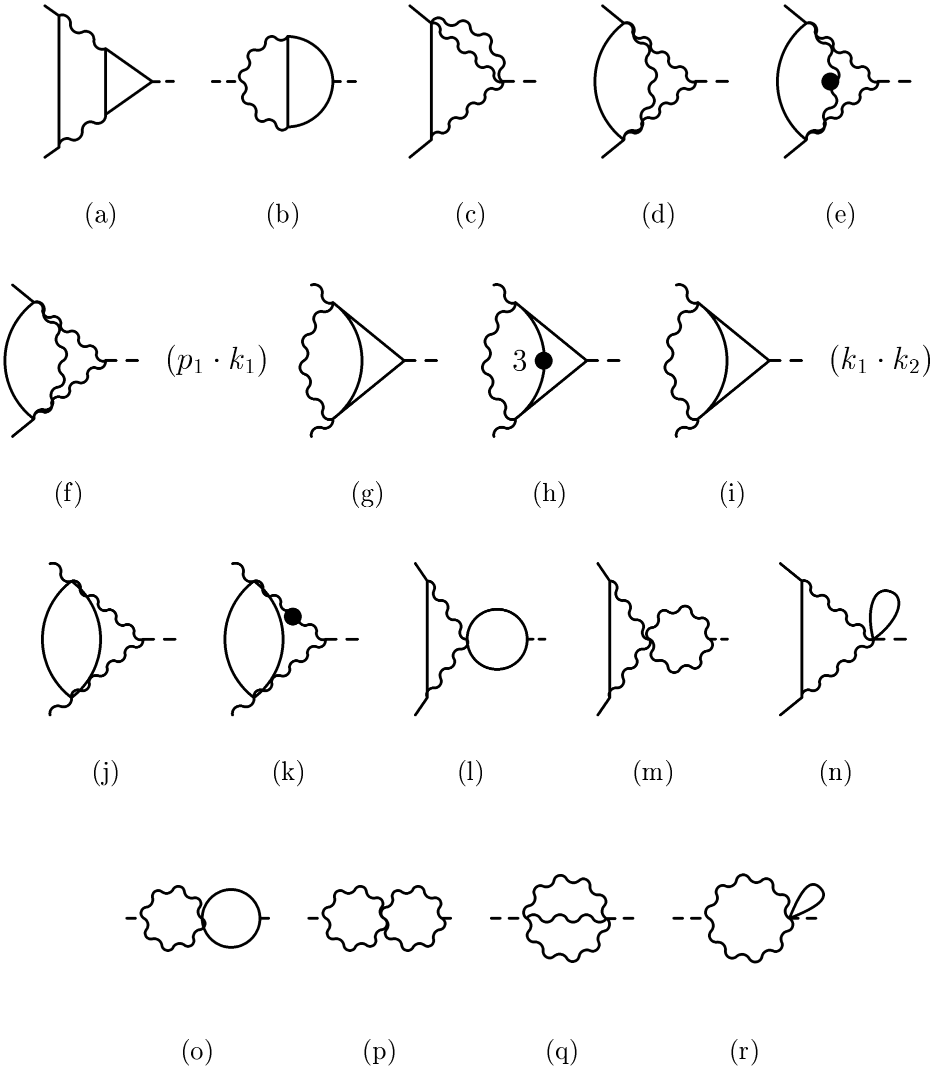


Figure 6: The 18 Master Integrals, calculated in the present work, represented as "decorated graphs". A simple dot on a propagator line, like in figures (e) and (k), means that in the integrand the corresponding denominator is squared; a dot labeled by "3", figure (h), means that the corresponding denominator is raised to the third power; an explicitly written scalar product, as in graphs (f) and (i), means that the corresponding integrand has that scalar product in the numerator.

defining

$$y = \frac{\sqrt{s} - \sqrt{s-4m^2}}{\sqrt{s} + \sqrt{s-4m^2}} \ , \qquad s = m^2 \frac{(1+y)^2}{y} \ , \tag{4}$$

y varies correspondingly from 1 to 0, and the analytic continuation is obtained by the replacement

$$x = -y + i\epsilon \ . \tag{5}$$

For the MIs (j) and (k) of Fig. 6, instead, we used the variable $\bar{x}$, defined as:

$$\bar{x} = \frac{\sqrt{Q^2} - \sqrt{Q^2-4m^2}}{\sqrt{Q^2} + \sqrt{Q^2-4m^2}} \ , \qquad Q^2 = m^2 \frac{(1+\bar{x})^2}{\bar{x}} \ . \tag{6}$$

When Q is space-like and Q^2 varies from $+\infty$ and $4m^2$, $\bar{x}$ varies from 0 to 1; when Q is still space-like, and Q^2 varies from $4m^2$ to 0, we can give to Q^2 a negative imaginary part $-i\epsilon$ (anticipating the prescription for the continuation to time-like values) so that $\bar{x}$ varies in the upper unit circle; finally, when Q is time-like and $Q^2 = -(s+i\epsilon)$ with $s>0$, defining

$$\bar{y} = \frac{\sqrt{s+4m^2} - \sqrt{s}}{\sqrt{s+4m^2} + \sqrt{s}} \ , \qquad s = m^2 \frac{(1-\bar{y})^2}{\bar{y}} \ , \tag{7}$$

$\bar{y}$ varies from 1, at $s=0$, to 0, at $s=+\infty$, and the analytic continuation is given by the replacement

$$\bar{x} = -\bar{y} + i\epsilon \ . \tag{8}$$

The denominators $\mathcal{D}'s$ appearing in the formulas are given in Appendix A. The loop integration measure in D continuous dimensions is defined as

$$\int \mathfrak{D}^D k = \frac{m^{(4-D)}}{C(D)} \int \frac{d^D k}{(2\pi)^{(D-2)}} \, , \tag{9}$$

(corresponding to the energy scale $\mu_0 = 1$), where $C(D)$ is the following function of the continuous dimension D

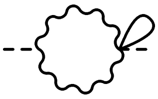
$$C(D) = (4\pi)^{\frac{(4-D)}{2}} \Gamma\left(3 - \frac{D}{2}\right) \, , \tag{10}$$

with the limiting value $C(4) = 1$ for $D = 4$. With that choice, the 1-loop tadpole with mass m reads

$$\int \mathfrak{D}^D k \, \frac{1}{k^2+m^2} = \frac{m^2}{(D-2)(D-4)} \ . \tag{11}$$

The explicit results follow in the next sections.

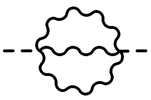
3.1 2-loop Topologies with 3 denominators



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_7 \mathcal{D}_{10}} \\
&= m^2 \sum_{i=-2}^{2} (D-4)^i \, F_1^{(i)}(x) + \mathcal{O}\left((D-4)^3\right) \,, \qquad (12)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_1^{(-2)}(x) &= -\frac{1}{4} \,, \qquad (13) \\
F_1^{(-1)}(x) &= \frac{1}{8}[3 + H(0;x) + 2H(1;x)] \,, \qquad (14) \\
F_1^{(0)}(x) &= -\frac{1}{16}[7 - \zeta(2) + 3H(0;x) + 6H(1;x) + H(0,0;x) + 2H(0,1;x) \\
&\quad +2H(1,0;x) + 4H(1,1;x)] \,, \qquad (15) \\
F_1^{(1)}(x) &= \frac{1}{32}\Big\{15 - 3\zeta(2) - 2\zeta(3) - (1 - 7\zeta(2))[H(0;x) + 2H(1;x)] \\
&\quad +3H(0,0;x) + 6H(0,1;x) + 6H(1,0;x) + 12H(1,1;x) \\
&\quad +H(0,0,0;x) + 2H(0,0,1;x) + 2H(0,1,0;x) \\
&\quad +4H(0,1,1;x) + 2H(1,0,0;x) + 4H(1,0,1;x) \\
&\quad +4H(1,1,0;x) + 16H(1,1,1;x)\Big\} \,, \qquad (16) \\
F_1^{(2)}(x) &= -\frac{1}{128}\Big\{62 - 14\zeta(2) - \frac{9}{5}\zeta(2)^2 - 12\zeta(3) + 2(15 - 3\zeta(2) \\
&\quad -2\zeta(3))[H(0;x) + 2H(1;x)] + 8(7 - \zeta(2))[4H(0,0;x) \\
&\quad +2H(0,1;x) + 2H(1,0;x) + H(1,1;x)] + 6H(0,0,0;x) \\
&\quad +12H(0,0,1;x) + 12H(0,1,0;x) + 24H(0,1,1;x) \\
&\quad +12H(1,0,0;x) + 24H(1,0,1;x) + 24H(1,1,0;x) \\
&\quad +48H(1,1,1;x) + 2H(0,0,0,0;x) + 4H(0,0,0,1;x) \\
&\quad +4H(0,0,1,0;x) + 8H(0,0,1,1;x) + 4H(0,1,0,0;x) \\
&\quad +8H(0,1,0,1;x) + 8H(0,1,1,0;x) + 16H(0,1,1,1;x) \\
&\quad +4H(1,0,0,0;x) + 8H(1,0,0,1;x) + 8H(1,0,1,0;x) \\
&\quad +16H(1,0,1,1;x) + 8H(1,1,0,0;x) + 16H(1,1,0,1;x) \\
&\quad +16H(1,1,1,0;x) + 32H(1,1,1,1;x)\Big\} \,. \qquad (17)
\end{aligned}
$$



$$
= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_9 \mathcal{D}_{23}}
$$

$$= \quad m^2 \sum_{i=-1}^{3} (D-4)^i \, F_2^{(i)}(x) + \mathcal{O}\left((D-4)^4\right) \,, \tag{18}$$

where:

$$F_2^{(-1)}(x) = \frac{1}{32}\Big[\frac{1}{x} + x - 2\Big] \,, \tag{19}$$

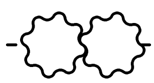
$$F_2^{(0)}(x) = -\Big[\frac{1}{x} + x - 2\Big]\Big[\frac{13}{128} + \frac{1}{32}H(0;x) + \frac{1}{16}H(1;x)\Big] \,, \tag{20}$$

$$\begin{aligned} F_2^{(1)}(x) = \; & \frac{1}{128}\Big[\frac{1}{x}+x-2\Big]\Big\{\frac{115}{4} - 2\zeta(2) + 13[H(0;x)+2H(1;x)] + 4H(0,0;x) \\ & +8H(0,1;x) + 8H(1,0;x) + 16H(1,1;x)\Big\} \,, \end{aligned} \tag{21}$$

$$\begin{aligned} F_2^{(2)}(x) = \; & -\frac{1}{128}\Big[\frac{1}{x}+x-2\Big]\Big\{\frac{865}{16} - \frac{13}{2}\zeta(2) - 5\zeta(3) + \Big(\frac{115}{2} - \frac{\zeta(2)}{4}\Big)[H(0;x) \\ & +2H(1;x)] + 13H(0,0;x) + 26H(0,1;x) + 26H(1,0;x) \\ & +52H(1,1;x) + 8H(0,0,0;x) + 16H(0,0,1;x) + 16H(0,1,0;x) \\ & +32H(0,1,1;x) + 16H(1,0,0;x) + 32H(1,0,1;x) + 32H(1,1,0;x) \\ & +64H(1,1,1;x)\Big\} \,, \end{aligned} \tag{22}$$

$$\begin{aligned} F_2^{(3)}(x) = \; & \frac{1}{512}\Big[\frac{1}{x}+x-2\Big]\Big\{\frac{5971}{8} - 115\zeta(2) - \frac{22}{5}\zeta^2(2) - 130\zeta(3) + \Big(\frac{865}{4} \\ & -26\zeta(2) - 20\zeta(3)\Big)[H(0;x) + 2H(1;x)] + (115 - 8\zeta(2))[H(0,0;x) \\ & +2H(0,1;x) + 2H(1,0;x) + 4H(1,1;x)] + 52[H(0,0,0;x) \\ & +2H(0,0,1;x) + 2H(0,1,0;x) + 2H(1,0,0;x) + 4H(0,1,1;x) \\ & +4H(1,0,1;x) + 4H(1,1,0;x) + 8H(1,1,1;x)] + 16[H(0,0,0,0;x) \\ & +2H(0,0,0,1;x) + 2H(0,0,1,0;x) + 4H(0,0,1,1;x) \\ & +2H(0,1,0,0;x) + 4H(0,1,0,1;x) + 4H(0,1,1,0;x) \\ & +8H(0,1,1,1;x) + 2H(1,0,0,0;x) + 4H(1,0,0,1;x) \\ & +4H(1,0,1,0;x) + 8H(1,0,1,1;x) + 4H(1,1,0,0;x) \\ & +8H(1,1,0,1;x) + 8H(1,1,1,0;x) + 16H(1,1,1,1;x)]\Big\} \,. \end{aligned} \tag{23}$$

3.2 2-loop Topologies with 4 denominators

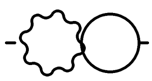


$$\begin{aligned} & = \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_7 \mathcal{D}_9 \mathcal{D}_{15}} \\ & = \sum_{i=-2}^{2} (D-4)^i \, F_3^{(i)}(x) + \mathcal{O}\left((D-4)^3\right) \,, \end{aligned} \tag{24}$$

where:

$$F_3^{(-2)}(x) = \frac{1}{4} \,, \tag{25}$$

$$
\begin{aligned}
F_3^{(-1)}(x) &= \frac{1}{2} - \frac{1}{4}H(0;x) - \frac{1}{2}H(1;x)\,, && (26)\\
F_3^{(0)}(x) &= \frac{1}{8}\Big[6 - \zeta(2) + 4H(0;x) + 8H(1;x) + 2H(0,0;x) + 4H(0,1;x)\\
&\quad +4H(1,0;x) + 8H(1,1;x)\Big]\,, && (27)\\
F_3^{(1)}(x) &= -1 + \frac{1}{8}\Big\{2\zeta(2) + \zeta(3) - (6-\zeta(2))[H(0;x) + 2H(1;x)]\\
&\quad -4H(0,0;x) - 8H(0,1;x) - 8H(1,0;x) - 16H(1,1;x)\\
&\quad -2H(0,0,0;x) - 4H(0,0,1;x) - 4H(0,1,0;x) - 8H(0,1,1;x)\\
&\quad -4H(1,0,0;x) - 8H(1,0,1;x) - 8H(1,1,0;x) - 16H(1,1,1;x)\,, && (28)\\
F_3^{(2)}(x) &= \frac{5}{4} - \frac{3}{8}\zeta(2) - \frac{1}{80}\zeta^2(2) - \frac{1}{4}\zeta(3) + \frac{1}{8}\Big[8 - 2\zeta(2) - \zeta(3)\Big][H(0;x)\\
&\quad +2H(1;x)] + \frac{1}{8}\Big[6 - \zeta(2)\Big][H(0,0;x) + 2H(0,1;x) + 2H(1,0;x)\\
&\quad +4H(1,1;x)] + \frac{1}{4}\Big[2H(0,0,0;x) + 4H(0,0,1;x) + 8H(0,1,1;x)\\
&\quad +4H(0,1,0;x) + 4H(1,0,0;x) + 8H(1,0,1;x) + 8H(1,1,0;x)\\
&\quad +16H(1,1,1;x) + H(0,0,0,0;x) + 2H(0,0,0,1;x)\\
&\quad +2H(0,0,1,0;x) + 4H(0,0,1,1;x) + 2H(0,1,0,0;x)\\
&\quad +4H(0,1,0,1;x) + 4H(0,1,1,0;x) + 8H(0,1,1,1;x)\\
&\quad +2H(1,0,0,0;x) + 4H(1,0,0,1;x) + 4H(1,0,1,0;x)\\
&\quad +8H(1,0,1,1;x) + 4H(1,1,0,0;x) + 8H(1,1,0,1;x)\\
&\quad +8H(1,1,1,0;x) + 16H(1,1,1,1;x)\Big]\,. && (29)
\end{aligned}
$$



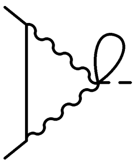
$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1\mathcal{D}_7\mathcal{D}_{10}\mathcal{D}_{16}}\\
&= \sum_{i=-2}^{2} (D-4)^i \, F_4^{(i)}(x) + \mathcal{O}\left((D-4)^3\right)\,, && (30)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_4^{(-2)}(x) &= \frac{1}{4}\,, && (31)\\
F_4^{(-1)}(x) &= -\frac{1}{2} - \frac{1}{4(1-x)}H(0;x) - \frac{1}{4}H(1;x)\,, && (32)\\
F_4^{(0)}(x) &= \frac{3}{4} - \frac{1}{8(1-x)}\Big[\zeta(2) + 4H(0;x) + 3H(0,0;x) - 2H(-1,0;x)\\
&\quad +2H(0,1;x) + 2H(1,0;x)\Big] + \frac{1}{8}\Big[H(1;x) - H(0,0;x)\\
&\quad +H(-1,0;x) + 2H(1,1;x)\Big]\,, && (33)
\end{aligned}
$$

$$
\begin{aligned}
F_4^{(1)}(x) \;=\;& -1+\frac{1}{16(1-x)}\Big\{4\zeta(2)+8\zeta(3)-(12-3\zeta(2))H(0;x)+2\zeta(2)[H(1;x)\\
&-H(-1;x)]-12H(0,0;x)+8H(-1,0;x)-8H(0,1;x)-8H(1,0;x)\\
&-7H(0,0,0;x)-4H(-1,-1,0;x)+6H(-1,0,0;x)+4H(-1,0,1;x)\\
&+4H(-1,1,0;x)+4H(0,-1,0;x)-6H(0,0,1;x)-6H(0,1,0;x)\\
&+4H(1,-1,0;x)-6H(1,0,0;x)-4H(1,0,1;x)-4H(0,1,1;x)\\
&-4H(1,1,0;x)\Big\}-\frac{1}{16}\Big\{\zeta(2)H(0;x)+12H(1;x)-\zeta(2)H(-1;x)\\
&-4H(0,0;x)+4H(-1,0;x)+8H(1,1;x)-3[H(0,0,0;x)\\
&-H(-1,0,0;x)]+2[H(-1,0,1;x)-H(-1,-1,0;x)+H(-1,1,0;x)\\
&+H(0,-1,0;x)-H(0,0,1;x)-H(0,1,0;x)+H(1,-1,0;x)\\
&-H(1,0,0;x)+2H(1,1,1;x)]\Big\}\,, \qquad (34)\\
F_4^{(2)}(x) \;=\;& \frac{5}{4}-\frac{1}{64}\zeta^2(2)-\frac{1}{8(1-x)}\Big[3\zeta(2)-\frac{1}{40}\zeta^2(2)+2\zeta(3)\Big]+\frac{1}{16}\Big[2\zeta(2)\\
&+\zeta(3)+\frac{16-6\zeta(2)-3\zeta(3)}{(1-x)}\Big]H(0;x)-\frac{1}{16}\Big[2\zeta(2)+\zeta(3)\\
&-\frac{2(2\zeta(2)+\zeta(3))}{(1-x)}\Big]H(-1;x)+\Big[1-\frac{(2\zeta(2)+\zeta(3))}{8(1-x)}\Big]H(1;x)\\
&-\frac{1}{32}\Big[12-3\zeta(2)-\frac{36-7\zeta(2)}{(1-x)}\Big]H(0,0;x)+\frac{1}{16}\Big[\zeta(2)+\frac{12-3\zeta(2)}{(1-x)}\Big]\times\\
&\times[H(1,0;x)+H(0,1;x)]-\frac{3\zeta(2)}{32}H(-1,0;x)+\frac{3}{16}\Big[2-\frac{4-\zeta(2)}{(1-x)}\Big]\times\\
&\times H(-1,0;x)+\frac{\zeta(2)}{16}\Big[1-\frac{2}{(1-x)}\Big][H(-1,-1;x)-H(1,-1;x)\\
&-H(0,-1;x)-H(-1,1;x)]+\frac{1}{4}\Big[3-\frac{1}{2(1-x)}\Big]H(1,1;x)\\
&-\frac{1}{8}\Big[1-\frac{2}{(1-x)}\Big]\Big\{3[H(0,0,0;x)-H(-1,0,0;x)]-2H(-1,0,1;x)\\
&-2H(-1,1,0,x)-2H(0,-1,0,x)-2H(1,-1,0,x)+2H(-1,-1,0,x)\Big\}\\
&-\frac{1}{4}\Big[1-\frac{3}{(1-x)}\Big]\Big\{H(0,0,1;x)+H(0,1,0;x)+H(1,0,0;x)\Big\}\\
&+\frac{1}{2}H(1,1,1;x)+\frac{1}{32(1-x)}\Big\{4[H(0,0,0;x)+4H(1,0,1;x)\\
&+4H(1,1,0;x)+4H(0,1,1;x)]+H(0,0,0,0;x)+2H(0,0,0,1;x)\\
&+4H(0,0,1,1;x)+2H(0,1,0,0;x)+4H(0,1,0,1;x)\\
&+4H(0,1,1,0;x)+16H(0,1,1,1;x)-12H(1,-1,0,0;x)\\
&+2H(1,0,0,0;x)+4H(1,0,0,1;x)+4H(1,0,1,0;x)\\
&+8H(1,0,1,1;x)+4H(1,1,0,0;x)+8H(1,1,0,1;x)\\
&+8H(1,1,1,0;x)\Big\}+\frac{1}{16}[4H(1,1,1,1;x)-3H(0,0,1,0;x)
\end{aligned}
$$

$$
\begin{aligned}
&+3H(1,-1,0,0;x)] - \frac{1}{32}\Big[1 - \frac{1}{(1-x)}\Big]\Big\{7[H(0,0,0,0;x) \\
&-H(-1,0,0,0;x)] + 4[H(-1,-1,0,1;x) + H(-1,-1,1,0;x) \\
&+H(-1,0,-1,0;x) - H(-1,0,1,1;x) + H(-1,1,-1,0;x) \\
&-H(-1,1,0,1;x) - H(-1,1,1,0;x) + H(0,-1,-1,0;x) \\
&-H(0,-1,0,1;x) - H(0,-1,1,0;x) - H(0,0,-1,0;x) \\
&+H(0,0,1,1;x) - H(0,1,-1,0;x) + H(0,1,0,1;x) \\
&+H(0,1,1,0;x) + H(1,-1,-1,0;x) - H(-1,-1,-1,0;x) \\
&-H(1,-1,0,1;x) - H(1,-1,1,0;x) - H(1,0,-1,0;x) \\
&+H(1,0,0,1;x) + H(1,0,1,0;x) - H(1,1,-1,0;x) \\
&+H(1,1,0,0;x)] + 6[H(-1,-1,0,0;x) - H(-1,0,0,1;x) \\
&-H(-1,0,1,0;x) - H(-1,1,0,0;x) - H(0,-1,0,0;x) \\
&+H(0,0,0,1;x) + H(0,1,0,0;x) + H(1,0,0,0;x)]\Big\}. \qquad (35)
\end{aligned}
$$

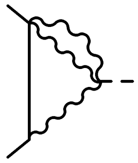


$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_5 \mathcal{D}_{10}} \\
&= \sum_{i=-1}^{1} (D-4)^i \, F_5^{(i)}(x) + \mathcal{O}\left((D-4)^2\right) , \qquad (36)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_5^{(-1)}(x) &= \frac{1}{16}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big][4\zeta(2) + H(0,0;x) + 2H(0,1;x)] , \qquad (37) \\
F_5^{(0)}(x) &= -\frac{1}{32}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{4\zeta(2) + \zeta(3) - \zeta(2)[H(0;x) - 8H(-1;x) \\
&-8H(1;x)] + H(0,0;x) + 2H(0,1;x) + H(0,0,0;x) + 2H(-1,0,0;x) \\
&+4H(-1,0,1;x) + 2H(0,0,1;x) + 2H(0,1,0;x) + 4H(0,1,1;x) \\
&+2H(1,0,0;x) + 4H(1,0,1;x)\Big\} , \qquad (38) \\
F_5^{(1)}(x) &= \frac{1}{64}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{4\zeta(2) + \frac{59}{10}\zeta^2(2) + 5\zeta(3) - [\zeta(2) \\
&+2\zeta(3)]H(0;x) + 2[4\zeta(2) + 5\zeta(3)][H(1;x) + H(-1;x)] \\
&+[1 - \zeta(2)][H(0,0;x) + 2H(0,1;x)] + 2\zeta(2)[8H(-1,1;x) \\
&-H(-1,0;x) + 8H(-1,-1;x) + 8H(1,-1;x) - H(1,0;x) \\
&+8H(1,1;x)] + H(0,0,0;x) + 2H(0,0,1;x) + 2H(-1,0,0;x) \\
&+4H(-1,0,1;x) + 2H(0,1,0;x) + 4H(0,1,1;x) + 2H(1,0,0;x) \\
&+4H(1,0,1;x) + H(0,0,0,0;x) + 4H(-1,-1,0,0;x) \\
&+8H(-1,-1,0,1;x) + 2H(-1,0,0,0;x) + 4H(-1,0,0,1;x)
\end{aligned}
$$

$$
\begin{aligned}
&+4H(-1,0,1,0;x)+8H(-1,0,1,1;x)+4H(-1,1,0,0;x)\\
&+8H(-1,1,0,1;x)+2H(0,0,0,1;x)+2H(0,0,1,0;x)\\
&+4H(0,0,1,1;x)+2H(0,1,0,0;x)+4H(0,1,0,1;x)\\
&+4H(0,1,1,0;x)+8H(0,1,1,1;x)+4H(1,-1,0,0;x)\\
&+8H(1,-1,0,1;x)+2H(1,0,0,0;x)+4H(1,0,0,1;x)\\
&+4H(1,0,1,0;x)+8H(1,0,1,1;x)+4H(1,1,0,0;x)\\
&+8H(1,1,0,1;x)\Big\}\,. \qquad (39)
\end{aligned}
$$

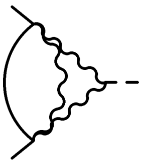


$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_2\mathcal{D}_5\mathcal{D}_9\mathcal{D}_{19}}\\
&= \sum_{i=-2}^{2} (D-4)^i \, F_6^{(i)}(x) + \mathcal{O}\left((D-4)^3\right)\,, \qquad (40)
\end{aligned}
$$

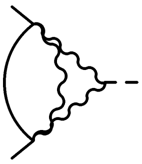
where:

$$
\begin{aligned}
F_6^{(-2)}(x) &= \frac{1}{8}\,, \qquad (41)\\
F_6^{(-1)}(x) &= -\frac{5}{16}\,, \qquad (42)\\
F_6^{(0)}(x) &= \frac{1}{32}\Big\{19+4\zeta(2)+2\Big[1-\frac{2}{(1+x)}\Big]\Big[4\zeta(2)+H(0,0;x)\\
&\quad +2H(0,1;x)\Big]\Big\}, \qquad (43)\\
F_6^{(1)}(x) &= -\frac{65}{64}-\frac{1}{16}\Big[15-\frac{20}{(1+x)}\Big]\zeta(3)-\frac{1}{32}\Big[11-\frac{14}{(1+x)}\Big]\zeta(3)\\
&\quad -\frac{1}{32}\Big[1-\frac{2}{(1+x)}\Big]\{\zeta(2)[H(0;x)+8H(-1;x)+16H(1;x)]\\
&\quad +5H(0,0;x)+10H(0,1;x)+3H(0,0,0;x)+2H(-1,0,0;x)\\
&\quad +4H(-1,0,1;x)+6H(0,0,1;x)+4H(0,1,0;x)+8H(0,1,1;x)\\
&\quad +4H(1,0,0;x)+8H(1,0,1;x)\}\,, \qquad (44)\\
F_6^{(2)}(x) &= \frac{211}{128}+\Big[\frac{57}{32}-\frac{19}{8(1+x)}\Big]\zeta(2)+\Big[\frac{267}{640}-\frac{139}{320(1+x)}\Big]\zeta^2(2)+\Big[\frac{55}{64}\\
&\quad -\frac{35}{32(1+x)}\Big]\zeta(3)+\frac{1}{64}\Big[1-\frac{2}{(1+x)}\Big]\Big\{(5\zeta(2)-2\zeta(3))H(0;x)\\
&\quad +(40\zeta(2)+14\zeta(3))H(-1;x)+(80\zeta(2)+28\zeta(3))H(1;x)+(19\\
&\quad -\zeta(2))H(0,0;x)+(38+12\zeta(2))H(0,1;x)+\zeta(2)[16H(-1,-1;x)\\
&\quad +64H(0,-1;x)+2H(-1,0;x)+32H(1,-1;x)+4H(1,0;x)\\
&\quad +64H(1,1;x)+32H(-1,1;x)]+15H(0,0,0;x)+10H(-1,0,0;x)
\end{aligned}
$$

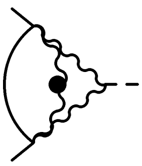
$$
\begin{aligned}
&+20H(-1,0,1;x)+30H(0,0,1;x)+20H(0,1,0;x)+40H(0,1,1;x)\\
&+20H(1,0,0;x)+40H(1,0,1;x)+7H(0,0,0,0;x)\\
&+4H(-1,-1,0,0;x)+8H(-1,-1,0,1;x)+6H(-1,0,0,0;x)\\
&+12H(-1,0,0,1;x)+8H(-1,0,1,0;x)+16H(-1,0,1,1;x)\\
&+8H(-1,1,0,0;x)+16H(-1,1,0,1;x)+2H(0,-1,0,0;x)\\
&+4H(0,-1,0,1;x)+14H(0,0,0,1;x)+12H(0,0,1,0;x)\\
&+24H(0,0,1,1;x)+12H(0,1,0,0;x)+24H(0,1,0,1;x)\\
&+16H(0,1,1,0;x)+32H(0,1,1,1;x)+8H(1,-1,0,0;x)\\
&+16H(1,-1,0,1;x)+12H(1,0,0,0;x)+24H(1,0,0,1;x)\\
&+16H(1,0,1,0;x)+32H(1,0,1,1;x)+16H(1,1,0,0;x)\\
&+32H(1,1,0,1;x)\Big\}\,. \qquad (45)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1\mathcal{D}_{10}\mathcal{D}_{19}\mathcal{D}_{21}}\\
&= \sum_{i=-2}^{1} (D-4)^i\, F_7^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (46)
\end{aligned}
$$



$$
\begin{aligned}
(p_1\cdot k_1) &= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{(p_1\cdot k_1)}{\mathcal{D}_1\mathcal{D}_{10}\mathcal{D}_{19}\mathcal{D}_{21}}\\
&= m^2 \sum_{i=-2}^{1} (D-4)^i\, F_8^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (47)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1^2\mathcal{D}_{10}\mathcal{D}_{19}\mathcal{D}_{21}}\\
&= \frac{1}{m^2} \sum_{i=-1}^{1} (D-4)^i\, F_9^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (48)
\end{aligned}
$$

where:

$$
F_7^{(-2)}(x) = \frac{1}{8}\,, \qquad (49)
$$

$$
F_7^{(-1)}(x) = -\frac{5}{16} + \frac{1}{8}H(0;x) + \frac{1}{4}H(1;x)\,, \qquad (50)
$$

$$
F_7^{(0)}(x) = \frac{1}{32}\Big\{19 - 8\zeta(2) + 10H(0;x) + 20H(1;x) + 2H(0,0;x) + 4H(0,1;x)
$$

$$
\begin{aligned}
& +6H(1,0;x)+12H(1,1;x)+\frac{1}{(1+x)}\Big[8\zeta(2)+4H(0,1;x) \\
& +2H(0,0;x)\Big]+\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\Big[-2\zeta(3)+2\zeta(2)H(0;x) \\
& +H(0,0,0;x)+2H(0,0,1;x)+2H(0,1,0;x)+4H(0,1,1;x)\Big]\bigg\}, \qquad (51)
\end{aligned}
$$

$$
\begin{aligned}
F_7^{(1)}(x) \;=\;& -\frac{65}{64}+\frac{1}{8}\zeta(3)-\frac{1}{32}\Big[19H(0;x)+(38-9\zeta(2))H(1;x)+10H(0,0;x) \\
& +20H(0,1;x)+15H(1,0;x)+30H(1,1;x)+4H(0,0,0;x) \\
& +8H(0,0,1;x)+10H(1,1,0;x)+6H(0,1,0;x)+12H(0,1,1;x) \\
& +20H(1,1,1;x)+2H(-1,0,0;x)+7H(1,0,0;x)+2H(-1,0,1;x) \\
& +14H(1,0,1;x)\Big]+\frac{1}{32}\Big[1-\frac{1}{(1+x)}\Big]\Big\{20\zeta(2)+9\zeta(3)-\zeta(2)[H(0;x) \\
& +4H(-1;x)+10H(1;x)]+5H(0,0;x)+10H(0,1;x)+2H(0,0,0;x) \\
& +4H(0,0,1;x)+2H(0,1,0;x)+4H(0,1,1;x)+2H(-1,0,0;x) \\
& +4H(1,0,0;x)+4H(-1,0,1;x)+8H(1,0,1;x)\Big\} \\
& +\frac{1}{64}\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]\Big\{\frac{32}{5}\zeta^2(2)+2\zeta(3)-(2\zeta(2)-3\zeta(3))H(0;x) \\
& +4\zeta(3)H(-1;x)-\zeta(2)[H(0,0;x)+4H(-1,0;x)+2H(0,1;x)] \\
& -H(0,0,0;x)+2H(0,0,1;x)+2H(0,1,0;x)+4H(0,1,1;x) \\
& +3H(0,0,0,0;x)+2H(-1,0,0,0;x)+4H(-1,0,0,1;x) \\
& +4H(-1,0,1,0;x)+8H(-1,0,1,1;x)+6H(0,0,0,1;x) \\
& +4H(0,0,1,0;x)+8H(0,0,1,1;x)+6H(0,1,0,0;x) \\
& +12H(0,1,0,1;x)+12H(0,1,1,0;x)+24H(0,1,1,1;x)\Big\}, \qquad (52)
\end{aligned}
$$

$$
F_8^{(-2)}(x) \;=\; -\frac{1}{64}\left[\frac{1}{x}+x+2\right], \qquad (53)
$$

$$
F_8^{(-1)}(x) \;=\; \frac{1}{256}\Big[\frac{1}{x}+x+2\Big]\Big[9+4H(0;x)+8H(1;x)\Big], \qquad (54)
$$

$$
\begin{aligned}
F_8^{(0)}(x) \;=\;& \frac{1}{1024}\bigg\{-126-63\frac{1}{x}-63x-16\zeta(2)+32\zeta(2)x+\frac{128\zeta(2)}{(1+x)}-\bigg[32+16\frac{1}{x} \\
& +8x-32\frac{1}{(1+x)}\bigg][H(0,0;x)+2H(0,1;x)]-\Big[40+\frac{36}{x}+36x\Big][H(0;x) \\
& +2H(1;x)]-\Big[32+\frac{24}{x}+24x\Big][H(1,0;x)+2H(1,1;x)] \\
& +\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right][16\zeta(3)-16\zeta(2)H(0;x)-8H(0,0,0;x) \\
& -16H(0,0,1;x)-16H(0,1,0;x)-32H(0,1,1;x)]\bigg\}, \qquad (55)
\end{aligned}
$$

$$
F_8^{(1)}(x) \;=\; \frac{405}{4096}\Big[2+\frac{1}{x}+x\Big]+\Big[\frac{23}{256}-\frac{9}{128}x\Big]\zeta(2)+\frac{1}{64}\Big[\frac{1}{2}-\frac{1}{x}-\frac{13}{4}x\Big]\zeta(3)
$$

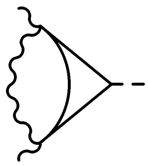
$$
\begin{aligned}
& -\frac{1}{4(1+x)}\zeta(2) + \frac{1}{256}\Big[\frac{1}{(1-x)} - \frac{29}{(1+x)}\Big]\zeta(3) + \frac{1}{1024}\Big[30 + \frac{63}{x} \\
& +(63+4\zeta(2))x\Big]H(0;x) + \frac{1}{64}\Big[4 + \frac{1}{x} - x\Big]\zeta(2)H(-1;x) + \frac{1}{512}\Big[30 \\
& +56\zeta(2) + \frac{(63+10\zeta(2))}{x} + (63-22\zeta(2))x\Big]H(1;x) \\
& -\frac{1}{64(1+x)}\Big\{\zeta(2)[H(0;x)+8H(-1;x)+16H(1;x)] + 4H(0,0;x) \\
& +8H(0,1;x)\Big\} + \frac{3}{512}\Big[8 + \frac{6}{x} + 3x\Big][H(0,0;x)+2H(0,1;x)] \\
& +\frac{1}{512}\Big[16 + \frac{27}{x} + 27x\Big][H(1,0;x)+2H(1,1;x)] + \frac{1}{512}\Big[16 + \frac{8}{x} + 4x \\
& -\frac{1}{(1-x)} - \frac{23}{(1+x)}\Big][H(0,0,0;x)+2H(0,0,1;x)] + \frac{1}{256}\Big[12 + \frac{12}{x} \\
& +4x - \frac{2}{(1-x)} - \frac{30}{(1+x)}\Big][H(0,1,0;x)+2H(0,1,1;x)] + \frac{1}{256}\Big[4 \\
& +\frac{1}{x} - x - \frac{8}{(1+x)}\Big][H(-1,0,0;x)+2H(-1,0,1;x)] + \frac{1}{256}\Big[12 + \frac{7}{x} \\
& +3x - \frac{16}{(1+x)}\Big][H(1,0,0;x)+2H(1,0,1;x)] + \frac{1}{128}\Big[4 + \frac{5}{x} + 5x\Big] \times \\
& \times[H(1,1,0;x)+2H(1,1,1;x)] - \frac{1}{256}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{\frac{5}{32}\zeta^2(2) \\
& +(\zeta(2)+3\zeta(3))H(0;x) + 4\zeta(3)H(-1;x) - \zeta(2)[H(0,0;x) \\
& +4H(-1,0;x) - H(0,1;x)] - 3H(0,0,0,0;x) - 2H(-1,0,0,0;x) \\
& -4H(-1,0,0,1;x) - 4H(-1,0,1,0;x) - 8H(-1,0,1,1;x) \\
& -6H(0,0,0,1;x) - 4H(0,0,1,0;x) - 8H(0,0,1,1;x) \\
& -6H(0,1,0,0;x) - 12H(0,1,0,1;x) - 12H(0,1,1,0;x) \\
& -24H(0,1,1,1;x)\Big\}\,,
\end{aligned}
\tag{56}
$$

$$
F_9^{(-1)}(x) = \frac{1}{16}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big][4\zeta(2) + H(0,0;x) + 2H(0,1;x)]\,,
\tag{57}
$$

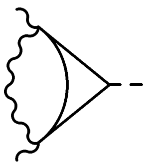
$$
\begin{aligned}
F_9^{(0)}(x) = & -\frac{1}{32}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big][9\zeta(3) + 8\zeta(2)H(-1;x) - \zeta(2)H(0;x) \\
& +16\zeta(2)H(1;x) + 2H(-1,0,0;x) + 4H(-1,0,1;x) + 2H(0,0,0;x) \\
& +4H(0,0,1;x) + 2H(0,1,0;x) + 4H(0,1,1;x) + 4H(1,0,0;x) \\
& +8H(1,0,1;x)]\,,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
F_9^{(1)}(x) = & \frac{1}{64}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{\frac{287}{10}\zeta^2(2) - \zeta(3)\Big[3H(0;x) - 32H(1;x) \\
& -18H(-1;x)\Big] + \zeta(2)\Big[14H(0,1;x) + 64H(1,1;x) + 8H(0,-1;x) \\
& -2H(-1,0;x) + 32H(-1,1;x) + 32H(1,-1;x) + 16H(-1,-1;x)\Big] \\
& +4H(0,0,0,0;x) + 4H(-1,-1,0,0;x) + 8H(-1,-1,0,1;x)
\end{aligned}
$$

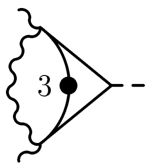
$$
\begin{aligned}
&+4H(-1,0,0,0;x)+8H(-1,0,0,1;x)+4H(-1,0,1,0;x)\\
&+8H(-1,0,1,1;x)+8H(-1,1,0,0;x)+16H(-1,1,0,1;x)\\
&+2H(0,-1,0,0;x)+4H(0,-1,0,1;x)+8H(0,0,0,1;x)\\
&+6H(0,0,1,0;x)+12H(0,0,1,1;x)+6H(0,1,0,0;x)\\
&+12H(0,1,0,1;x)+4H(0,1,1,0;x)+8H(0,1,1,1;x)\\
&+8H(1,-1,0,0;x)+16H(1,-1,0,1;x)+10H(1,0,0,0;x)\\
&+20H(1,0,0,1;x)+12H(1,0,1,0;x)+24H(1,0,1,1;x)\\
&+16H(1,1,0,0;x)+32H(1,1,0,1;x)\Big\}\,. \qquad (59)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1\mathcal{D}_{10}\mathcal{D}_{20}\mathcal{D}_{21}}\\
&= \sum_{i=-2}^{1}(D-4)^i\, F_{10}^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (60)
\end{aligned}
$$



$$
\begin{aligned}
(k_1\cdot k_2) &= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{(k_1\cdot k_2)}{\mathcal{D}_1\mathcal{D}_{10}\mathcal{D}_{20}\mathcal{D}_{21}}\\
&= m^2\sum_{i=-2}^{1}(D-4)^i\, F_{11}^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (61)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1^3\mathcal{D}_{10}\mathcal{D}_{20}\mathcal{D}_{21}}\\
&= \frac{1}{m^4}\sum_{i=0}^{2}(D-4)^i\, F_{12}^{(i)}(x) + \mathcal{O}\left((D-4)^3\right), \qquad (62)
\end{aligned}
$$

where:

$$
F_{10}^{(-2)}(x) = \frac{1}{8}\,, \qquad (63)
$$

$$
F_{10}^{(-1)}(x) = -\frac{5}{16} + \frac{1}{8}\left[1-\frac{2}{(1-x)}\right]H(0;x)\,, \qquad (64)
$$

$$
\begin{aligned}
F_{10}^{(0)}(x) = {}& \frac{19}{32} - \frac{1}{16(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{4\zeta(3)+2\zeta(2)H(0;x)+H(0,0,0;x)\\
&+2H(0,1,0;x)\Big\} - \frac{1}{16}\left[1-\frac{2}{(1-x)}\right]\Big\{5H(0;x)+H(0,0;x)\\
&-2H(-1,0;x)+H(1,0;x)\Big\} + \frac{1}{16(1-x)}H(0,0;x)\,, \qquad (65)
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{10}(x) \;=\;& -\frac{65}{64}+\frac{1}{32(1-x)}\Big\{5\zeta(3)+\zeta(2)H(0;x)-5H(0,0;x)-2H(0,0,0;x)\\
&+2H(0,-1,0;x)\Big\}+\frac{1}{16}H(0,1,0;x)+\frac{1}{32}\left[1-\frac{2}{(1-x)}\right]\Big\{\zeta(3)\\
&+(19+\zeta(2))H(0;x)+\zeta(2)H(1;x)+5H(0,0;x)+5H(1,0;x)\\
&-10H(-1,0;x)+2H(0,0,0;x)+4H(-1,-1,0;x)-3H(-1,0,0;x)\\
&-2H(-1,1,0;x)-2H(0,-1,0;x)-2H(1,-1,0;x)+H(1,0,0;x)\\
&+2H(1,1,0;x)\Big\}+\frac{1}{32(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{4\zeta(3)-\frac{22}{5}\zeta^2(2)+(2\zeta(2)\\
&-3\zeta(3))H(0;x)-8\zeta(3)H(1;x)+\zeta(2)\Big[H(0,0;x)+2H(0,1;x)\\
&-4H(0,-1;x)-4H(1,0;x)\Big]+H(0,0,0;x)-2H(0,1,0;x)\\
&+3H(0,0,0,0;x)-2H(0,-1,0,0;x)-4H(0,-1,1,0;x)\\
&-2H(0,0,-1,0;x)+2H(0,0,1,0;x)-4H(0,1,-1,0;x)\\
&+4H(0,1,0,0;x)+4H(0,1,1,0;x)-2H(1,0,0,0;x)\\
&-4H(1,0,1,0;x)\Big\}, \qquad (66)\\
F^{(-2)}_{11}(x) \;=\;& -\frac{1}{16}+\frac{1}{32}\left[x+\frac{1}{x}\right], \qquad (67)\\
F^{(-1)}_{11}(x) \;=\;& \frac{9}{64}-\frac{1}{128}\left[x+\frac{1}{x}\right]\Big\{9+4H(0;x)\Big\}-\frac{1}{16}\left[1-\frac{2}{(1-x)}\right]H(0;x), \qquad (68)\\
F^{(0)}_{11}(x) \;=\;& -\frac{71}{256}+\frac{63}{256}x+\frac{1}{64}\left[\frac{1}{x}-\frac{2}{(1-x)}\right]H(0,0;x)+\frac{1}{512}\left[x-\frac{1}{x}\right]\Big\{63\\
&+18H(0;x)-8H(0,0;x)+16H(-1,0;x)-8H(1,0;x)\Big\}\\
&+\frac{1}{32}\left[1-\frac{2}{(1-x)}\right]\Big\{4H(0;x)+H(0,0;x)-2H(-1,0;x)\\
&+H(1,0;x)\Big\}+\frac{1}{32(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{4\zeta(3)+2\zeta(2)H(0;x)\\
&+H(0,0,0;x)+2H(0,1,0;x)\Big\}, \qquad (69)\\
F^{(1)}_{11}(x) \;=\;& +\frac{525}{1024}-\frac{1}{64}\zeta(3)-\frac{1}{64}\left[1-\frac{1}{(1-x)^2}\right]\zeta(2)H(0;x)-\frac{x}{1024}\Big[405\\
&-40\zeta(3)-8\zeta(2)H(0;x)+36H(0,0;x)+16H(0,0,0;x)\\
&+16H(1,0,0;x)-16H(0,-1,0;x)\Big]-\frac{1}{128(1-x)}\Big[6\zeta(3)\\
&-11H(0,0;x)-4H(0,0,0;x)-4H(1,0,0;x)+4H(0,-1,0;x)\Big]\\
&+\frac{1}{2048}\left[x-\frac{1}{x}\right]\Big\{405-4\Big[16\zeta(3)-63H(0;x)-4\zeta(2)H(1;x)\\
&-36H(0,0;x)-18H(1,0;x)+36H(-1,0;x)\Big]-16\Big[4H(0,0,0;x)
\end{aligned}
$$

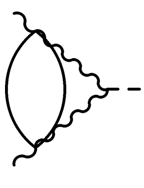
$$
\begin{aligned}
&+2H(0,1,0;x) - 3H(-1,0,0;x) - 2H(-1,1,0;x) + 3H(1,0,0;x) \\
&+2H(1,1,0;x) + 4H(-1,-1,0;x) - 2H(1,-1,0;x) - 4H(0,-1,0;x)\Big]\Big\} \\
&-\frac{1}{128}\left[1-\frac{2}{(1-x)}\right]\Big\{\zeta(2) + 27H(0;x) + 2\zeta(2)H(1;x) + 7H(0,0;x) \\
&+9H(1,0;x) - 16H(-1,0;x) + 4H(0,0,0;x) + 4H(0,1,0;x) \\
&-6H(-1,0,0;x) - 4H(-1,1,0;x) + 2H(1,0,0;x) + 4H(1,1,0;x) \\
&+8H(-1,-1,0;x) - 4H(1,-1,0;x) - 4H(0,-1,0;x)\Big\} \\
&-\frac{1}{64(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{2\zeta(3) - \frac{22}{5}\zeta^2(2) - \zeta(3)[3H(0;x) + 8H(1;x)] \\
&+\zeta(2)[H(0,0;x) - 4H(1,0;x) + 2H(0,1;x) - 4H(0,-1;x)] \\
&+\frac{1}{2}H(0,0,0;x) + H(0,1,0;x) + 3H(0,0,0,0;x) \\
&-2H(0,-1,0,0;x) - 4H(0,-1,1,0;x) - 2H(0,0,-1,0;x) \\
&+2H(0,0,1,0;x) - 4H(0,1,-1,0;x) + 4H(0,1,0,0;x) \\
&+4H(0,1,1,0;x) - 2H(1,0,0,0;x) - 4H(1,0,1,0;x)\Big\}, \qquad (70)
\end{aligned}
$$

$$
F_{12}^{(0)}(x) = -\frac{1}{32(1-x)}\left[1-\frac{1}{(1-x)}\right]H(0,0;x)\,, \qquad (71)
$$

$$
\begin{aligned}
F_{12}^{(1)}(x) &= -\frac{1}{64(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{9\zeta(3) + 3\zeta(2)H(0;x) - H(0,0,0;x) \\
&+2H(0,-1,0;x) + 2H(0,1,0;x) + 2H(1,0,0;x)\Big\}\,, \qquad (72)
\end{aligned}
$$

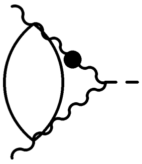
$$
\begin{aligned}
F_{12}^{(2)}(x) &= -\frac{1}{256(1-x)}\left[1-\frac{1}{(1-x)}\right]\Big\{11\zeta^2(2) + 12\zeta(3)H(1;x) \\
&-2\zeta(2)\Big[H(0,0;x) - 6(H(0,-1;x) + H(1,0;x)) + 2H(0,1;x)\Big] \\
&+2H(0,0,0,0;x) - 4\Big[H(0,0,-1,0;x) + H(0,1,0,0;x)\Big] \\
&+8\Big[H(0,-1,-1,0;x) + H(0,-1,1,0;x) + H(0,1,-1,0;x) \\
&-H(0,1,1,0;x) - H(1,0,-1,0;x) + 2H(1,0,0,0;x) \\
&+2H(1,0,1,0;x) + H(1,1,0,0;x)\Big]\Big\}\,, \qquad (73)
\end{aligned}
$$

As we already pointed out, the following MIs, involved only in the calculation of the topology (h) in Fig. 3, are given in terms of HPLs of the variable $\bar{x}$ defined in Eq. (6). In agreement with [18], we find:



$$
= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_2\mathcal{D}_{10}\mathcal{D}_{19}\mathcal{D}_{21}}
$$

$$= \sum_{i=-2}^{1} (D-4)^i \, F_{13}^{(i)}(\bar{x}) + \mathcal{O}\left((D-4)^2\right), \tag{74}$$



$$= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_2 \mathcal{D}_{10} \mathcal{D}_{19}^2 \mathcal{D}_{21}}$$

$$= \frac{1}{m^2} \sum_{i=-2}^{2} (D-4)^i \, F_{14}^{(i)}(\bar{x}) + \mathcal{O}\left((D-4)^3\right), \tag{75}$$

where:

$$F_{13}^{(-2)}(\bar{x}) = \frac{1}{8}, \tag{76}$$

$$F_{13}^{(-1)}(\bar{x}) = -\frac{5}{16} - \frac{1}{8}\Big[H(0;\bar{x}) - 2H(-1;\bar{x})\Big], \tag{77}$$

$$\begin{aligned}
F_{13}^{(0)}(\bar{x}) = {} & \frac{19}{32} - \frac{1}{8}\left[1 - \frac{1}{(1+\bar{x})}\right]\zeta(2) + \frac{5}{16}\Big[H(0;\bar{x}) - 2H(-1;\bar{x})\Big] \\
& -\frac{1}{8}\Big[H(-1,0;\bar{x}) - 2H(-1,-1;\bar{x})\Big] + \frac{1}{8(1+\bar{x})}\Big[H(0,0;\bar{x}) \\
& -2H(0,-1;\bar{x})\Big] - \frac{1}{8(1+\bar{x})}\left[1 - \frac{1}{(1+\bar{x})}\right]\Big[2\zeta(3) - \zeta(2)H(0;\bar{x}) \\
& -H(0,0,0;\bar{x}) + 2H(0,0,-1;\bar{x})\Big],
\end{aligned} \tag{78}$$

$$\begin{aligned}
F_{13}^{(1)}(\bar{x}) = {} & -\frac{1}{64}\Big\{65 + 38[H(0;\bar{x}) - 2H(-1;\bar{x})] + 8\zeta(2)H(1;\bar{x}) - 20H(-1,0;\bar{x}) \\
& +40H(-1,-1;\bar{x}) + 8H(-1,-1,0;\bar{x}) - 16H(-1,-1,-1;\bar{x})\Big\} \\
& +\frac{1}{16(1+\bar{x})}\Big[2\zeta(3) - 5H(0,0;\bar{x}) + 10H(0,-1;\bar{x}) - 2H(0,0,0;\bar{x}) \\
& -4H(-1,0,-1;\bar{x}) + 2H(-1,0,0;\bar{x}) + 4H(0,0,-1;\bar{x}) \\
& +4H(1,0,-1;\bar{x}) - 2H(1,0,0;\bar{x}) - 4H(0,-1,-1;\bar{x}) \\
& +2H(0,-1,0;\bar{x})\Big] + \frac{1}{16}\left[1 - \frac{1}{(1+\bar{x})}\right]\Big\{5\zeta(2) - 2\zeta(3) + 2\zeta(2)\Big[H(0;\bar{x}) \\
& -H(-1;\bar{x}) + 2H(1;\bar{x})\Big] + 2H(0,0,0;\bar{x}) - 4H(0,0,-1;\bar{x}) \\
& -4H(1,0,-1;\bar{x}) + 2H(1,0,0;\bar{x})\Big\} - \frac{1}{32(1+\bar{x})}\left[1 - \frac{1}{(1+\bar{x})}\right]\Big\{\zeta^2(2) \\
& -12\zeta(3) + 6\zeta(2)H(0;\bar{x}) - 4\zeta(3)\Big[H(0;\bar{x}) + 2H(-1;\bar{x})\Big] \\
& +2\zeta(2)\Big[H(0,0;\bar{x}) - 2H(0,-1;\bar{x}) + 2H(-1,0;\bar{x}) + 4H(0,1;\bar{x})\Big] \\
& +6H(0,0,0;\bar{x}) - 12H(0,0,-1;\bar{x}) + 6H(0,0,0,0;\bar{x}) \\
& -8H(-1,0,0,-1;\bar{x}) + 4H(-1,0,0,0;\bar{x}) + 8H(0,-1,0,-1;\bar{x}) \\
& -4H(0,-1,0,0;\bar{x}) + 8H(0,0,-1,-1;\bar{x}) - 4H(0,0,-1,0;\bar{x})
\end{aligned}$$

$$-12H(0,0,0,-1;\bar{x}) - 16H(0,1,0,-1;\bar{x}) + 8H(0,1,0,0;\bar{x})\Big\}, \quad (79)$$

$$F_{14}^{(-2)}(\bar{x}) = -\frac{1}{4(1+\bar{x})}\left[1-\frac{1}{(1+\bar{x})}\right], \quad (80)$$

$$F_{14}^{(-1)}(\bar{x}) = \frac{1}{8(1+\bar{x})}\left[1-\frac{1}{(1+\bar{x})}\right]\Big[H(0;\bar{x}) - 2H(-1;\bar{x})\Big], \quad (81)$$

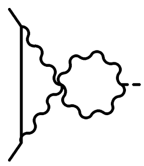
$$\begin{aligned}
F_{14}^{(0)}(\bar{x}) = {} & \frac{1}{8(1+\bar{x})}\left[1-\frac{2}{(1+\bar{x})}+\frac{1}{(1+\bar{x})^2}\right]\zeta(2) - \frac{1}{8(1+\bar{x})}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{2 \\
& + H(0;\bar{x}) - 2H(-1;\bar{x}) + 2H(-1,-1;\bar{x}) - H(-1,0;\bar{x})\Big\} \\
& - \frac{1}{8(1+\bar{x})^2}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{H(0,0;\bar{x}) - 2H(0,-1;\bar{x})\Big\}, \quad (82)
\end{aligned}$$

$$\begin{aligned}
F_{14}^{(1)}(\bar{x}) = {} & \frac{1}{8(1+\bar{x})}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{H(0;\bar{x}) - 2H(-1;\bar{x}) + 2H(-1,-1;\bar{x}) \\
& - H(-1,0;\bar{x}) - 2H(-1,-1,-1;\bar{x})\Big\} - \frac{1}{8(1+\bar{x})^2}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{\zeta(3) \\
& - \zeta(2)H(1;\bar{x}) - H(0,0;\bar{x}) + 2H(0,-1;\bar{x}) - H(0,0,0;\bar{x}) \\
& - H(1,0,0;\bar{x}) - H(-1,-1,0;\bar{x}) + 2H(0,0,-1;\bar{x}) + 2H(1,0,-1;\bar{x}) \\
& - 2H(-1,0,-1;\bar{x}) + H(-1,0,0;\bar{x}) - 2H(0,-1,-1;\bar{x}) \\
& + H(0,-1,0;\bar{x})\Big\} - \frac{1}{8(1+\bar{x})}\left[1-\frac{2}{(1+\bar{x})}+\frac{1}{(1+\bar{x})^2}\right]\Big\{\zeta(3) \\
& + \zeta(2)\Big[H(0;\bar{x}) - H(-1;\bar{x}) + H(1;\bar{x})\Big] + H(0,0,0;\bar{x}) \\
& - 2H(0,0,-1;\bar{x}) - 2H(1,0,-1;\bar{x}) + H(1,0,0;\bar{x})\Big\}, \quad (83)
\end{aligned}$$

$$\begin{aligned}
F_{14}^{(2)}(\bar{x}) = {} & -\frac{1}{20(1+\bar{x})}\zeta^2(2) - \frac{1}{16(1+\bar{x})}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{4 - \frac{3}{5}\zeta^2(2) - 2\zeta(3) \\
& + 2(1-\zeta(3))H(0;\bar{x}) - 4H(-1;\bar{x}) + 2(\zeta(2)+2\zeta(3))H(1;\bar{x}) \\
& - \zeta(2)H(-1,0;\bar{x}) - 2H(-1,0;\bar{x}) + 2H(1,0;\bar{x}) - 2H(1,-1;\bar{x}) \\
& + 4H(1,1;\bar{x}) - 2H(-1,1;\bar{x}) + 4H(-1,-1;\bar{x}) + 2H(0,0,0;\bar{x}) \\
& - 4H(0,0,-1;\bar{x}) - 4H(1,0,-1;\bar{x}) + 2H(1,0,0;\bar{x}) + 2H(-1,-1,0;\bar{x}) \\
& - 4H(-1,-1,-1;\bar{x}) - 4H(0,0,0,0;\bar{x}) - 2H(-1,-1,-1,0;\bar{x}) \\
& + 4H(-1,-1,-1,-1;\bar{x})\Big\} - \frac{1}{16(1+\bar{x})^2}\left[1-\frac{1}{(1+\bar{x})}\right]\Big\{4\zeta(3)H(-1;\bar{x}) \\
& + 2H(0,0;\bar{x}) - 4H(0,-1;\bar{x}) + 2H(0,1;\bar{x}) - 2H(0,-1,0;\bar{x}) \\
& + 4H(0,-1,-1;\bar{x}) - 2H(-1,0,0;\bar{x}) + 4H(-1,0,-1;\bar{x}) \\
& + 6H(0,0,0,0;\bar{x}) - 4H(-1,-1,0,-1;\bar{x}) + 2H(-1,-1,0,0;\bar{x}) \\
& - 4H(-1,0,-1,-1;\bar{x}) + 2H(-1,0,-1,0;\bar{x}) - 4H(0,-1,-1,-1;\bar{x}) \\
& + 2H(0,-1,-1,0;\bar{x}) + 6H(-1,0,0,-1;\bar{x}) - 3H(-1,0,0,0;\bar{x}) \\
& + 4H(-1,1,0,-1;\bar{x}) + 4H(0,-1,0,-1;\bar{x}) - 2H(-1,1,0,0;\bar{x}) \\
& - 2H(0,-1,0,0;\bar{x}) + 4H(0,0,-1,-1;\bar{x}) - 2H(0,0,-1,0;\bar{x})
\end{aligned}$$

$$
\begin{aligned}
& -4H(0,1,0,-1;\bar{x}) - 4H(0,0,0,-1;\bar{x}) + 2H(0,1,0,0;\bar{x}) \\
& +4H(1,-1,0,-1;\bar{x}) - 2H(1,-1,0,0;\bar{x}) + 4H(1,0,-1,-1;\bar{x}) \\
& -2H(1,0,-1,0;\bar{x}) - 8H(1,0,0,-1;\bar{x}) + 4H(1,0,0,0;\bar{x}) \\
& -8H(1,1,0,-1;\bar{x}) + 4H(1,1,0,0;\bar{x})\Big\} + \frac{1}{32(1+\bar{x})}\Big[1 - \frac{2}{(1+\bar{x})} \\
& + \frac{1}{(1+\bar{x})^2}\Big]\Big\{4\zeta(2) + \zeta^2(2) - 8\zeta(3) + 4[\zeta(2) - 2\zeta(3)][H(0;\bar{x}) \\
& +2H(1;\bar{x})] - 4\zeta(2)H(-1;\bar{x}) + 2\zeta(2)[H(0,0;\bar{x}) - 2H(-1,0;\bar{x}) \\
& -2H(0,-1;\bar{x}) + 2H(-1,-1;\bar{x}) + 4H(1,0;\bar{x}) + 4H(0,1;\bar{x}) \\
& -4H(1,-1;\bar{x}) + 8H(1,1;\bar{x}) - 4H(-1,1;\bar{x})] + 8H(0,0,0;\bar{x}) \\
& -16H(0,0,-1;\bar{x}) - 16H(1,0,-1;\bar{x}) + 8H(1,0,0;\bar{x}) \\
& -2H(0,0,0,0;\bar{x}) + 4H(-1,0,0,-1;\bar{x}) - 2H(-1,0,0,0;\bar{x}) \\
& +8H(-1,1,0,-1;\bar{x}) - 4H(-1,1,0,0;\bar{x}) + 8H(0,-1,0,-1;\bar{x}) \\
& -4H(0,-1,0,0;\bar{x}) + 8H(0,0,-1,-1;\bar{x}) - 4H(0,0,-1,0;\bar{x}) \\
& -12H(0,0,0,-1;\bar{x}) - 16H(0,1,0,-1;\bar{x}) + 8H(0,1,0,0;\bar{x}) \\
& +8H(1,-1,0,-1;\bar{x}) - 4H(1,-1,0,0;\bar{x}) + 8H(1,0,-1,-1;\bar{x}) \\
& -4H(1,0,-1,0;\bar{x}) - 16H(1,0,0,-1;\bar{x}) + 4H(1,0,0,0;\bar{x}) \\
& -16H(1,1,0,-1;\bar{x}) + 8H(1,1,0,0;\bar{x})\Big\} . \qquad (84)
\end{aligned}
$$

3.3 2-loop Topologies with 5 denominators



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_2\mathcal{D}_3\mathcal{D}_5\mathcal{D}_9\mathcal{D}_{15}} \\
&= \frac{1}{m^2}\sum_{i=-1}^{1}(D-4)^i \, F_{15}^{(i)}(x) + \mathcal{O}\left((D-4)^2\right) , \qquad (85)
\end{aligned}
$$

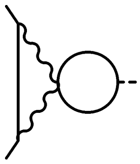
where:

$$
F_{15}^{(-1)}(x) = -\frac{1}{16}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big][4\zeta(2) + H(0,0;x) + 2H(0,1;x)] , \qquad (86)
$$

$$
\begin{aligned}
F_{15}^{(0)}(x) = & \frac{1}{32}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{8\zeta(2) + 5\zeta(3) + \zeta(2)[3H(0;x) + 8H(-1;x) \\
& +16H(1;x)] + 2H(0,0;x) + 4H(0,1;x) + 4H(0,0,0;x) \\
& +2H(-1,0,0;x) + 4H(-1,0,1;x) + 8H(0,0,1;x) + 6H(0,1,0;x) \\
& +12H(0,1,1;x) + 4H(1,0,0;x) + 8H(1,0,1;x)\Big\} , \qquad (87)
\end{aligned}
$$

$$
\begin{aligned}
F_{15}^{(1)}(x) = & -\frac{1}{64}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{16\zeta(2) + \frac{19}{10}\zeta^2(2) + 10\zeta(3) + 3[2\zeta(2) \\
& +\zeta(3)]H(0;x) + [16\zeta(2) + 10\zeta(3)][2H(1;x) + H(-1;x)]
\end{aligned}
$$

$$
\begin{aligned}
&+4H(0,0;x)+2[4+5\zeta(2)]H(0,1;x)+2\zeta(2)[6H(1,0;x)\\
&+3H(-1,0;x)+4H(0,-1;x)+32H(1,1;x)+16H(-1,1;x)\\
&+16H(1,-1;x)+8H(-1,-1;x)]+4[2H(0,0,0;x)+4H(0,0,1;x)\\
&+2H(1,0,0;x)+3H(0,1,0;x)+H(-1,0,0;x)+4H(1,0,1;x)\\
&+6H(0,1,1;x)+2H(-1,0,1;x)]+11H(0,0,0,0;x)\\
&+4H(-1,-1,0,0;x)+8H(-1,-1,0,1;x)+8H(-1,0,0,0;x)\\
&+16H(-1,0,0,1;x)+12H(-1,0,1,0;x)+24H(-1,0,1,1;x)\\
&+8H(-1,1,0,0;x)+16H(-1,1,0,1;x)+2H(0,-1,0,0;x)\\
&+4H(0,-1,0,1;x)+22H(0,0,0,1;x)+20H(0,0,1,0;x)\\
&+40H(0,0,1,1;x)+18H(0,1,0,0;x)+36H(0,1,0,1;x)\\
&+28H(0,1,1,0;x)+56H(0,1,1,1;x)+8H(1,-1,0,0;x)\\
&+16H(1,-1,0,1;x)+16H(1,0,0,0;x)+32H(1,0,0,1;x)\\
&+24H(1,0,1,0;x)+48H(1,0,1,1;x)+16H(1,1,0,0;x)\\
&+32H(1,1,0,1;x)\Big\}\,. \qquad (88)
\end{aligned}
$$

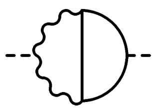


$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_2\mathcal{D}_3\mathcal{D}_5\mathcal{D}_{10}\mathcal{D}_{16}}\\
&= \frac{1}{m^2}\sum_{i=-1}^{1}(D-4)^i\, F_{16}^{(i)}(x)+\mathcal{O}\left((D-4)^2\right)\,, \qquad (89)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_{16}^{(-1)}(x) &= -\frac{1}{16}\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big][4\zeta(2)+H(0,0;x)+2H(0,1;x)]\,, \qquad (90)\\
F_{16}^{(0)}(x) &= -\frac{1}{16(1-x)}\Big[1-\frac{1}{(1-x)}\Big]\Big[4\zeta(2)H(0;x)+3H(0,0,0;x)\\
&\quad +4H(0,0,1;x)+2H(0,1,0;x)\Big]+\frac{1}{32}\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]\Big\{8\zeta(2)\\
&\quad +5\zeta(3)+\zeta(2)[H(0;x)+8H(-1;x)+8H(1;x)]+2H(0,0;x)\\
&\quad +4H(0,1;x)+H(0,0,0;x)+2H(0,0,1;x)+2H(0,1,0;x)\\
&\quad +2H(-1,0,0;x)+4H(-1,0,1;x)+4H(0,1,1;x)+2H(1,0,0;x)\\
&\quad +4H(1,0,1;x)\Big\}\,, \qquad (91)\\
F_{16}^{(1)}(x) &= -\frac{1}{32(1-x)}\Big[1-\frac{1}{(1-x)}\Big]\Big\{4\zeta^2(2)-[8\zeta(2)+5\zeta(3)]H(0;x)\\
&\quad -\zeta(2)[H(0,0;x)+6H(0,1;x)+8H(1,0;x)+8H(0,-1;x)]\\
&\quad -6H(0,0,0;x)-8H(0,0,1;x)-4H(0,1,0;x)+2H(0,-1,0,0;x)\\
&\quad +4H(0,-1,1,0;x)+2H(0,0,-1,0;x)+4H(0,1,-1,0;x)
\end{aligned}
$$

$$
\begin{aligned}
&-10H(0,0,0,0;x) - 12H(0,0,0,1;x) - 10H(0,0,1,0;x) \\
&-8H(0,0,1,1;x) - 8H(0,1,0,0;x) - 8H(0,1,0,1;x) \\
&-4H(0,1,1,0;x) - 6H(1,0,0,0;x) - 8H(1,0,0,1;x) \\
&-4H(1,0,1,0;x)\Big\} - \frac{1}{64}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{16\zeta(2) + \frac{59}{10}\zeta^2(2) \\
&+10\zeta(3) - 2[\zeta(2)+\zeta(3)]H(0;x) + 2[8\zeta(2)+5\zeta(3)][H(1;x) \\
&+H(-1;x)] + [4-\zeta(2)][H(0,0;x) + 2H(0,1;x)] \\
&+2\zeta(2)[8H(1,-1;x) - H(1,0;x) + 8H(1,1;x) - H(-1,0;x) \\
&+8H(-1,1;x) + 8H(-1,-1;x)] + 2H(0,0,0;x) + 4H(0,0,1;x) \\
&+4H(0,1,0;x) + 8H(0,1,1;x) + 4H(1,0,0;x) + 8H(1,0,1;x) \\
&+4H(-1,0,0;x) + 8H(-1,0,1;x) + H(0,0,0,0;x) \\
&+2H(0,0,0,1;x) + 2H(0,0,1,0;x) + 4H(0,0,1,1;x) \\
&+2H(0,1,0,0;x) + 4H(0,1,0,1;x) + 4H(0,1,1,0;x) \\
&+2H(1,0,0,0;x) + 4H(1,0,0,1;x) + 4H(1,0,1,0;x) \\
&+4H(-1,-1,0,0;x) + 8H(-1,-1,0,1;x) + 2H(-1,0,0,0;x) \\
&+4H(-1,0,0,1;x) + 4H(-1,0,1,0;x) + 8H(-1,0,1,1;x) \\
&+4H(-1,1,0,0;x) + 8H(-1,1,0,1;x) + 8H(0,1,1,1;x) \\
&+4H(1,-1,0,0;x) + 8H(1,-1,0,1;x) + 8H(1,0,1,1;x) \\
&+4H(1,1,0,0;x) + 8H(1,1,0,1;x)\Big\}. \qquad (92)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1\mathcal{D}_7\mathcal{D}_{10}\mathcal{D}_{18}\mathcal{D}_{24}} \\
&= \frac{1}{m^2}\sum_{i=0}^{1}(D-4)^i\, F_{17}^{(i)}(x) + \mathcal{O}\left((D-4)^2\right), \qquad (93)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_{17}^{(0)}(x) &= \frac{1}{8}\Big[\frac{1}{(1-x)^2} - \frac{1}{(1-x)}\Big]\Big[3\zeta(3) + H(0,0,1;x) + H(0,1,0;x) \\
&\quad -2H(1,0,0;x)\Big], \qquad (94) \\
F_{17}^{(1)}(x) &= \frac{1}{80}\Big[\frac{1}{(1-x)} - \frac{1}{(1-x)^2}\Big]\Big[30\zeta(3) + 9\zeta^2(2) + 30\zeta(3)H(0;x) \\
&\quad +60\zeta(3)H(1;x) - 5\zeta(2)H(0,1;x) + 10\zeta(2)H(1,0;x) \\
&\quad -20H(1,0,0;x) + 10H(0,1,0;x) + 10H(0,0,1;x) \\
&\quad -10H(0,-1,0,0;x) - 10H(0,-1,0,1;x) - 10H(0,-1,1,0;x) \\
&\quad +10H(0,0,1,0;x) + 10H(0,0,1,1;x) - 10H(0,1,-1,0;x)
\end{aligned}
$$

$$+20H(0,0,0,1;x)+15H(0,1,0,0;x)+10H(0,1,0,1;x)$$
$$+10H(0,1,1,0;x)+60H(1,0,-1,0;x)+25H(0,0,-1,0;x)$$
$$-30H(1,0,0,0;x)+10H(1,0,0,1;x)-10H(1,0,1,0;x)\Big] . \qquad (95)$$

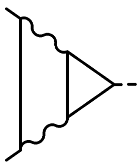
Here, the zeroth order term is in agreement with I_{12} of [19] and the first order in $(D-4)$ is in agreement with F_{10101} of [20]. For carrying out the comparison, it is necessary to account for the difference in normalization and notation; the results are found to agree by multiplying the formulas of [19, 20] by the overall factor

$$\left(\frac{i}{4}\right)^2 \Big(1-\zeta(2)\epsilon^2+\mathcal{O}(\epsilon^3)\Big) , \qquad (96)$$

by replacing their variable y by our x, and finally by recalling $(D-4)=-2\epsilon$.

3.4 2-loop Topologies with 6 denominators

We find only one MI with six propagators, associated to Fig. 2 (a); all the integrals associated to the other six propagator topologies can be expressed in terms of MIs of their subtopologies. The six propagator MI is



$$= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_2\mathcal{D}_3\mathcal{D}_5\mathcal{D}_{10}\mathcal{D}_{20}\mathcal{D}_{22}}$$
$$= \frac{1}{m^4} F^{(0)}_{18}(x) + \mathcal{O}(D-4), \qquad (97)$$

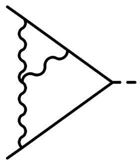
where:

$$F^{(0)}_{18}(x) = \frac{1}{128}\Big[\frac{4}{(1-x)^3} - \frac{6}{(1-x)^2} + \frac{1}{(1-x)} + \frac{1}{(1+x)}\Big]\Big\{\frac{6}{5}\zeta^2(2)$$
$$+2\zeta(3)H(0;x)+4\zeta(2)H(0,0;x)+8H(0,0,-1,0;x)$$
$$+H(0,0,0,0;x)+6H(0,0,0,1;x)+4H(0,1,0,0;x)\Big\} . \qquad (98)$$

4 6-denominator reducible integrals

Besides the 6-denominator MI of the previous Section, there are no other MIs corresponding to the remaining 6-denominator topologies of Fig. 2, *i.e.* it turns out that all the related scalar integrals can be entirely expressed in terms of the MIs of their subtopologies, given in the previous sections.

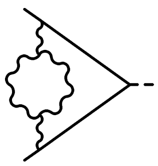
For completeness and for easiness of comparison with the results of other authors, we list now the explicit values of a few reducible 6-denominator integrals (*i.e.* integrals which can be expressed in terms of MIs of their subtopologies).



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{17}\mathcal{D}_{20}\mathcal{D}_{22}} \\
&= \frac{1}{m^4}\sum_{i=-2}^{0}(D-4)^i\, F_{19}^{(i)}(x) + \mathcal{O}(D-4), \qquad (99)
\end{aligned}
$$

where:

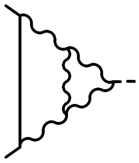
$$
\begin{aligned}
F_{19}^{(-2)}(x) &= \frac{1}{32}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]H(0;x)\,, && (100)\\
F_{19}^{(-1)}(x) &= \frac{1}{16}\Big\{\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]H(0;x)+\Big[\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\Big]H(0,0;x)\Big\}, && (101)\\
F_{19}^{(0)}(x) &= \frac{1}{64}\Big\{2\zeta(3)\Big[\frac{1}{(1-x)}+\frac{1}{(1+x)^2}-\frac{2}{(1+x)}\Big] + \Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]\times \\
&\quad \times[8H(0;x) -2\zeta(2)H(-1;x)+7\zeta(2)H(0;x)-4H(-1,-1,0;x)] \\
&\quad +\Big[\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\Big][8H(0,0;x) - 4H(0,1,0;x)\Big] + \Big[\frac{2}{(1-x)} \\
&\quad -\frac{8}{(1+x)^2} + \frac{6}{(1+x)}\Big][H(-1,0,0;x) + H(0,-1,0;x)] \\
&\quad +\Big[\frac{7}{(1-x)} + \frac{10}{(1+x)^2} - \frac{17}{(1+x)}\Big]H(0,0,0;x)\Big\}\,. && (102)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1^2\mathcal{D}_4\mathcal{D}_6\mathcal{D}_9\mathcal{D}_{17}} \\
&= \frac{1}{m^4}\sum_{i=-1}^{0}(D-4)^i\, F_{20}^{(i)}(x) + \mathcal{O}(D-4), \qquad (103)
\end{aligned}
$$

where:

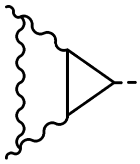
$$
\begin{aligned}
F_{20}^{(-1)}(x) &= \frac{3}{128}\Big\{4\Big[\frac{1}{(1+x)^2} - \frac{1}{(1+x)}\Big] + \Big[\frac{1}{(1-x)} + \frac{4}{(1+x)^3} \\
&\quad -\frac{6}{(1+x)^2} + \frac{1}{(1+x)}\Big]H(0;x)\Big\}\,, && (104)\\
F_{20}^{(0)}(x) &= \frac{3}{128}\Big\{8\Big[\frac{1}{(1+x)^2}-\frac{1}{(1+x)}\Big]+\Big[\frac{1}{(1-x)}+\frac{4}{(1+x)^3}-\frac{6}{(1+x)^2} \\
&\quad +\frac{1}{(1+x)}\Big]\Big[\zeta(2) + 2H(-1,0;x) - H(0,0;x)\Big]\Big\}\,. && (105)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_2\mathcal{D}_3\mathcal{D}_5\mathcal{D}_9\mathcal{D}_{19}\mathcal{D}_{21}} \\
&= \frac{1}{m^4} F_{21}^{(0)}(x) + \mathcal{O}(D-4), \qquad (106)
\end{aligned}
$$

where:

$$
\begin{aligned}
F_{21}^{(0)}(x) \;=\;& \frac{1}{128}\left[\frac{4}{(1-x)^3} - \frac{6}{(1-x)^2} + \frac{1}{(1-x)} + \frac{1}{(1+x)}\right]\Bigg[\frac{56}{5}\zeta^2(2) \\
& +4\zeta(2)H(0,0;x) + H(0,0,0,0;x) + 2H(0,0,0,1;x)\Bigg]. \qquad (107)
\end{aligned}
$$



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1\mathcal{D}_3\mathcal{D}_5\mathcal{D}_{10}\mathcal{D}_{20}\mathcal{D}_{22}} \\
&= \frac{1}{m^4} \sum_{i=-2}^{0} (D-4)^i \, F_{22}^{(i)}(x) + \mathcal{O}(D-4), \qquad (108)
\end{aligned}
$$

where:

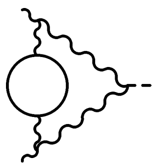
$$
\begin{aligned}
F_{22}^{(-2)}(x) \;=\;& \frac{1}{4(1-x)^2}\left[1 - \frac{2}{(1-x)} + \frac{1}{(1-x)^2}\right] H(0,0;x)\,, \qquad (109) \\
F_{22}^{(-1)}(x) \;=\;& -\frac{1}{8(1-x)^2}\left[1 - \frac{2}{(1-x)} + \frac{1}{(1-x)^2}\right] \Bigg\{ 3\zeta(3) + \zeta(2)H(0;x) \\
& +2\Big[H(0,0,0;x) + H(0,-1,0;x) + H(0,0,1;x) + H(0,1,0;x) \\
& +H(1,0,0;x)\Big]\Bigg\}\,, \qquad (110) \\
F_{22}^{(0)}(x) \;=\;& \frac{1}{16(1-x)^2}\left[1 - \frac{2}{(1-x)} + \frac{1}{(1-x)^2}\right] \Bigg\{ \frac{27}{10}\zeta^2(2) + \zeta(3)\Big[11H(0;x) \\
& +6H(1;x)\Big] + 2\zeta(2)\Big[2H(0,0;x) + H(0,1;x) + H(1,0;x) \\
& -3H(0,-1;x)\Big] + 2\Big[2H(0,0,0,0;x) - 14H(0,-1,-1,0;x) \\
& +7H(0,-1,0,0;x) - 2H(0,-1,0,1;x) + 2H(0,-1,1,0;x) \\
& +9H(0,0,-1,0;x) + 4H(0,0,0,1;x) + 2H(0,0,1,0;x) \\
& +2H(0,0,1,1;x) + 2H(0,1,-1,0;x) + 2H(0,1,0,0;x) \\
& +2H(0,1,0,1;x) + 2H(0,1,1,0;x) + 2H(1,0,-1,0;x) \\
& +2H(1,0,0,0;x) + 2H(1,0,0,1;x) + 2H(1,0,1,0;x) \\
& +2H(1,1,0,0;x)\Big]\Bigg\}\,. \qquad (111)
\end{aligned}
$$

The imaginary part of the Feynman diagram in Fig. 1 (k), relative to the topology in Fig. 2 (e), was firstly evaluated in [21].

The corresponding scalar diagram Eq. (108) was firstly calculated in [22] using the small momentum expansion, then in [19] in terms of binomial sums. The same diagram was also considered in [20], and expressed in terms of generalized Nielsen's polylogarithms and harmonic polylogarithms.

Our expression, given in Eqs. (109–111), agrees with the expression of the amplitude P_{126} given in Eq. (4.11) of [20] (for the comparison see the discussion following the diagram in Eq. (93)) and with the numerical checks provided by the TOPSIDE collaboration [8].

The next scalar diagram is expressed in terms of the variable $\bar{x}$, as in the case of the 4-denominator MIs of Fig. 6 (j) and (k). In agreement with [18], we find:



$$
\begin{aligned}
&= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \frac{1}{\mathcal{D}_1^2 \mathcal{D}_3 \mathcal{D}_5 \mathcal{D}_{10} \mathcal{D}_{18}} \\
&= \frac{1}{m^4} \sum_{i=-2}^{0} (D-4)^i \, F_{23}^{(i)}(\bar{x}) + \mathcal{O}(D-4), \qquad (112)
\end{aligned}
$$

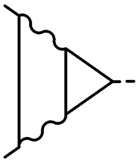
where:

$$
\begin{aligned}
F_{23}^{(-2)}(\bar{x}) &= -\frac{1}{24(1+\bar{x})}\left[1+\frac{11}{(1+\bar{x})}-\frac{24}{(1+\bar{x})^2}+\frac{12}{(1+\bar{x})^3}\right], \qquad (113)\\
F_{23}^{(-1)}(\bar{x}) &= -\frac{1}{4(1+\bar{x})^2}\left[1-\frac{2}{(1+\bar{x})}+\frac{1}{(1+\bar{x})^2}\right]-\frac{1}{24(1+\bar{x})}\left[1+\frac{11}{(1+\bar{x})}\right.\\
&\quad \left.-\frac{24}{(1+\bar{x})^2}+\frac{12}{(1+\bar{x})^3}\right]\Big\{H(0;\bar{x})-2H(-1;\bar{x})\Big\}, \qquad (114)\\
F_{23}^{(0)}(\bar{x}) &= \frac{1}{(1+\bar{x})^2}\left[\frac{3}{16}-\frac{7}{16(1+\bar{x})}+\frac{1}{3(1+\bar{x})^2}-\frac{1}{12(1+\bar{x})^3}\right]\zeta(2)\\
&\quad +\frac{1}{6(1+\bar{x})}\left[\frac{13}{36}-\frac{19}{9(1+\bar{x})}+\frac{7}{2(1+\bar{x})^2}-\frac{7}{4(1+\bar{x})^3}\right]\\
&\quad +\frac{1}{72(1+\bar{x})}\left[2+\frac{1}{(1+\bar{x})}-\frac{6}{(1+\bar{x})^2}+\frac{3}{(1+\bar{x})^3}\right]\Big\{H(0;\bar{x})-2H(-1;\bar{x})\Big\}\\
&\quad +\frac{1}{48(1+\bar{x})}\left[1+\frac{11}{(1+\bar{x})}-\frac{24}{(1+\bar{x})^2}+\frac{12}{(1+\bar{x})^3}\right]\Big\{H(-1,0;\bar{x})\\
&\quad -2H(-1,-1;\bar{x})\Big\}-\frac{1}{48(1+\bar{x})}\left[1+\frac{2}{(1+\bar{x})}-\frac{3}{(1+\bar{x})^2}-\frac{4}{(1+\bar{x})^3}\right.\\
&\quad \left.+\frac{4}{(1+\bar{x})^4}\right]\Big\{H(0,0;\bar{x})-2H(0,-1;\bar{x})\Big\}. \qquad (115)
\end{aligned}
$$

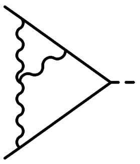
5 Expansion for $Q^2 \gg m^2$

We list, in this Section, the asymptotic expansion of the 6-denominator vertex diagrams given in the previous sections, in order to show their behaviour for momentum transfer larger than the mass.

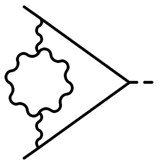
Putting $L = \ln(Q^2/m^2)$ and keeping only the leading term in m^2/Q^2, we find:

m^4 

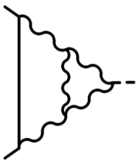
$$\simeq \left(\frac{m^2}{Q^2}\right)^2 \left\{ \frac{3}{40}\zeta^2(2) - \frac{1}{8}\zeta(3)\,L + \frac{1}{8}\zeta(2)\,L^2 + \frac{1}{384}\,L^4 \right\}. \tag{116}$$

m^4 

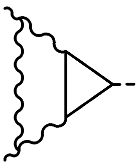
$$\begin{aligned} \simeq & -\frac{1}{(D-4)^2}\left(\frac{m^2}{Q^2}\right)\frac{1}{16}L - \frac{1}{(D-4)}\left(\frac{m^2}{Q^2}\right)\left\{\frac{1}{8}\,L - \frac{1}{32}\,L^2\right\} \\ & + \left(\frac{m^2}{Q^2}\right)\left\{\frac{1}{32}\zeta(3) - \frac{1}{4}\left[1 + \frac{7}{8}\zeta(2)\right]L + \frac{1}{16}\zeta(2)\,L^2 \right. \\ & \left. -\frac{1}{96}\,L^3\right\}. \end{aligned} \tag{117}$$

m^4 

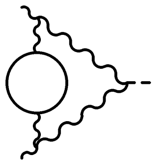
$$\simeq -\frac{1}{(D-4)}\left(\frac{m^2}{Q^2}\right)\frac{3}{32} - \left(\frac{m^2}{Q^2}\right)\frac{3}{16}. \tag{118}$$

m^4 

$$\simeq \left(\frac{m^2}{Q^2}\right)^2 \left\{ \frac{7}{10}\zeta^2(2) + \frac{1}{8}\zeta(2)\,L^2 + \frac{1}{384}\,L^4 \right\}. \tag{119}$$

m^4 

$$\begin{aligned} \simeq & \frac{1}{(D-4)^2}\left(\frac{m^2}{Q^2}\right)^2\frac{1}{8}\,L^2 \\ & -\frac{1}{(D-4)}\left(\frac{m^2}{Q^2}\right)^2\left\{\frac{3}{8}\zeta(3) - \frac{1}{8}\zeta(2)\,L - \frac{1}{24}\,L^3\right\} \\ & + \left(\frac{m^2}{Q^2}\right)^2\left\{\frac{27}{160}\zeta^2(2) - \frac{11}{16}\zeta(3)\,L + \frac{1}{8}\zeta(2)\,L^2 \right. \\ & \left. +\frac{1}{96}\,L^4\right\}. \end{aligned} \tag{120}$$

m^4 

$$\simeq -\frac{1}{(D-4)^2}\left(\frac{m^2}{Q^2}\right)\frac{1}{24} - \frac{1}{(D-4)}\left(\frac{m^2}{Q^2}\right)\frac{1}{48}L + \left(\frac{m^2}{Q^2}\right)\left\{\frac{13}{216} + \frac{1}{48}\zeta(2) - \frac{1}{36}L\right\}. \qquad (121)$$

6 Summary

We presented in this paper the explicit analytic values of all the MIs occurring in the 2-loop QCD corrections to the forward-backward asymmetry of the production of a quark-antiquark pair in e^+e^- annihilation processes. We keep the full dependence on the squared momentum transfer Q^2 and on the mass m of the heavier quark of each diagram. The results are given for space-like Q; the time-like region can be recovered by standard analytic continuation.

Out of the 35 MIs of the problem, 17 were already evaluated in a previous paper. Therefore, in this work we gave only the results concerning the 18 new MIs, which come from the the topologies related to the non-abelian Feynman diagrams and from the diagrams contributing to the axial form factors.

All the integrals are regularized within the continuous D-dimensional regularization scheme, where both UV and IR divergences appear as poles in $(D-4)$.

The method used for the reduction to the MIs is based on the IBPs, LI and general symmetry relations, while the calculation of the MIs was performed by means of the differential equations method. The results were given as a Laurent series expansion around $D=4$ up to the term in $(D-4)$ containing 1-dimensional harmonic polylogarithms with maximum weight $w=4$.

For completeness and easiness of comparison with other results in the literature, we presented also the result for a few 6-denominator scalar integrals, although they are not genuine MIs, but can be expressed in terms of MIs of their subtopologies, giving as well their expansions for large momentum transfer.

7 Acknowledgments

We kindly acknowledge discussions with W. Bernreuter and T. Gehrmann, R. Heinesch and T. Leineweber. We wish to thank the TOPSIDE collaboration, and in particular S. Uccirati, for useful discussions and for the numerical checks of some diagrams. We are grateful to J. Vermaseren for his kind assistance in the use of the algebra manipulating program `FORM` [23], by which all our calculations were carried out. P.M. wishes to thank B. Feucht and A. V. Kotikov for useful discussions. We acknowledge a complete agreement with U. Aglietti for the results of the MIs in Eqs. (74,75) and of the scalar 6-denominator diagram in Eq. (112).

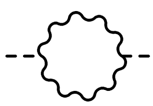
A Propagators

We list here the denominators of the integral expressions appeared in the paper.

$$\begin{aligned}
\mathcal{D}_1 &= k_1^2 \, , & (122)\\
\mathcal{D}_2 &= [k_1^2 + m^2] \, , & (123)\\
\mathcal{D}_3 &= (p_1 - k_1)^2 \, , & (124)\\
\mathcal{D}_4 &= [(p_1 - k_1)^2 + m^2] \, , & (125)\\
\mathcal{D}_5 &= (p_2 + k_1)^2 \, , & (126)\\
\mathcal{D}_6 &= [(p_2 + k_1)^2 + m^2] \, , & (127)\\
\mathcal{D}_7 &= (p_1 + p_2 - k_1)^2 \, , & (128)\\
\mathcal{D}_8 &= [(p_1 + p_2 - k_1)^2 + m^2] \, , & (129)\\
\mathcal{D}_9 &= k_2^2 \, , & (130)\\
\mathcal{D}_{10} &= [k_2^2 + m^2] \, , & (131)\\
\mathcal{D}_{11} &= (p_1 - k_2)^2 \, , & (132)\\
\mathcal{D}_{12} &= [(p_1 - k_2)^2 + m^2] \, , & (133)\\
\mathcal{D}_{13} &= (p_2 + k_2)^2 \, , & (134)\\
\mathcal{D}_{14} &= [(p_2 + k_2)^2 + m^2] \, , & (135)\\
\mathcal{D}_{15} &= (p_1 + p_2 - k_2)^2 \, , & (136)\\
\mathcal{D}_{16} &= [(p_1 + p_2 - k_2)^2 + m^2] \, , & (137)\\
\mathcal{D}_{17} &= (k_1 + k_2)^2 \, , & (138)\\
\mathcal{D}_{18} &= [(k_1 + k_2)^2 + m^2] \, , & (139)\\
\mathcal{D}_{19} &= (p_1 - k_1 - k_2)^2 \, , & (140)\\
\mathcal{D}_{20} &= [(p_1 - k_1 - k_2)^2 + m^2] \, , & (141)\\
\mathcal{D}_{21} &= (p_2 + k_1 + k_2)^2 \, , & (142)\\
\mathcal{D}_{22} &= [(p_2 + k_1 + k_2)^2 + m^2] \, , & (143)\\
\mathcal{D}_{23} &= (p_1 + p_2 - k_1 - k_2)^2 \, , & (144)\\
\mathcal{D}_{24} &= [(p_1 + p_2 - k_1 - k_2)^2 + m^2] \, . & (145)
\end{aligned}$$

B One-loop topologies

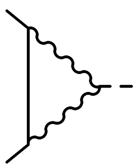
In this Appendix we give the results of the 1-loop scalar integrals entering in the computation of the MIs of Fig. 6 (l), (m), (n), (o) and (p). The tadpole and the 1-loop bubble with two equal masses, which enter as well in the MIs (l) and (n), are given in [7].



$$= \int \mathfrak{D}^D k_1 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_7} = \sum_{i=-1}^{3} (D-4)^i \, G_1^{(i)}(x) + \mathcal{O}\left((D-4)^4\right) \, , \qquad (146)$$

where:

$$
\begin{aligned}
G_1^{(-1)}(x) &= -\frac{1}{2}\,, && (147)\\
G_1^{(0)}(x) &= \frac{1}{2}+\frac{1}{4}H(0;x)+\frac{1}{2}H(1;x)\,, && (148)\\
G_1^{(1)}(x) &= -\frac{1}{2}+\frac{1}{8}\zeta(2)-\frac{1}{4}H(0;x)-\frac{1}{2}H(1;x)-\frac{1}{8}H(0,0;x)-\frac{1}{4}H(0,1;x)\\
&\quad -\frac{1}{4}H(1,0;x)-\frac{1}{2}H(1,1;x)\,, && (149)\\
G_1^{(2)}(x) &= \frac{1}{16}\Big\{8-2(\zeta(2)+\zeta(3))+(4-\zeta(2))[H(0;x)+2H(1;x)]\\
&\quad +2H(0,0;x)+4H(1,0;x)+4H(0,1;x)+8H(1,1;x)+H(0,0,0;x)\\
&\quad +2H(0,0,1;x)+2H(0,1,0;x)+4H(0,1,1;x)+2H(1,0,0;x)\\
&\quad +4H(1,0,1;x)+4H(1,1,0;x)+8H(1,1,1;x)\Big\}\,, && (150)\\
G_1^{(3)}(x) &= \frac{1}{32}\Big\{\frac{9}{10}\zeta^2(2)-[4-\zeta(2)-\zeta(3)][4+2H(0;x)+4H(1;x)]\\
&\quad -(4+\zeta(2))[H(0,0;x)+2H(0,1;x)+2H(1,0;x)+4H(1,1;x)]\\
&\quad -2H(0,0,0;x)-4H(0,0,1;x)-4H(0,1,0;x)-8H(0,1,1;x)\\
&\quad -4H(1,0,0;x)-8H(1,0,1;x)-8H(1,1,0;x)-16H(1,1,1;x)\\
&\quad -H(0,0,0,0;x)-2H(0,0,0,1;x)-2H(0,0,1,0;x)\\
&\quad -4H(0,0,1,1;x)-2H(0,1,0,0;x)-4H(0,1,0,1;x)\\
&\quad -4H(0,1,1,0;x)-8H(0,1,1,1;x)-2H(1,0,0,0;x)\\
&\quad -4H(1,0,0,1;x)-4H(1,0,1,0;x)-8H(1,0,1,1;x)\\
&\quad -4H(1,1,0,0;x)-8H(1,1,0,1;x)-8H(1,1,1,0;x)\\
&\quad -16H(1,1,1,1;x)\Big\}\,. && (151)
\end{aligned}
$$



$$
= \int \mathfrak{D}^D k_1 \, \frac{1}{\mathcal{D}_2\mathcal{D}_3\mathcal{D}_5} = \frac{1}{m^2}\sum_{i=0}^{2}(D-4)^i\, G_2^{(i)}(x) + \mathcal{O}\left((D-4)^3\right)\,, \qquad (152)
$$

where:

$$
\begin{aligned}
G_2^{(0)}(x) &= \frac{1}{8}\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]\Big[4\zeta(2)+H(0,0;x)+2H(0,1;x)\Big]\,, && (153)\\
G_2^{(1)}(x) &= -\frac{1}{16}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\Big\{5\zeta(3)+\zeta(2)[8H(-1;x)-H(0;x)\\
&\quad +8H(1;x)]+H(0,0,0;x)+2[H(-1,0,0;x)+2H(-1,0,1;x)\\
&\quad +H(1,0,0;x)+H(0,1,0;x)+H(0,0,1;x)+2H(1,0,1;x)\\
&\quad +2H(0,1,1;x)]\Big\}, && (154)
\end{aligned}
$$

$$
\begin{aligned}
G_2^{(2)}(x) \quad = \quad & \frac{1}{32}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big\{\frac{59}{10}\zeta^2(2) - \zeta(3)[2H(0;x) - 10H(1;x) \\
& -10H(-1;x)] - \zeta(2)[H(0,0;x) + 2H(0,1;x) + 2H(1,0;x) \\
& -16H(1,1;x) + 2H(-1,0;x) - 16H(-1,1;x) - 16H(1,-1;x) \\
& -16H(-1,-1;x)] + H(0,0,0,0;x) + 4H(-1,-1,0,0;x) \\
& +8H(-1,-1,0,1;x) + 2H(-1,0,0,0;x) + 4H(-1,0,0,1;x) \\
& +4H(-1,0,1,0;x) + 8H(-1,0,1,1;x) + 4H(-1,1,0,0;x) \\
& +8H(-1,1,0,1;x) + 2H(0,0,0,1;x) + 2H(0,0,1,0;x) \\
& +4H(0,0,1,1;x) + 2H(0,1,0,0;x) + 4H(0,1,0,1;x) \\
& +4H(0,1,1,0;x) + 8H(0,1,1,1;x) + 4H(1,-1,0,0;x) \\
& +8H(1,-1,0,1;x) + 2H(1,0,0,0;x) + 4H(1,0,0,1;x) \\
& +4H(1,0,1,0;x) + 8H(1,0,1,1;x) + 4H(1,1,0,0;x) \\
& +8H(1,1,0,1;x)\Big\}. \qquad (155)
\end{aligned}
$$

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