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Planar box diagram for the ($N_F = 1$) 2-loop QED virtual corrections to Bhabha scattering

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Abstract

In this paper we present the master integrals necessary for the analytic calculation of the box diagrams with one electron loop ($N_F = 1$) entering in the 2-loop (α^3) QED virtual corrections to the Bhabha scattering amplitude of the electron. We consider on-shell electrons and positrons of finite mass m, arbitrary squared c.m. energy s, and momentum transfer t; both UV and soft IR divergences are regulated within the continuous D-dimensional regularization scheme. After a brief overview of the method employed in the calculation, we give the results, for s and t in the Euclidean region, in terms of 1- and 2-dimensional harmonic polylogarithms, of maximum weight 3. The corresponding results in the physical region can be recovered by analytical continuation. For completeness, we also provide the analytic expression of the 1-loop scalar box diagram including the first order in $(D-4)$.

Key words: Feynman diagrams, Multi-loop calculations, Box diagrams
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1 Introduction

The Bhabha scattering process plays a key role in the phenomenology of particle physics, since it is employed in order to determine the luminosity of electron-positron colliders. In particular, the small angle Bhabha scattering has been used to measure the luminosity at high energy colliders, such as LEP and SLC. The large angle Bhabha scattering is instead employed in measuring the luminosity of flavor factories (BABAR, BELLE, BEPC/BES, DAΦNE, VEPP-2M).

The accuracy of the luminosity measurements depends on the precision of the theoretical predictions for the Bhabha scattering cross section, since the luminosity is defined as the ratio of the number of events observed and the theoretical cross section for the Bhabha process.

For this reason, in the last three decades, a number of publications have been devoted to the study of the radiative corrections to this process. At the one-loop level, the complete set of radiative corrections to the Bhabha scattering cross section, in the framework of the Standard Model of the electroweak interactions, has been calculated several years ago [1], some results, namely for the factorisable 2-loop subset [2] have been obtained, but a complete and exact evaluation of the 2-loop QED quantum corrections is still missing.

Within the approximation of only keeping contributions enhanced by factors of $\ln(s/m_e^2)$ a large amount of work has already been done. In this approximation the 2-loop corrections to the large angle Bhabha scattering cross-section containing electron-positron pairs were considered in [3]. This calculation includes corrections coming from the interference of the graph in Fig. 1 with the tree level diagrams. The contributions without the pair production, virtual or real, were considered in [4], where, however, the 2-loop box graphs were ignored.

Recently, the full set of 2-loop virtual QED corrections to Bhabha scattering has been calculated, but with the approximation of neglecting the electron mass [5]. The second order logarithmic corrections to the large angle Bhabha scattering cross section, coming from graphs that do not involve vacuum polarization insertions, have been calculated in [6].

The 2-loop 4-point Feynman diagrams are one of the essential ingredients in the calculation of the next-to-next-to-leading order QED corrections to the Bhabha cross section; not surprisingly, their evaluation represents one of the main technical difficulties encountered in calculating these corrections. With the exception of cases presenting specific kinematic configurations, the calculation of 2-loop box diagrams is an open problem. In the recent past, the complete evaluation of the master integrals for the 2-loop 4-point functions, with massless propagators, has been carried out; this has been done both in the case of massless external legs [7, 8, 9, 10], and in the case of three massless and one off-shell external legs [11, 12]. For what concerns 4-point diagrams with massive propagators, as far as the authors know, the only available result is in [13], where the scalar double box integral with four massive and three massless internal lines, and external legs on their mass-shell, is evaluated in terms of Nielsen's polylogarithms (of non-simple arguments and maximum weight 3).

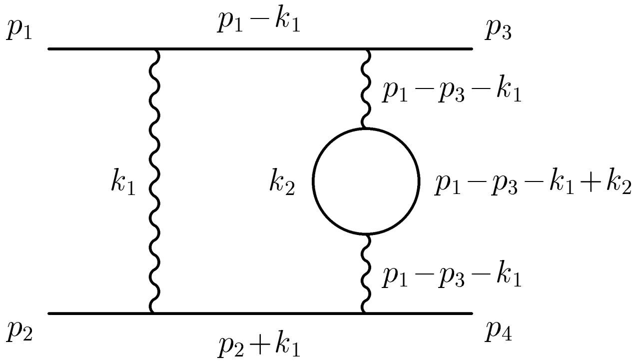


Figure 1: 2-loop box diagram relevant for the purpose of the paper; the momenta p_1 and p_2 are incoming, p_3 and p_4 outgoing.

In the present paper, we evaluate the master integrals (MIs) necessary for the calculation of the box diagrams with one closed electron loop ($N_F = 1$) entering the 2-loop virtual corrections to the electron Bhabha scattering amplitude in QED; the calculation is carried out without neglecting the electron mass m. Performing the calculation without considering the electron massless allows to control the collinear singularities.

The relevant t-channel Feynman diagram is shown on Fig. 1 (the remaining t-channel diagrams and the s-channel diagrams can be recovered by crossing), where we consider the scattering of an incoming electron of momentum p_1 and a positron of momentum p_2, into an outgoing electron and a positron of momenta p_3 and p_4, respectively. All the external legs are on their mass-shell, $p_i^2 = -m^2$; we further define

$$P = p_1 + p_2 \, , \quad Q = p_1 - p_3 \, , \quad s = -P^2 \, , \quad t = -Q^2 \, . \tag{1}$$

We carry out our calculation in the non-physical region $P^2, Q^2 > 0$; the physical region for the Bhabha scattering, $s > 4m^2$, $t < 0$ is to be recovered by analytical continuation.

The interference of this class of diagrams with the tree-level amplitude (which provides the $\mathcal{O}(\alpha^4)$ contribution to the cross-section we are interested in) can be expressed in terms of a large number of 2-loop scalar integrals associated with the considered graphs. Following a by now standard approach, we express all the scalar integrals that appear in the problem as combinations of a small number of independent scalar integrals, the so-called Master Integrals (MIs) of the diagrams under consideration. The reduction procedure that allows to express the generic scalar integral in terms of MIs has been discussed extensively in [8, 14], and it is based on the use of the Integration by Parts Identities (IBPs) [15], the Lorentz Invariance Identities (LI) [8], and the symmetry properties [14] of the scalar integrals encountered in the problem. The analytic calculation of the MIs is then performed by means of the Differential Equations Method [16, 17, 18, 8].

All the integrals considered in this work are Euclidean, regularized within the dimensional regularization scheme [19], in which both UV and IR divergences are regulated by the same parameter D, the (continuous) number of space-time dimensions. The results are given as a Laurent series in $(D-4)$, and the coefficients of these series are expressed in terms of generalized 1- and 2-dimensional harmonic polylogarithms (HPLs) [20, 21, 22, 23, 24], a suitable generalization of the Nielsen's polylogarithms [25, 26, 27, 28]. As we work in the unphysical Euclidean region, where $s \leq 0$, all the integrals are real.

The present paper is structured as follows: in Section 2 we review briefly the procedure that allows to express a scalar integral in terms of the MIs, focusing our attention on the specific case under consideration; in Section 3 we review the method of differential equations for the calculation of the MIs, giving an explicit example in Subsection 3.1, where the solution of the system of differential equations for a 5-denominator four-point function is discussed. In Section 4 we provide the expression of the 6-denominator scalar integral associated to the (unrenormalized) graph of Fig. 1. Appendix A contains the definitions of the propagators in terms of the loop momenta; in Appendix B we give the expression of the 1-loop QED box scalar diagram (two massless and two massive internal lines) up to the first order in $(D-4)$ included. Finally, in Appendix C, we briefly review the formalism of 1- and 2-dimensional harmonic polylogarithms.

The complete expression of the contributions of the two-loop box diagrams to the corrections of $\mathcal{O}(\alpha^3)$ to the Bhabha scattering amplitude will be given elsewhere.

2 The reduction to the Master Integrals

In this section, we give a brief overview of the reduction procedure that has been employed in order to express, in terms of MIs, the scalar integrals involved in the calculation of the α^3 contributions to the Bhabha scattering amplitude. This topic is discussed in greater detail in [8, 14].

At first, we remind the reader that, according to the definition given in [14], there is just one 6-denominator topology (i. e. a graph in which all the propagators are different and all the numerators are equal to 1) related to the non-renormalized diagram of Fig. 1; this topology is shown in Fig. 2.

In the calculation, one also encounters the so-called subtopologies, corresponding to the topologies that can be obtained, from a given topology, by removing one or more propagator lines in all the possible ways. Starting from the topology of Fig. 2 and progressively removing one propagator, one finds the 4 independent 5-denominator topologies of Fig. 3; then the 6 independent 4-denominator topologies of Fig. 4; the 6 independent 3-denominator topologies of Fig. 5 and finally the only 2-denominator non-vanishing topology, the one shown in Fig. 6.

In the graphical representation of subtopologies, internal straight lines correspond to propagators with mass m, internal wavy lines to massless (photon) propagators; external straight lines correspond to on mass-shell particles of mass m, while the momentum carried by external wavy lines is indicated explicitly. Note that the topology of Fig. 2 and the topologies (c), (d) of Fig. 3 depend on both variables P^2 and Q^2, the vertex subtopologies depend only on the square of the momentum written in the corresponding figures, while the subtopologies (e) of Fig. 4, (d), (e),

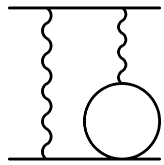
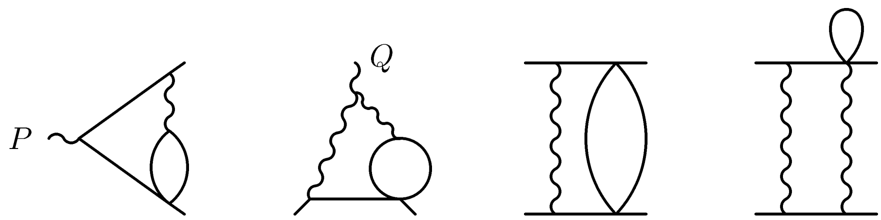


Figure 2: The 6-denominator topology.



(a) (b) (c) (d)

Figure 3: The set of four 5-denominator topologies. The last one is the product of a 1-loop box and a tadpole.

and (f) of Fig. 5, and the one in Fig. 7 are constants (i.e. independent of P^2, Q^2).

The number of the subtopologies of any given topology can be large (as shown by the previous discussion); but different topologies can have common subtopologies, which amounts to say that many subtopologies can be known from independent previous work on graphs of different topologies. That is the case in the present calculation; as will be discussed later in more detail, most of the subtopologies of the current problem were indeed encountered and already worked out in [14] (which deals with vertex topologies).

2.1 The MIs

As it is well known [8, 14], the scalar integrals associated to any topology or subtopology are not all independent, as it is possible to establish several relations that link them among each other. As we will recall shortly in the following, there are (at least) three ways for writing such relations: using the Integration By Parts (IBPs) identities, exploiting the Lorentz structure of the integrals (LI) and relying on the symmetry properties (if any) of the integrals.

We will use the following definition of the loop integration measure in D continuous dimensions,

$$\int \mathfrak{D}^D k = \frac{m^{(4-D)}}{C(D)} \int \frac{d^D k}{(2\pi)^{(D-2)}} \,, \tag{2}$$

(corresponding to the energy scale $\mu_0 = 1$), where $C(D)$ is the following function of

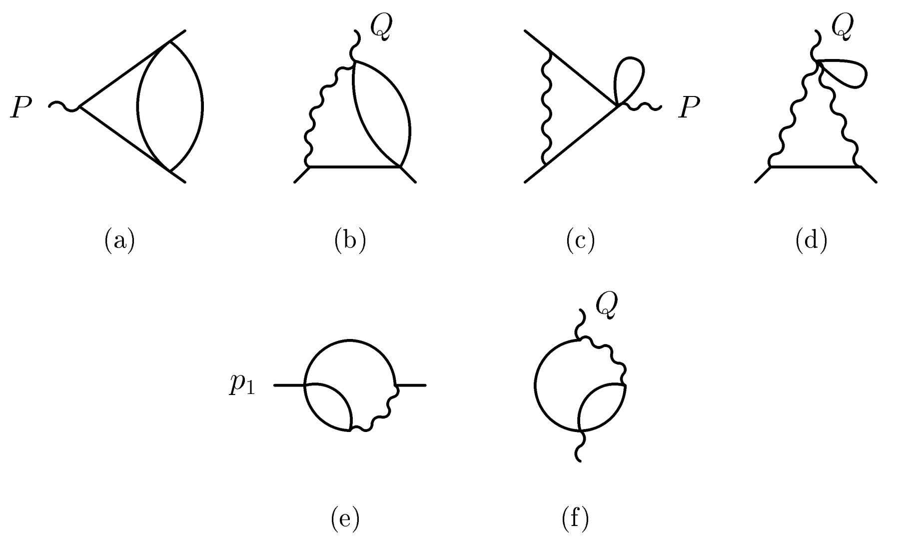


Figure 4: The set of six independent 4-denominator topologies.

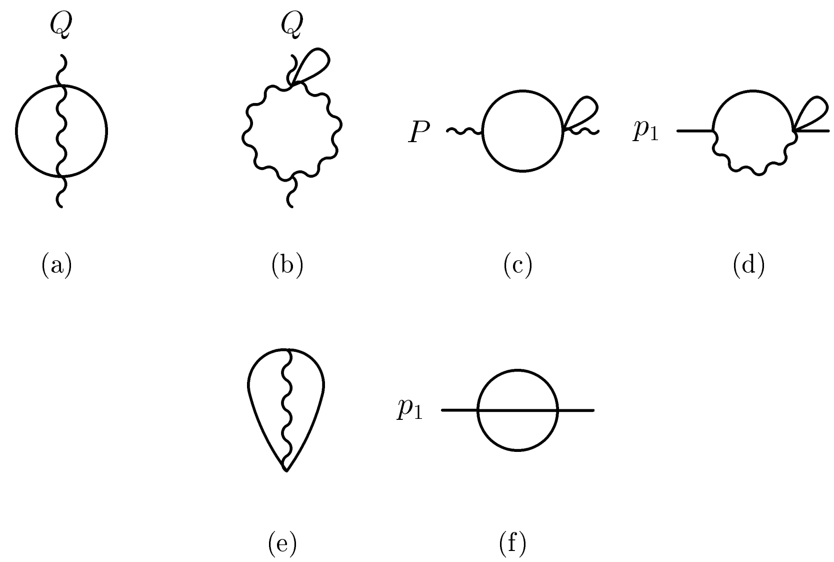


Figure 5: The set of six independent 3-denominator topologies.

Figure 6: The 2-denominator topology product of two 1-loop massive tadpoles.

the space-time dimension D:

$$C(D) = (4\pi)^{\frac{(4-D)}{2}} \Gamma\left(3 - \frac{D}{2}\right) , \tag{3}$$

with the limiting value $C(4) = 1$ at $D = 4$.

With this choice, the 1-loop tadpole with mass m reads

$$\int \mathfrak{D}^D k \, \frac{1}{k^2 + m^2} = \frac{m^2}{(D-2)(D-4)} . \tag{4}$$

The most generic scalar integral associated to any 2-loop topology (or subtopology) can then be written as

$$I(p_i) = I(p_1, p_2, p_3, p_4) \; = \; \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_\tau^{m_\tau}} , \tag{5}$$

where τ represents the number of different denominators $D_1, ..., D_\tau$; the i-th denominator is raised to the integer power m_i, with $m_i \geq 1$. In the numerator, there are q scalar products $S_1, ..., S_q$, which involve one of the independent external momenta and one integration momentum k_j, or two integration momenta. Since in the problem at hand there are 3 independent external momenta and 2 integration momenta, there are 9 possible scalar products depending on the integration momenta; τ of the scalar products can be simplified (or reduced) against the τ propagators, so that the number of the "irreducible" scalar products remaining in the numerator is $q = 9 - \tau$.

Having established the notation, we can describe the relations which hold for the integrals.

- *Integration by Parts Identities.*

  Given any of the integrals defined in Eq. (5), its integrand can be used for writing the following 2 sets of identities [15]:

$$\int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{\partial}{\partial k_1^\mu} v^\mu \left\{ \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_\tau^{m_\tau}} \right\} \;\; = \;\; 0 \, , \tag{6}$$

$$\int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{\partial}{\partial k_2^\mu} v^\mu \left\{ \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_\tau^{m_\tau}} \right\} \;\; = \;\; 0 \, , \tag{7}$$

where the vector v^μ can be any of the 5 independent vectors of the problem: the momenta of integration k_1 and k_2, and the external momenta p_1, p_2 and p_3. In the context of dimensional regularization, every quantity appearing in Eqs. (6,7) is well defined, and the identities are always meaningful. Once the derivative acting on the integrands has been explicitly evaluated, Eqs. (6,7) lead to a set of 10 identities for each initial integrand. These identities involve, as a rule, other integrals associated to the same topology as the initial one, where, at most, one of the powers n_i of the scalar products and one of the powers m_j of the denominators can be one unit larger, while all the other powers remain the same or are decreased by one. In particular, it can happen that one of the propagators, raised to the first power in the original integrand, disappears as a consequence of the algebraic simplification against some reducible scalar product, generated by the differentiation; the resulting integrals are then associated to the subtopology where that propagator is missing.

- *Lorentz Invariance Identities.*

  Another class of identities can be obtained from the fact that the integrals of Eq. (5) are Lorentz scalars [8], and therefore they are invariant under infinitesimal Lorentz transformations of the external momenta $p_i \to p_i + \delta p_i$, with $\delta p_i^\mu = \epsilon^\mu_\nu p_i^\nu$, where the infinitesimal tensor ϵ^μ_ν is antisymmetric but otherwise arbitrary. That gives

$$\sum_n \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) \quad = \quad 0 \, . \tag{8}$$

  With the three independent external momenta of a four-point function, one can build three antisymmetric tensors of rank two; by saturating the above equation with the three tensors, one obtains the three identities

$$\left(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu \right) \sum_n \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) \quad = \quad 0 \, , \tag{9}$$

$$\left(p_1^\mu p_3^\nu - p_1^\nu p_3^\mu \right) \sum_n \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) \quad = \quad 0 \, , \tag{10}$$

$$\left(p_2^\mu p_3^\nu - p_2^\nu p_3^\mu \right) \sum_n \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) \quad = \quad 0 \, . \tag{11}$$

  Within the framework of dimensional regularization, it is possible in Eqs. (9–11) to differentiate directly the integrand of the integral $I(p_i)$, before integration. Eqs. (9–11) then give three identities of a form similar to those obtained by the Integration by Parts method.

- *General Symmetry Relation Identities.*

  Further identities among integrals can arise when the topology has some degree of symmetry. This can happen in cases in which, among the internal lines of a given topology, two or more represent particles of the same mass. In such

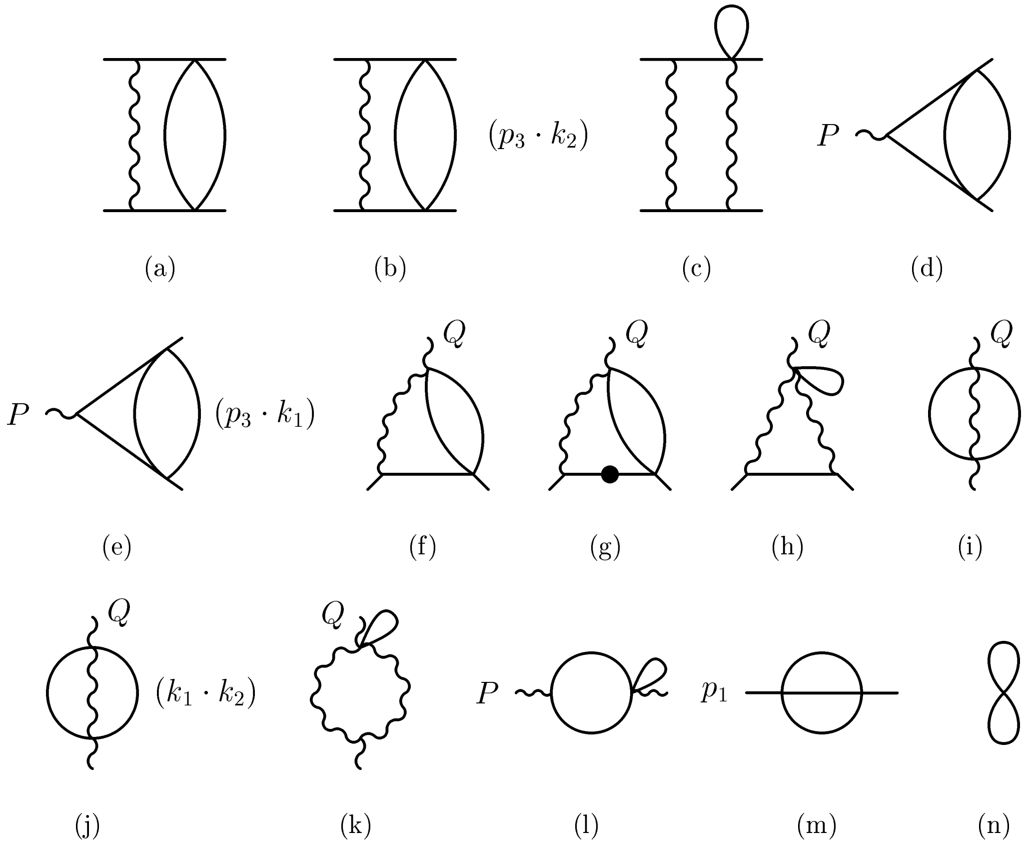


Figure 7: The set of 14 master integrals involved in the calculation of the diagram of Fig. 1.

a case, there can be a transformation of the integration momenta which does not change the value of the integral, but changes the form of the integrand. By imposing the identity of the initial integral with the combination of integrals resulting from the transformation of the integration momenta, one can obtain additional identities relating integrals associated with the given topology and its subtopologies. An explicit example, involving the topology (b) in Fig. 4, is discussed in Section 2.0.3 of [14].

For each topology, one can systematically write the above described identities starting from the integrand with all the powers n_i of the scalar products equal to zero and all the powers of the denominators m_j equal to one, then in the case of all the integrands with $N = \sum_i n_i = 1$ and $M = \sum_j (m_j - 1) = 0$ (with $m_j > 0$), then for all the integrands having $N = 0$ and $M = 1$ (and always $m_j > 0$), then for all the integrands with $N = 1$ and $M = 1$ and so on. One finds that the number of the equations grows faster than the number of the involved integrals, until one obtains an apparently over-constrained set of linear equations for the integrals themselves. (For a more detailed discussion see for instance [8]). The problem of solving such a linear system (whose coefficients are polynomials in D, the masses and the external Mandelstam variables) is in principle trivial, but, due to the size of the system, it can be very demanding from the algebraic point of view (technical details on the computer programs developed in order to solve the system of linear equations are discussed in [14]). As a result, one can identify a small number of so-called Master Integrals (MIs) for the considered problem, such that all the other integrals appearing in the considered identities are expressed as linear combinations of those MIs, with coefficients which are ratios of polynomials in D, masses and Mandelstam variables. It may also happen that all the integrals associated to a given topology can be expressed entirely in terms of the MIs of its subtopologies.

For the purpose of this paper it is sufficient to say that all the scalar integrals that are necessary for the calculation of the $\mathcal{O}(\alpha^3)$ contributions to the Bhabha scattering amplitude from the considered Feynman graphs can be expressed in terms of 14 independent MIs only. There is some freedom in the choice of the integrals to be promoted to the role of MIs of the problem; we choose the set of MIs which are shown in Fig. 7 as "decorated graphs". Each of the decorated graphs of Fig. 7 stands for a specific Master Integral; the denominator of the integrand is read from the lines (each line corresponding to a propagator raised to the first power, a line with a dot indicating that the corresponding propagator is squared), the numerator is given by the "decoration" ($(p_3 \cdot k_2)$, $(p_3 \cdot k_1)$, etc. for graphs (b), (e), etc., or simply 1 when there are no other decorations).

As a first remark, in Fig. 7 there is no 6-denominator MI: that means that all the scalar integrals associated to the 6-denominator topology of graph (c) in Fig. 2 can be expressed in terms of the MIs of its subtopologies with 5 or less denominators. Among the MIs of Fig. 7, the MIs (d)–(n) have already been calculated in [14]. As a consequence, in the present paper we focus our attention on graphs (a) and (b) of Fig. 7. Concerning graph (c) of Fig. 7, it is the product of a massive tadpole, Eq. (4), and the 1-loop box graph. As the tadpole is singular as $1/(D-4)$, one needs the 1-loop graph up to the first order in $(D-4)$ included; its complete calculation is provided in Appendix B. Let us observe that diagrams (a), (b), and (c) are the only MIs of the problem that depend on both the independent Mandelstam variables $s = -P^2$ and $t = -Q^2$ of Eq. (1) (as well as on the electron squared mass m^2), while all the other graphs of Fig. 7 are functions of either P^2 or Q^2 (and m^2) only.

3 The differential equations method

The calculation of the MIs is performed by means of the differential equation method [16, 17, 18, 8]. In this section, the main features of the method are recalled.

Each of the MIs depends, in general, on all the independent Mandelstam variables of the problem, which we indicate by $s_i = -p_i^2$, $(i = 1, 2, 3, 4)$, $s_5 = -(p_1 - p_3)^2 = -Q^2$, and $s_6 = -(p_1 + p_2)^2 = -P^2$ (the invariants p_1^2, p_2^2, p_3^2, and p_4^2 are constrained to be on the mass shell, $p_i^2 = -m^2$, but that plays no role here).

When acting on any function of the Mandelstam variables, say $M(s_r)$, one has

$$p_j^\mu \frac{\partial}{\partial p_k^\mu} M(s_r) = p_j^\mu \sum_{\xi=1}^{6} \frac{\partial s_\xi}{\partial p_k^\mu} \frac{\partial}{\partial s_\xi} M(s_r) \,, \tag{12}$$

where $j,k = 1,2,3$, while $r = 1,\cdots,6$. The $\partial s_\xi/\partial p_k^\mu$ are linear in the external momenta, so that the factors $p_j^\mu \partial s_\xi/\partial p_k^\mu$ in the Eqs. (12) are linear in the Mandelstam variables. Eqs. (12) can then be solved by expressing the $\partial/\partial s_\xi$ in terms of the $p_j^\mu \partial/\partial p_k^\mu$.

One obtains in this way

$$\begin{aligned}
P^2 \frac{\partial}{\partial P^2} M(s_r) \;=\; & \left[\frac{1}{2} \frac{Q^2+4m^2}{P^2+Q^2+4m^2} \left(p_1^\mu \frac{\partial}{\partial p_1^\mu} - p_3^\mu \frac{\partial}{\partial p_3^\mu} \right) \right. \\
& + \frac{1}{2} \left(1 + \frac{P^2}{P^2+Q^2+4m^2} \right) p_2^\mu \frac{\partial}{\partial p_2^\mu} \\
& \left. + \frac{m^2}{P^2+Q^2+4m^2} (p_1^\mu + p_3^\mu) \left(\frac{\partial}{\partial p_3^\mu} - \frac{\partial}{\partial p_1^\mu} + \frac{\partial}{\partial p_2^\mu} \right) \right] M(s_r), \qquad (13) \\
Q^2 \frac{\partial}{\partial Q^2} M(s_r) \;=\; & \left[\frac{1}{2} \frac{P^2+4m^2}{P^2+Q^2+4m^2} \left(p_1^\mu \frac{\partial}{\partial p_1^\mu} - p_2^\mu \frac{\partial}{\partial p_2^\mu} \right) \right. \\
& + \frac{1}{2} \left(1 + \frac{Q^2}{P^2+Q^2+4m^2} \right) p_2^\mu \frac{\partial}{\partial p_2^\mu} \\
& \left. + \frac{m^2}{P^2+Q^2+4m^2} (p_2^\mu - p_1^\mu) \left(\frac{\partial}{\partial p_1^\mu} + \frac{\partial}{\partial p_2^\mu} + \frac{\partial}{\partial p_3^\mu} \right) \right] M(s_r). \qquad (14)
\end{aligned}$$

We take one of the above equations, say Eq. (13) for definiteness, and we replace the generic function $M(s_r)$ by any of the MIs, say $M_i(s_r)$. The l.h.s. is nothing but $P^2(\partial M_i(s_r)/\partial P^2)$; in the r.h.s., we write $M_i(s_r)$ in its representation as an integral over the loop internal momenta, and we carry out the derivatives with respect to the momenta p_j^μ in the integrand. In this way, we obtain a combination of scalar integrals associated to the same topology as $M_i(s_r)$, which, according to the previous discussion and the very definition of the MIs, can be expressed in terms of the MIs themselves. The result is a set of first order linear differential equations, of the form

$$\begin{aligned}
P^2 \frac{\partial}{\partial P^2} M_i(D,m^2,P^2,Q^2) \;=\; & \sum_j A_{ij}^{(1)}(D,m^2,P^2,Q^2) M_j(D,m^2,P^2,Q^2) \\
& + \sum_k B_{ik}^{(1)}(D,m^2,P^2,Q^2) N_k(D,m^2,P^2,Q^2), \qquad (15) \\
Q^2 \frac{\partial}{\partial Q^2} M_i(D,m^2,P^2,Q^2) \;=\; & \sum_j A_{ij}^{(2)}(D,m^2,P^2,Q^2) M_j(D,m^2,P^2,Q^2) \\
& + \sum_k B_{ik}^{(2)}(D,m^2,P^2,Q^2) N_k(D,m^2,P^2,Q^2), \qquad (16)
\end{aligned}$$

where $M_j(D,m^2,P^2,Q^2)$ are the MIs of the topology under consideration, while $N_k(D,m^2,P^2,Q^2)$ represent the MIs of the sub-topologies; finally, the coefficients

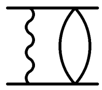
$A_{ij}^{(l)}(D,m^2,P^2,Q^2)$ and $B_{ik}^{(l)}(D,m^2,P^2,Q^2)$ are ratios of polynomials in D,P^2,Q^2 and m^2. Note that the partial derivatives with respect to P^2 and Q^2 never mix within a same equation, so that the equations are in fact differential equations in a single variable. The two sets of differential equations in P^2 and Q^2 are therefore somehow redundant, and the redundancy can be used in the calculations as an *a posteriori* check.

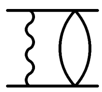
With some additional qualitative information on the MIs, the equations can also be exploited in order to fix the boundary conditions for the solution of the differential equations. We know that the considered box amplitudes are regular at $P^2 = 0$; therefore, by setting $P^2 = 0$ in Eq. (16) (for arbitrary Q^2) the l.h.s. vanishes, while the r.h.s. gives a relation between the values of the considered $M_i(s_r)$ at $P^2 = 0$ and the values, at that same point, of the MIs of the subtopologies (which, in general, are simpler to obtain and supposedly known from previous calculations).

While, in principle, the system of differential equations can be studied for arbitrary values of the parameter D, we restrict our interest to the Laurent expansion of the MIs in powers of $(D-4)$. Therefore, we expand systematically in $(D-4)$ each of the MIs and of the coefficients that appear in Eqs. (15,16), and solve the system directly for the Laurent coefficients of $M_i(s_r)$. An explicit example of the procedure for the solution of the equations and the evaluation of the MIs is given in the following Section.

3.1 The calculation of the 5-denominator box MIs

The topology of Fig. 3 (c) is one of the topologies that present MIs which have not been already considered in [14] (the second topology is the product of the 1-loop box, given in Appendix B, and the tadpole). The two scalar integrals which have been chosen as the MIs for this topology are shown in Fig. 7 (a) and (b); according to the previous discussion, their explicit forms as loop integrals are

$F_1(D,m^2,P^2,Q^2) =$  $= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1\mathcal{D}_3\mathcal{D}_4\mathcal{D}_5\mathcal{D}_6}\,, \qquad (17)$

$F_2(D,m^2,P^2,Q^2) =$  $(p_3\cdot k_2) = \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{p_3\cdot k_2}{\mathcal{D}_1\mathcal{D}_3\mathcal{D}_4\mathcal{D}_5\mathcal{D}_6}\,. \qquad (18)$

By following the procedure outlined in the previous paragraphs, and dropping for ease of notation the arguments on which the two master integrals depend, one finds that the first order linear differential equations for F_1, F_2 in the variable P^2 can be written as

$$\frac{\partial F_1}{\partial P^2} = -\frac{1}{2}\left[\frac{1}{P^2} - \frac{D-5}{P^2+4m^2} + \frac{D-4}{P^2+Q^2+4m^2}\right] F_1 + \Omega_1(D,m^2,P^2,Q^2), \qquad (19)$$

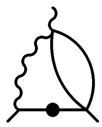
$$\frac{\partial F_2}{\partial P^2} = -\frac{1}{2}\left[\frac{1}{P^2} - \frac{D-5}{P^2+4m^2} + \frac{D-4}{P^2+Q^2+4m^2}\right] F_2 + \Omega_2(D,m^2,P^2,Q^2), \qquad (20)$$

where $\Omega_1(D,m^2,P^2,Q^2)$ and $\Omega_2(D,m^2,P^2,Q^2)$ stand for lengthy combinations (not listed here for brevity) of the MIs corresponding to the graphs (c)–(n) of Fig. 7, with coefficients given by ratios of polynomials in $D, P^2, Q^2\ m^2$.

The two equations of the system, Eqs. (19,20), are completely decoupled, even for arbitrary value of D. The task of solving the system is then reduced to the integration of two independent first-order linear differential equations. The associated homogeneous equation (which plays a key role in the solution of the equations) is exactly the same for both the MIs, a fact which further simplifies the explicit calculations.

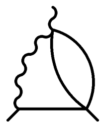
As already observed, the initial conditions are easily obtained by the equations themselves by imposing the analyticity of the solutions at $P^2 = 0$; in the case of Eqs. (19,20), we can multiply both sides by P^2 and then take the $P^2 = 0$ limit. The l.h.s. vanishes, while in the r.h.s., as the known terms Ω_1, Ω_2 possess also a polar singularity $1/P^2$, we are left with $(-1/2)$ times the values of the F_i at $P^2 = 0$ and the residua of the singularities of the Ω_i; the explicit calculation gives

$$F_1(D, m^2, P^2 = 0, Q^2) \; = $$

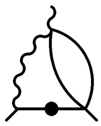


$$, \qquad (21)$$

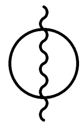
$$F_2(D, m^2, P^2 = 0, Q^2) \; = \; -\left\{ \frac{1}{4} - \frac{(D-4)}{8} \frac{Q^2}{m^2} \right\}$$



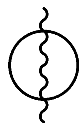
$$+\frac{1}{4} Q^2$$



$$+\left\{ \frac{(5D-12)}{8} \frac{1}{m^2} - \frac{(3D-7)}{2} \frac{1}{(Q^2+4m^2)} \right\}$$

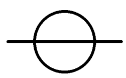


$$+\frac{3(D-2)}{2} \frac{1}{m^2(Q^2+4m^2)}$$



$$(k_1 \cdot k_2)$$

$$-\frac{(3D-8)}{32} \frac{1}{m^2}$$

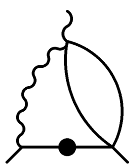


$$+\left\{ \frac{3(D-2)^2}{64(D-3)} \frac{1}{m^4} - (D-2) \frac{1}{m^2(Q^2+4m^2)} \right\}$$

$$. \qquad (22)$$

Let us observe that, as a check, one may obtain the two quantities $F_i(D, m^2, P^2 = 0, Q^2)$ by setting $p_1 = -p_2$ (which implies $P^2 = 0$) directly in the definitions Eqs. (19,20) of the two MIs. That leads to the following relations:

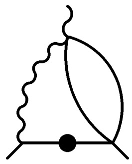
$$F_1(D, m^2, P^2 = 0, Q^2) =$$



$$= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_3 \mathcal{D}_5^2 \mathcal{D}_6} \, , \qquad (23)$$

which is exactly Eq. (21), and

$$F_2(D, m^2, P^2 = 0, Q^2) =$$



$$(p_3 \cdot k_2) \quad = \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{(p_3 \cdot k_2)}{\mathcal{D}_1 \mathcal{D}_3 \mathcal{D}_5^2 \mathcal{D}_6} \, ; \quad (24)$$

where a dot on a propagator indicates that the corresponding denominator in the integrand is raised to the 2nd power. By expressing the above 4-denominator integral in terms of the MIs of Fig. 7, Eq. (22) is recovered.

As pointed out in the previous section, we are interested in the Laurent expansion of the MIs F_1 and F_2 with respect to $(D-4)$; it is known that they have at most double poles in $(D-4)$, so that

$$F_1(D, m^2, P^2, Q^2) \;=\; \sum_{k=-2}^{0} (D-4)^k F_1^{(k)}(m^2, P^2, Q^2) + \mathcal{O}(D-4) \, , \qquad (25)$$

$$F_2(D, m^2, P^2, Q^2) \;=\; \sum_{k=-2}^{0} (D-4)^k F_2^{(k)}(m^2, P^2, Q^2) + \mathcal{O}(D-4) \, . \qquad (26)$$

By expanding in the same way also the inhomogeneous (known) terms, Eqs. (19,20) generate a set of nested equations for the coefficients $F_i^{(k)}(m^2, P^2, Q^2)$ of the expansion in $(D-4)$:

$$\frac{\partial F_i^{(k)}(m^2, P^2, Q^2)}{\partial P^2} = -\frac{1}{2}\left[\frac{1}{P^2} - \frac{1}{P^2 + 4m^2}\right] F_i^{(k)}(m^2, P^2, Q^2) + \Psi_i^{(k)}(m^2, P^2, Q^2). \qquad (27)$$

where, due to D-dependence of the homogeneous part of Eqs. (19,20),

$$\begin{aligned} \Psi_i^{(k)}(m^2, P^2, Q^2) \;&=\; \frac{1}{2}\left(\frac{1}{P^2 + 4m^2} - \frac{1}{P^2 + Q^2 + 4m^2}\right) F_i^{(k-1)}(m^2, P^2, Q^2) \\ &+\; \Omega_i^{(k)}(m^2, P^2, Q^2) \, , \end{aligned}$$

with $\Omega_i^{(k)}(m^2, P^2, Q^2)$ equal to the coefficient of order k in the Laurent-expansion of $\Omega_i(D, m^2, P^2, Q^2)$ in powers of $(D-4)$; note that the complete inhomogeneous part of Eq. (27) contains (for $k > -2$) also terms in $F_1^{(k-1)}(m^2, P^2, Q^2)$.

The solution of the differential Eqs. (19,20), once written in the expanded form of Eq. (27), is built, order by order in $(D-4)$, by repeatedly using Euler's method of the variation of the constants, see Eq. (30) below.

Euler's method requires the knowledge of the solution of the associated homogeneous equation, which is the same for any order k of the expansion in $(D-4)$; in the case at hand it reads

$$\frac{\partial f(r)}{\partial r} = -\frac{1}{2}\left[\frac{1}{r} + \frac{1}{(r + 4m^2)}\right] f(r) \, . \qquad (28)$$

whose solution, up to an irrelevant multiplicative constant, is

$$f(r) = \frac{1}{\sqrt{r(r+4m^2)}} \, . \tag{29}$$

The Euler's method then gives the solution of Eq. (27) in the form

$$F_i^{(k)}(m^2, P^2, Q^2) = \frac{1}{\sqrt{P^2(P^2+4m^2)}} \left\{ K_i^{(k)} + \int^{P^2} dr \sqrt{r(r+4m^2)} \Psi_i^{(k)}(m^2, r, Q^2) \right\} \; ; \tag{30}$$

the $K_i^{(k)}$ are the integration constants, which are fixed by imposing the initial conditions Eqs. (21,22) at $P^2 = 0$.

As we are interested in the expansion up to the finite part in $(D-4)$ and the expansion starts from $1/(D-4)^2$, we need the first three terms of the expansion, i.e. we have to use repeatedly Eq. (30) for $k = -2, -1, 0$. It is actually convenient to replace the Mandelstam variables P^2 and Q^2, by the dimensionless quantities x and y, defined as

$$x = \frac{\sqrt{P^2+4m^2} - \sqrt{P^2}}{\sqrt{P^2+4m^2} + \sqrt{P^2}} \, , \qquad y = \frac{\sqrt{Q^2+4m^2} - \sqrt{Q^2}}{\sqrt{Q^2+4m^2} + \sqrt{Q^2}} \, , \tag{31}$$

and to introduce the functions

$$F_i^{(k)}(x, y) = F_i^{(k)}(m^2, P^2, Q^2) \; . \tag{32}$$

The result can then be expressed in terms of 1– and 2–dimensional HPLs of argument x and y and maximum weight 3; omitting for simplicity the dependence of the functions $F_i^{(k)}(x, y)$ on their arguments, and following the notation of Appendix C, we have

$$m^2 F_1^{(-2)} = \frac{1}{8}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0;x) \, , \tag{33}$$

$$\begin{aligned} m^2 F_1^{(-1)} = {} & \frac{1}{16}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \left\{ \zeta(2) - \left[2 - \left(1 - \frac{2}{(1-y)}\right) H(0;y)\right] H(0;x) \right. \\ & \left. - H(0,0;x) + 2H(-1,0;x) \right\} \, , \end{aligned} \tag{34}$$

$$\begin{aligned} m^2 F_1^{(0)} = {} & -\frac{1}{16}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \Big\{ \zeta(2) + \zeta(3) - 2H(0;x) - \zeta(2)H(-1;x) \\ & - H(0,0;x) + 2H(-1,0;x) - H(0,0,0;x) + H(-1,0,0;x) \\ & + H(0,0;y)\, H(0;x) - 2H(-1,-1,0;x) + H(0,-1,0;x) \\ & + H(0,0;y)H(0;x) - \frac{1}{2}\left[1 - \frac{2}{(1-y)}\right] \Big[4\zeta(2)H(0;y) - H(0,0,0;y) \\ & + \Big(\zeta(2) - 2H(0;y) - 4H(0,0;y) - 2H(1,0;y) + 6H(-1,0;y)\Big) H(0;x) \\ & + \Big(3\zeta(2) + H(0,0;y)\Big)\Big(G(-y;x) - G(-1/y;x)\Big) + 2H(0;y)H(-1,0;x) \end{aligned}$$

$$
\begin{aligned}
& -H(0;y)\Big(G(-y,0;x)+G(-1/y,0;x)\Big)+G(-y,0,0;x) \\
& -G(-1/y,0,0;x)\Big]\Bigg\}\,, \qquad (35)\\
F_2^{(-2)} &= \frac{1}{32}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\left[\frac{1}{y}-2+y\right]H(0;x)\,, \qquad (36)\\
F_2^{(-1)} &= \frac{1}{64}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\Bigg\{\left[\frac{1}{y}-2+y\right]\Big[\zeta(2)-4H(0;x)-H(0,0;x) \\
& +2H(-1,0;x)\Big]-\left[\frac{1}{y}-y\right]H(0;y)H(0;x)\Bigg\} \\
& +\frac{1}{32}\left[1-\frac{2}{(1+x)}\right]H(0;x), \qquad (37)\\
F_2^{(0)} &= \frac{1}{64}\Bigg\{2\left[2-\frac{1}{(1-x)}\right]\zeta(2)-2\left[\frac{1}{(1-y)}-\frac{1}{(1+y)}\right]\Big[\zeta(2)H(0;y) \\
& +H(0,0,0;y)\Big]+H(0,0;y)-5H(0;x)+2H(-1,0;x) \\
& +\frac{2}{(1-x)}\Big[5H(0;x)+H(0,0;x)-2H(-1,0;x)\Big]+\Bigg[\frac{1}{(1-x)} \\
& -\frac{1}{(1+x)}\Bigg]\Big[(\zeta(2)-2H(0,0;y))H(0;x)+H(0,0,0;x)\Big]\Bigg\} \\
& -\frac{1}{128}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\Bigg\{\left[\frac{1}{y}-2+y\right]\Big[4\zeta(2)+2\zeta(3)-2\zeta(2)H(-1;x) \\
& -(14+H(0,0,0;y))H(0;x)-4H(0,0;x)+8H(-1,0;x) \\
& -2H(0,0,0;x)+2H(-1,0,0;x)+2H(0,-1,0;x)-4H(-1,-1,0;x)\Big] \\
& +\left[\frac{1}{y}-y\right]\Big[4\zeta(2)H(0;y)-H(0,0,0;y)-2(2H(0;y)-\frac{1}{2}\zeta(2) \\
& +H(0,0;y)+H(1,0;y)-3H(-1,0;y))H(0;x) \\
& +(3\zeta(2)+H(0,0;y))(G(-y;x)-G(-1/y;x))+2H(0;y)H(-1,0;x) \\
& -H(0;y)G(-y,0;x)-H(0;y)G(-1/y,0;x)+G(-y,0,0;x) \\
& -G(-1/y,0,0;x)\Big]\Bigg\}\,. \qquad (38)
\end{aligned}
$$

As a check of the results reported above, the expressions for the MIs $F_1^{(k)}(x,y)$ and $F_2^{(k)}(x,y)$ have been inserted in the corresponding differential equations with respect to the variable Q^2 (Eq. 16); those equations were found to be satisfied.

3.2 Asymptotic expansions

In this Subsection we provide the asymptotic expansions, in various kinematic regions, of the coefficients $F_1^{(k)}$ and $F_2^{(k)}$ ($k=-2,-1,0$) introduced above.

The first region of interest is the one in which $P^2 \sim Q^2 \gg m^2$, $(x \sim y \ll 1)$, relevant for the large angle Bhabha scattering. Employing the definitions

$$L_r = \ln\left(\frac{m^2}{P^2}\right), \quad L_w = \ln\left(\frac{m^2}{Q^2}\right), \tag{39}$$

and keeping only the leading terms, we find

$$\begin{aligned}
m^2 F_1^{(-2)} &= \frac{1}{4}\frac{m^2}{P^2} L_r\,, &(40)\\
m^2 F_1^{(-1)} &= \frac{1}{16}\frac{m^2}{P^2}\left[2\zeta(2) - 4L_r - L_r^2 - 2L_rL_w\right], &(41)\\
m^2 F_1^{(0)} &= -\frac{1}{96}\frac{m^2}{P^2}\Bigg[12\zeta(2) + 12\zeta(3) + 18\zeta(2)\ln\left(1+\frac{Q^2}{P^2}\right) + 6\zeta(2)L_r - 24L_r\\
&\quad -6L_r^2 + 3L_r^2\ln\left(1+\frac{Q^2}{P^2}\right) - 2L_r^3 - 12L_rL_w - 6L_rL_w\ln\left(1+\frac{Q^2}{P^2}\right)\\
&\quad -6L_rL_w^2 + 6L_rLi_2\left(-\frac{Q^2}{P^2}\right) + 24\zeta(2)L_w + 3L_w^2\ln\left(1+\frac{Q^2}{P^2}\right) + L_w^3\\
&\quad -6L_wLi_2\left(-\frac{Q^2}{P^2}\right) + 6Li_3\left(-\frac{Q^2}{P^2}\right)\Bigg]; &(42)\\
F_2^{(-2)} &= \frac{1}{16}\frac{Q^2}{P^2}L_r\,, &(43)\\
F_2^{(-1)} &= -\frac{1}{32}L_r + \frac{1}{64}\frac{Q^2}{P^2}\Big[2\zeta(2) - 8L_r - L_r^2 - 2L_rL_w\Big], &(44)\\
F_2^{(0)} &= \frac{1}{128}\Big[4\zeta(2) + 10L_r + 2L_r^2 + L_w^2\Big] - \frac{1}{128}\frac{Q^2}{P^2}\Bigg[8\zeta(2) + 4\zeta(3)\\
&\quad +6\zeta(2)\ln\left(1+\frac{Q^2}{P^2}\right) - 14L_r + 2\zeta(2)L_r - 4L_r^2 + L_r^2\ln\left(1+\frac{Q^2}{P^2}\right)\\
&\quad -\frac{2}{3}L_r^3 - 8L_rL_w - 2L_rL_w\ln\left(1+\frac{Q^2}{P^2}\right) - 2L_rL_w^2 + 2L_rLi_2\left(-\frac{Q^2}{P^2}\right)\\
&\quad +8\zeta(2)L_w + L_w^2\ln\left(1+\frac{Q^2}{P^2}\right) + \frac{1}{3}L_w^3 - 2L_wLi_2\left(-\frac{Q^2}{P^2}\right)\\
&\quad +2Li_3\left(-\frac{Q^2}{P^2}\right)\Bigg]. &(45)
\end{aligned}$$

The case $P^2 \gg Q^2 \gg m^2$ can be immediately obtained from the previous equations.

The second region of interest is the one in which again $P^2 \gg m^2$, while Q^2 is much smaller than P^2 ($Q^2 \ll P^2$), but otherwise arbitrary, so that $x \ll y \leq 1$. Keeping only the leading terms we find:

$$\begin{aligned}
m^2 F_1^{(-2)} &= \frac{1}{4}\frac{m^2}{P^2}L_r\,, &(46)\\
m^2 F_1^{(-1)} &= \frac{1}{16}\frac{m^2}{P^2}\left\{2\zeta(2) - 2\left[2 + \sqrt{1+\frac{4m^2}{Q^2}}H(0;y)\right]L_r - L_r^2\right\}, &(47)
\end{aligned}$$

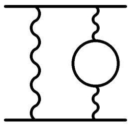
$$
\begin{aligned}
m^2 F_1^{(0)} &= -\frac{1}{16}\frac{m^2}{P^2}\Bigg\{2\zeta(2)+2\zeta(3)+\sqrt{1+\frac{4m^2}{Q^2}}\Big[4\zeta(2)H(0;y)+H(0,0,0;y)\Big] \\
&\quad +\Bigg[4-2H(0,0;y)-\sqrt{1+\frac{4m^2}{Q^2}}\Big(\zeta(2)-2H(0;y)-4H(0,0;y) \\
&\quad +6H(-1,0;y)-2H(1,0;y)\Big)\Bigg]L_r+L_r^2+\frac{1}{3}L_r^3\Bigg\}\,, \qquad (48) \\
F_2^{(-2)} &= \frac{1}{16}\frac{Q^2}{P^2}L_r \qquad (49) \\
F_2^{(-1)} &= -\frac{1}{32}L_r+\frac{1}{64}\frac{Q^2}{P^2}\Bigg\{4(1-L_r)+\frac{Q^2}{m^2}\Big[2\zeta(2)-8L_r-L_r^2\Big] \\
&\quad -4\sqrt{\frac{Q^2}{4m^2}\left(1+\frac{Q^2}{4m^2}\right)}H(0;y)\,L_r\Bigg\}\,, \qquad (50) \\
F_2^{(0)} &= \frac{1}{64}\Bigg\{2\zeta(2)+H(0,0;y)+\frac{2}{\sqrt{\frac{Q^2}{4m^2}\left(1+\frac{Q^2}{4m^2}\right)}}\Big[\zeta(2)H(0;y)+H(0,0,0;y)\Big] \\
&\quad +5L_r+L_r^2\Bigg\}-\frac{1}{32}\frac{m^2}{P^2}\Bigg\{4+\zeta(2)-\Big[2+\zeta(2)-2H(0,0;y)\Big]L_r-\frac{1}{2}L_r^2 \\
&\quad -\frac{1}{6}L_r^3+\frac{Q^2}{m^2}\Big[2\zeta(2)+\zeta(3)-(7-H(0,0;y))L_r-L_r^2-\frac{1}{6}L_r^3\Big] \\
&\quad +\sqrt{\frac{Q^2}{4m^2}\left(1+\frac{Q^2}{4m^2}\right)}\Big[8\zeta(2)H(0;y)+2H(0,0,0;y)+\Big(2\zeta(2)-8H(0;y) \\
&\quad -8H(0,0;y)-4H(1,0;y)+12H(-1,0;y)\Big)L_r\Big]\Bigg\}\,. \qquad (51)
\end{aligned}
$$

In the extreme case $y \to 1$, i.e. $m^2 > Q^2 > 0$, the previous expansion gives:

$$
\begin{aligned}
m^2 F_{-2}^{(1)} &= \frac{1}{4}\frac{m^2}{P^2}L_r\,, \qquad (52) \\
m^2 F_{-1}^{(1)} &= \frac{1}{16}\frac{m^2}{P^2}\Big[2\zeta(2)-L_r^2\Big]\,, \qquad (53) \\
m^2 F_{0}^{(1)} &= \frac{1}{48}\frac{m^2}{P^2}\Big[18\zeta(2)-6\zeta(3)+12L_r+3L_r^2+L_r^3\Big]\,, \qquad (54) \\
F_{-2}^{(2)} &= 0\,, \qquad (55) \\
F_{-1}^{(2)} &= -\frac{1}{32}L_r+\frac{1}{16}\frac{m^2}{P^2}\Big(1-L_r\Big)\,, \qquad (56) \\
F_{0}^{(2)} &= \frac{1}{64}\Big[4\zeta(2)+5L_r+L_r^2\Big]-\frac{1}{192}\frac{m^2}{P^2}\Big[24+6\zeta(2)-12L_r-6\zeta(2)L_r-3L_r^2 \\
&\quad -L_r^3\Big]+\frac{1}{384}\frac{Q^2}{m^2}\Big[5-2\zeta(2)\Big]\,. \qquad (57)
\end{aligned}
$$

4 The scalar 6-denominator integral

We have already observed that there is no MI associated to the 6-denominator topology (c) in Fig. 2. The scalar integral associated to the original graph in Fig. 1 (which has 6 propagators, one of them raised to the second power), therefore, is not a MI and can be expressed in terms of the 5, 4, 3 and 2-denominator MIs of Fig. 7. For completeness, we report its definition and analytic value:



$$= \int \mathfrak{D}^D k_1 \mathfrak{D}^D k_2 \, \frac{1}{\mathcal{D}_1 \mathcal{D}_2^2 \mathcal{D}_3 \mathcal{D}_4 \mathcal{D}_5 \mathcal{D}_6} \,, \tag{58}$$

$$= \sum_{k=-2}^{0} (D-4)^k A^{(k)}(x,y) + \mathcal{O}\Big((D-4)\Big) \,, \tag{59}$$

where the coefficients $A^{(k)}(x,y)$ of the Laurent expansion in powers of $(D-4)$ are given (dropping again for simplicity the arguments x, y) by:

$$m^6 A^{(-2)} = \frac{1}{4(1-y)^2}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big[1 - \frac{2}{(1-y)} + \frac{1}{(1-y)^2}\Big] H(0;x), \tag{60}$$

$$\begin{aligned}
m^6 A^{(-1)} = & -\frac{1}{8(1+x)(1-y)}\Big[1 - \frac{1}{(1+x)}\Big]\Big[1 - \frac{1}{(1-y)}\Big] \\
& -\Big\{ \frac{1}{16(1+x)^2(1-y)}\Big[3 - \frac{2}{(1+x)}\Big]\Big[1 - \frac{1}{(1-y)}\Big] \\
& -\frac{3}{8(1-y)^3}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big[2 - \frac{1}{(1-y)}\Big] \\
& -\frac{1}{48(1-x)(1-y)}\Big[1 - \frac{19}{(1-y)}\Big] - \frac{1}{24(1+x)(1-y)}\Big[1 + \frac{8}{(1-y)}\Big] \\
& +\frac{1}{16(1-y)^2}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big[1 - \frac{3}{(1-y)^2} + \frac{2}{(1-y)^3}\Big] H(0;y) \\
& +\frac{1}{4(1-y)^2}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big]\Big[1 - \frac{2}{(1-y)} \\
& +\frac{1}{(1-y)^2}\Big] H(1;y) \Big\} H(0;x) \,,
\end{aligned} \tag{61}$$

$$\begin{aligned}
m^6 A^{(0)} = & \frac{1}{(1-y)}\Big[\frac{1}{(1-x)} - \frac{1}{(x+1)}\Big]\Big[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\Big]\Big\{ -\frac{1}{48} H(0;x)H(0;y) \\
& -\frac{1}{8}\Big[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\Big]\Big[H(0;y)\Big(2\zeta(2) + H(-1,0;x) - \frac{5}{6}H(0;x)\Big) \\
& -\frac{1}{2}H(0;y)\Big(G(-1/y,0;x) + G(-y,0;x) + \frac{1}{3}H(0;x) + 4\zeta(2)\Big)\Big]\Big\} \\
& +\Big[\frac{1}{(1-x)} - \frac{1}{(x+1)}\Big]\Big[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\Big]\Big\{ -\frac{1}{48}\Big(3\zeta(2) + H(0,0;x)\Big) \\
& +\frac{1}{8}\Big[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\Big]\Big[\Big(G(-1/y,0;x) + G(-y,0;x) + \frac{1}{3}H(0;x)
\end{aligned}$$

$$
\begin{aligned}
&+4\zeta(2)\Big)\Big(\frac{1}{4}H(0;y)+H(1;y)\Big)+\frac{1}{16}\Big(3\zeta(2)+H(0,0;x)\Big)\Big]\Big]\Big\}\\
&-\frac{1}{48}\Big[\frac{1}{(1-y)}-\frac{1}{(1-y)^2}\Big]\Big[\frac{1}{(x+1)}-\frac{1}{(x+1)^2}\Big]\Big[3\zeta(2)+H(0,0;x)\Big]\times\\
&\times\Big\{\Big[1+\frac{1}{(x+1)}\Big]\Big[1-\frac{2}{(x+1)^2}\Big]+6\Big[\frac{1}{(1-y)}-\frac{1}{(1-y)^2}\Big]\Big\}\\
&+\Big[\frac{1}{(1-x)}-\frac{1}{(x+1)}\Big]\Big[\frac{1}{(1-y)}-\frac{1}{(1-y)^2}\Big]\Big\{\frac{1}{24}\Big[H(-1,0;x)+2\zeta(2)\\
&-\frac{5}{6}H(0;x)\Big]-\frac{1}{192}H(0;x)\Big(1+4H(0;y)+4H(1;y)\Big)\\
&-\frac{1}{16}\Big[\frac{1}{(1-y)}-\frac{1}{(1-y)^2}\Big]\Big[\Big(2\zeta(2)+H(-1,0;x)-\frac{5}{6}H(0;x)\Big)\Big(2+\\
&H(0;y)+4H(1;y)\Big)+\frac{1}{16}\frac{1}{(1-y)}\Big(H(0;x)\Big(-6H(-1,0;y)-6\zeta(2)\\
&+\frac{16}{3}H(0;y)+4H(0,0;y)+2H(1,0;y)\Big)-4\zeta(2)H(0;y)\\
&-H(0,0;y)\Big(G(-y;x)-G(-1/y;x)\Big)-H(0,0,0;y)-3\zeta(2)G(-y;x)\\
&-G(-y,0,0;x)+3\zeta(2)G(-1/y;x)+G(-1/y,0,0;x)\Big)\\
&+\frac{1}{48}\Big(6\zeta(3)-6\zeta(2)H(-1;x)-12H(-1,-1,0;x)+H(0;x)\Big(11-\frac{3}{2}\zeta(2)\\
&+9H(-1,0;y)-8H(0;y)+3H(1,0;y)+12H(1,1;y)\\
&+6H(0,1;y)H(0;x)-3H(0,0;y)H(0;x)\Big)+6H(-1,0,0;x)\\
&-2\zeta(2)H(0;y)-\frac{3}{2}G(-y;x)\Big(3\zeta(2)+H(0,0;y)+4H(0,1;y)\Big)\\
&+\frac{3}{2}G(-1/y;x)\Big(7\zeta(2)+H(0,0;y)+4H(0,1;y)\Big)-\frac{3}{2}H(0,0,0;y)\\
&-6H(0,0,1;y)+6G(-y,-1,0;x)-\frac{3}{2}G(-y,0,0;x)+6G(-1/y,-1,0;x)\\
&-\frac{9}{2}G(-1/y,0,0;x)\Big)\Big]\Big\}-\Big[\frac{1}{(1-y)}-\frac{1}{(1-y)^2}\Big]\Big[\frac{1}{(x+1)}-\frac{1}{(x+1)^2}\Big]\times\\
&\times\Big\{\frac{1}{24}-\frac{1}{16}\Big[1-\frac{2}{(x+1)}\Big]\Big[2\zeta(2)-\frac{5}{6}H(0;x)+H(-1,0;x)\Big]\Big\}. \qquad (62)
\end{aligned}
$$

5 Summary

The aim of the present paper was to identify and evaluate the MIs which are necessary for the calculation of the contribution of two-loop box diagrams with one electron loop ($N_F = 1$) to the corrections of $\mathcal{O}(\alpha^3)$ to the Bhabha scattering amplitude in QED. The result was obtained without any approximation, so that our expressions keep the full dependence of the MIs on the electron mass m and on the Mandelstam variables s, t.

It has been shown that all the scalar integrals occurring in the problem can be expressed in terms of a set of 14 MIs; the reduction procedure has been carried out with the by now standard approach based on the use of the IBPs, LIs, and symmetry identities.

Out of the 14 MIs of the set, 11 had already been calculated in a previous work. For what concerns the remaining three, one is simply the product of two 1-loop integrals, while the other two are genuine 2-loop integrals; they all depend on both the Mandelstam variables s and t.

The central part of the paper has been devoted to the evaluation of the two MIs not already known in the literature. The calculation has been performed with the method of the differential equations with respect to the Mandelstam variables. While a two by two system of differential equations was expected on general grounds for the two MIs, it was found that the two equations of the system are in fact decoupled, greatly simplifying the task of finding the solution. The boundary conditions for the solutions of the differential equations corresponding to the considered Feynman graph integrals have been explicitly obtained from the equations themselves and the qualitative knowledge of the analytical properties of the MIs, namely the regularity of the MIs at $P^2 = 0$.

The analytic expression for the MIs has been given as a Laurent series in powers of $(D-4)$, where D is the dimensional-regulator for both IR and UV divergences. The coefficients of the Laurent series of order $1/(D-4)^2$, $1/(D-4)$, and zeroth order in $(D-4)$, have been written in closed analytic form in terms of 1- and 2-dimensional harmonic polylogarithms, of maximum weight $w = 3$.

For completeness, the Laurent expansion for the scalar 6-denominator integral associated to the graph in Fig. 1, is also given up to the same order in $(D-4)$.

With these results, it is now possible to evaluate the contributions of the two-loop box diagrams with one electron loop to the amplitude of the Bhabha scattering in QED, at $\mathcal{O}(\alpha^3)$. The explicit expression for the $\mathcal{O}(\alpha^3)$ contribution to the amplitude will be given elsewhere.

6 Acknowledgment

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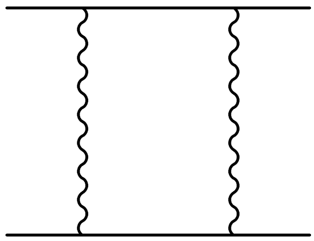


Figure 8: The 1-loop box.

A Propagators

In this Appendix we list the dependence on the loop momenta of the denominators of the propagators appearing in the various integrals.

$$\mathcal{D}_1 = k_1^2 \, , \tag{63}$$
$$\mathcal{D}_2 = (p_1 - p_3 - k_1)^2 \, , \tag{64}$$
$$\mathcal{D}_3 = [k_2^2 + m^2] \, , \tag{65}$$
$$\mathcal{D}_4 = [(p_1 - k_1)^2 + m^2] \, , \tag{66}$$
$$\mathcal{D}_5 = [(p_2 + k_1)^2 + m^2] \, , \tag{67}$$
$$\mathcal{D}_6 = [(p_1 - p_3 - k_1 + k_2)^2 + m^2] \, . \tag{68}$$

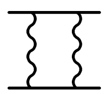
B One-loop results

Among the 14 MIs of Fig. 7, there are 4 MIs which are simply products of two one-loop diagrams: the MI shown in Fig. 7 (c) is the product of a box diagram and a tadpole; the graph (h) is the product of a vertex in the t-channel and a tadpole; the one in (k) is the product of a tadpole and a bubble in the t-channel with massless internal lines. Finally (l) is the product of a tadpole and a a two-point function in the s-channel with massive propagators.

While the value of the tadpole was given in Eq. (4), we refer to [14] for the explicit expressions of the massless and massive bubble diagrams and the vertex correction. We discuss here the calculation of the one-loop box diagram, up to the first order in $(D-4)$ included. The considered topology (which in this case coincides with the MI itself) is shown in Fig. 8.

By applying the reduction procedure outlined in Section 2, it is found (as expected) that the only MI for that topology is the scalar integral $B(D, m^2, P^2, Q^2)$:

$$B(D, m^2, P^2, Q^2) =$$



$$= \int \mathfrak{D}^D k_1 \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_4 \mathcal{D}_5}, \tag{69}$$

where, as usual, $P^2 = (p_1 + p_2)^2$ and $Q^2 = (p_1 - p_3)^2$, such that $P^2 = -s$ and $Q^2 = -t$.

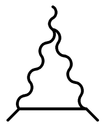
By following the method outlined in Section 3, we find that $B(D,m^2,P^2,Q^2)$ obeys the following first-order linear differential equations in P^2 and Q^2 :

$$\frac{\partial B}{\partial P^2} = -\frac{1}{2}\Big[\frac{1}{P^2}-\frac{(D-5)}{(P^2+4m^2)}+\frac{(D-4)}{(P^2+Q^2+4m^2)}\Big]B+\Omega_1(D,m^2,P^2,Q^2), \quad (70)$$

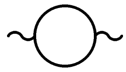
$$\frac{\partial B}{\partial Q^2} = \frac{1}{2}\Big[\frac{(D-6)}{Q^2}-\frac{(D-4)}{(P^2+Q^2+4m^2)}\Big]B+\Omega_2(D,m^2,P^2,Q^2), \quad (71)$$

where the homogeneous part of the differential equation in P^2 is the same as in Eq. (19), and the non-homogeneous terms Ω_1 and Ω_2 (not to be confused with the inhomogeneous terms of Eqs. (19,20) !) are given by

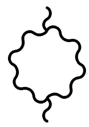
$$\Omega_1(D,m^2,P^2,Q^2) = -(D-4)\Bigg[\frac{1}{4m^2P^2}-\frac{(Q^2+4m^2)}{4m^2Q^2(P^2+4m^2)}+ +\frac{1}{Q^2(P^2+Q^2+4m^2)}\Bigg]$$



$$+\frac{2(D-3)}{Q^2}\Bigg[\frac{1}{(P^2+4m^2)^2}-\frac{1}{Q^2(P^2+4m^2)}+ +\frac{1}{Q^2(P^2+Q^2+4m^2)}\Bigg]$$



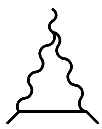
$$-\frac{(D-3)}{2m^2Q^2}\Bigg[\frac{1}{P^2}-\frac{1}{(P^2+4m^2)}\Bigg]$$



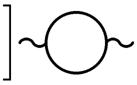
$$+\frac{(D-2)}{m^2Q^2}\Bigg[\frac{1}{(P^2+4m^2)^2}-\frac{1}{Q^2(P^2+4m^2)} +\frac{1}{Q^2(P^2+Q^2+4m^2)}\Bigg]$$

$, \quad (72)$

$$\Omega_2(D,m^2,P^2,Q^2) = -\frac{(D-4)}{(P^2+4m^2)}\Bigg[\frac{1}{Q^2}-\frac{1}{(P^2+Q^2+4m^2)}\Bigg]$$



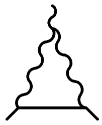
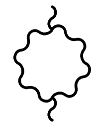
$$-\frac{2(D-3)}{(P^2+4m^2)^2}\Bigg[\frac{1}{Q^2}-\frac{1}{(P^2+Q^2+4m^2)}\Bigg]$$



$$-\frac{(D-2)}{m^2(P^2+4m^2)^2}\Bigg[\frac{1}{Q^2}-\frac{1}{(P^2+Q^2+4m^2)}\Bigg]$$

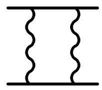
$. \quad (73)$

Each of the two equations Eqs. (70,71) is sufficient to obtain the explicit value of $B(D,m^2,P^2,Q^2)$; we use the equation in P^2 for definiteness. Following the lines of Section 3.1 we find as boundary condition at $P^2 = 0$

$$B(D, m^2, P^2 = 0, Q^2) = -\frac{(D-4)}{2m^2}$$  $$-\frac{(D-3)}{m^2 Q^2}$$  $$. \tag{74}$$

The differential equation Eq. (70) can be explicitly solved by the method of the variation of the constants of Euler outlined in Section 3.1. The homogeneous equation, as already observed, is the same as in Eq. (19), so that the coefficients of the Laurent expansion in $(D-4)$ of $B(D, m^2, P^2, Q^2)$ can be obtained as in Eq. (30).

In terms of the dimensionless variables x and y defined in Eq. (31), the Laurent expansion in $(D-4)$, which begins with a single pole, is

 $$= \sum_{k=-1}^{1} (D-4)^k B^{(k)}(x,y) + \mathcal{O}\left((D-4)^{(n+1)}\right) . \tag{75}$$

Since the one-loop scalar box graph is multiplied by a tadpole, which is singular as $1/(D-4)$, in the MI of Fig. 7 (c), we need its Laurent expansion up to the first order in $(D-4)$ included.

Dropping for simplicity the dependence on (x,y) of the coefficients of the Laurent expansion $B^{(k)}(x,y)$ we find the following results

$$m^4 B^{(-1)} = \frac{1}{2}\left[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\right]\left\{\frac{1}{(1-x)} - \frac{1}{(1+x)}\right\} H(0;x) , \tag{76}$$

$$\begin{aligned} m^4 B^{(0)} = {} & \frac{1}{4}\left[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\right]\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\Big\{2H(1;y) \\ & - H(0;y)\Big\} H(0;x) , \end{aligned} \tag{77}$$

$$\begin{aligned} m^4 B^{(1)} = {} & -\frac{1}{8}\left[\frac{1}{(1-y)} - \frac{1}{(1-y)^2}\right]\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\Big\{-2\zeta(3) \\ & + 2\zeta(2)H(-1;x) + 4H(-1,-1,0;x) \\ & + 2H(-1,0;x)\left[H(0;y) + 2H(1;y)\right] \\ & - 2H(-1,0,0;x) + H(0;x)\left[\zeta(2) - 2H(1,0;y) - 4H(1,1;y)\right] \\ & + H(0;y)\left[4\zeta(2) - G(-y,0;x) - G(-1/y,0;x)\right] \\ & - \left[H(0,0;y) + 2H(0,1;y)\right]\left[H(0;x) - G(-y;x) + G(-1/y;x)\right] \\ & + H(0,0,0;y) + 2H(0,0,1;y) \\ & - 2H(1;y)\left[G(-y,0;x) + G(-1/y,0;x)\right] + 3\zeta(2)G(-y;x) \\ & - 2G(-y,-1,0;x) + G(-y,0,0;x) - 5\zeta(2)G(-1/y;x) \\ & - 2G(-1/y,-1,0;x) + G(-1/y,0,0;x)\Big\} . \end{aligned} \tag{78}$$

Eqs. (76–78) are valid in the non-physical region $P^2 = -s \geq 0$; the corresponding expressions for the physical region are recovered by standard analytical continuation.

C Harmonic Polylogarithms

In this Appendix we briefly review some of the properties of the Harmonic Polylogarithms of one variable, x, (HPLs), introduced in [20] as an extension of Nielsen's polylogarithms [25, 26, 27, 28], as well as of the Harmonic Polylogarithms of two variables x and y (2dHPLs), introduced in [12].

C.1 One-dimensional Harmonic Polylogarithms

One starts by defining the following set of algebraic factors

$$f(-1;x) = \frac{1}{(1+x)}\,, \tag{79}$$

$$f(0;x) = \frac{1}{x}\,, \tag{80}$$

$$f(1;x) = \frac{1}{(1-x)}\,. \tag{81}$$

The one-dimensional HPL, $H(\mathbf{m}_w;x)$, can then be defined as the set of functions generated by the repeated integrations

$$\int_0^x dz\,\{f(-1;z); f(0;z); f(1;z)\}\ H(\mathbf{m}_w;z)\,; \tag{82}$$

with

$$H(-1;x) = \int_0^x \frac{dz}{(1+z)} = \ln(1+x)\,, \tag{83}$$

$$H(0;x) = \ln x\,, \tag{84}$$

$$H(1;x) = \int_0^x \frac{dz}{(1-z)} = -\ln(1-x)\,, \tag{85}$$

and where $\mathbf{m}_w$ is a w-dimensional vector whose components can assume the values 1, 0 or -1; w is said the weight of the corresponding HPL.

The following relations are valid:

$$H(\mathbf{0}_w;x) = \frac{1}{w!}\ln^w x\,, \tag{86}$$

$$H(a,\mathbf{m}_{w-1};x) = \int_0^x dz f(a;z) H(\mathbf{m}_{w-1};z)\,, \tag{87}$$

$$\frac{d}{dx}H(a,\mathbf{m}_{w-1};x) = f(a;x)H(\mathbf{m}_{w-1};x)\,, \tag{88}$$

where $\mathbf{0}_w$ stands for the vector whose w components are all equal to 0.

The set of the HPLs fulfills an algebra; the product of two HPLs of the same argument x and of weights w_1 and w_2 is a suitable combination of HPLs of the same argument and weight $w = w_1 + w_2$:

$$H(\mathbf{p};x)H(\mathbf{q};x) = \sum_{\mathbf{r}=\mathbf{p}+\mathbf{q}} H(\mathbf{r};x)\,, \tag{89}$$

where $\mathbf{r}$ is a $(w_p + w_q)$-dimensional vector constituted by all mergers of $\mathbf{p}$ and $\mathbf{q}$ in which the relative orders of the elements of $\mathbf{p}$ and $\mathbf{q}$ are preserved. In the case $w_p = 1$, for example, we have:

$$\begin{aligned} H(a;x)H(m_1,\cdots,m_q;x) &= H(a,m_1,\cdots,m_q;x) + H(m_1,a,m_2,\cdots,m_q;x) \\ &\quad + \cdots + H(m_1,\cdots,m_q,a;x)\,. \end{aligned} \tag{90}$$

Moreover, by subsequent integrations by parts on the definition itself, the following identities between HPLs hold:

$$\begin{aligned} H(m_1,\cdots,m_q;x) &= H(m_1;x)H(m_2,\cdots,m_q;x) \\ &\quad -H(m_2,m_1;x)H(m_3,\cdots,m_q;x) \\ &\quad +H(m_3,m_2,m_1;x)H(m_4,\cdots,m_q;x) \\ &\quad -\cdots-(-1)^q H(m_q,\cdots,m_1;x)\,, \end{aligned} \tag{91}$$

For a more complete treatment (in particular for the analytical continuation) and the numerical evaluation of the HPLs we refer the reader to [20] and [21].

C.2 Two-dimensional Harmonic Polylogarithms

We recall here shortly the definition and properties of the 2 dimensional HPLs already used in Eqs. (35,38). As observed in [22], they can be simply obtained by replacing the $f(i;x)$ of Eqs. (79–81) by a generalized set of factors

$$g(i;x) = \frac{1}{x-i}\,,$$

where in the present calculation the "index" i spans the enlarged set of values 0, -1, $-y$ and $-1/y$:

$$g(-1;x) = \frac{1}{(1+x)}\,, \tag{92}$$

$$g(0;x) = \frac{1}{x}\,, \tag{93}$$

$$g(-y;x) = \frac{1}{(x+y)}\,, \tag{94}$$

$$g(-1/y;x) = \frac{1}{\left(x+\frac{1}{y}\right)}\,. \tag{95}$$

2dHPLs are then defined as the set of functions generated by the repeated integrations

$$\int_0^x dz\,\{g(j;z)\}\,G(\mathbf{m}_w;z)\,, \tag{96}$$

where j and the components of $\mathbf{m}_w$ can take the values 0, -1, $-y$, and $-1/y$. In particular

$$G(-1;x) = \ln(1+x) = H(-1;x)\,, \tag{97}$$

$$G(0;x) = \ln x = H(0;x)\,, \tag{98}$$

$$G(-y;x) = \int_0^x \frac{dz}{(z+y)} = \ln\left(1+\frac{x}{y}\right), \tag{99}$$

$$G(-1/y;x) = \int_0^x \frac{dz}{\left(z+\frac{1}{y}\right)} = \ln(1+xy)\ . \tag{100}$$

It is not difficult to see that the 2dHPLs involving the subset of indices 0, $(-1/y)$, (-1) and argument x can be re-expressed in terms of 1dHPLs and 2dHPLs of argument xy and indices 0, $(-y)$ and (-1); in particular we have

$$
\begin{aligned}
G(-1/y;x) &= H(-1;xy)\,, && (101)\\
G(-1/y,0;x) &= H(-1,0;xy) - H(0;y)H(-1;xy)\,, && (102)\\
G(-1/y,0,0;x) &= H(-1,0,0;xy) + H(0,0;y)H(-1;xy) \\
&\quad -H(0;y)H(-1,0;xy)\,, && (103)\\
G(-1/y,-1,0;x) &= G(-1,-y,0;xy) - H(0;y)G(-1,-y;xy)\,. && (104)
\end{aligned}
$$

The analytical continuation of the 2dHPLs listed above can be obtained by following the lines of [23].

Concerning their numerical evaluation, the results of [24] cannot be used here as they apply to a different set of indices of the 2dHPLs. For that reason, we give now their expressions in terms of Nielsen's polylogarithms of non-trivial argument. Let us remind here that such expressions are by no means unique, as those polylogarithms of non-trivial argument can satisfy several identities, often quite involved; an elementary example is

$$\mathrm{Li}_2(1-z) + \mathrm{Li}_2(z) + \log z \log(1-z) - \zeta(2) = 0\,.$$

The expressions in terms of 2dHPLs of suitable indices and argument x do not suffer of that drawback.

We find

$$
\begin{aligned}
G(-y,0;x) &= \ln(x)\ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)\,, && (105)\\
G(-1/y,0;x) &= \ln(x)\ln(1+xy) + \mathrm{Li}_2(-xy), && (106)\\
G(-y,0,0;x) &= \int_0^x \frac{dx'}{x'+y}\,\frac{1}{2}\ln^2 x' && (107)\\
&= \frac{\ln^2(x)}{2}\ln\left(1+\frac{x}{y}\right) - \ln(y)\mathrm{Li}_2\left(-\frac{x}{y}\right) - \mathrm{Li}_3\left(-\frac{x}{y}\right), && (108)\\
G(-1/y,0,0;x) &= \int_0^x \frac{dx'}{x'+\frac{1}{y}}\,\frac{1}{2}\ln^2 x' && (109)\\
&= \frac{\ln^2 x}{2}\ln(1+xy) - \ln y\,\mathrm{Li}_2(-xy) - \mathrm{Li}_3(-xy), && (110)\\
G(-y,-1,0;x) &= \int_0^x \frac{dx'}{x'+y}\int_0^{x'}\frac{dx''}{x''+1}\ln x'' && (111)
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\ln(x)+\frac{3}{2}\ln(1+x)-\frac{3}{2}\ln\left(1+\frac{x}{y}\right)-\ln(y)\right]\ln(1+x)\ln\left(1+\frac{x}{y}\right)\\
&\quad+\left[\ln(1+x)\ln(y)-\frac{1}{2}\ln(x)\ln(1+x)-\frac{1}{3}\ln^2(1+x)\right.\\
&\quad\left.-\frac{1}{2}\ln(1+x)\ln(1-y)+\frac{1}{2}\ln^2(1-y)-\ln(1-y)\ln(y)\right]\ln(1+x)\\
&\quad+\left[\frac{1}{2}\ln(x)\ln\left(1+\frac{x}{y}\right)+\frac{1}{2}\ln\left(1+\frac{x}{y}\right)\ln(1-y)-\frac{1}{2}\ln^2(1-y)\right.\\
&\quad\left.-\frac{1}{2}\ln\left(1+\frac{x}{y}\right)\ln(y)+\ln(1-y)\ln(y)\right]\ln\left(1+\frac{x}{y}\right)\\
&\quad-\left[\ln(x)-\ln\left(1+\frac{x}{y}\right)+\ln(1-y)-\ln(y)\right]\mathrm{Li}_2(-x)\\
&\quad+\left[\ln(x)+\ln(1-y)-\ln(y)\right]\mathrm{Li}_2\left(-\frac{x}{y}\right)-\ln\left(1+\frac{x}{y}\right)\mathrm{Li}_2\left(\frac{y}{x+y}\right)\\
&\quad-\ln(y)\mathrm{Li}_2(y)-\left[\ln(1+x)-\ln\left(1+\frac{x}{y}\right)\right]\mathrm{Li}_2\left(\frac{(1+x)y}{x+y}\right)\\
&\quad-\left[\ln(1+x)-\ln\left(1+\frac{x}{y}\right)-\ln(y)\right]\mathrm{Li}_2\left(\frac{x+y}{1+x}\right)\\
&\quad+\left[\ln(x)-\ln(1+x)+\ln(1-y)-\ln(y)\right]\mathrm{Li}_2\left(\frac{x(1-y)}{x+y}\right)\\
&\quad+\mathrm{Li}_3(-x)-\mathrm{Li}_3\left(-\frac{x}{y}\right)+\mathrm{Li}_3\left(-\frac{x(1-y)}{(1+x)y}\right)+\mathrm{Li}_3(y)\\
&\quad-\mathrm{Li}_3\left(\frac{y}{x+y}\right)+\mathrm{Li}_3\left(\frac{(1+x)y}{x+y}\right)-\mathrm{Li}_3\left(\frac{x+y}{1+x}\right)-\mathrm{S}_{1,2}(y)\\
&\quad+\mathrm{S}_{1,2}\left(\frac{x+y}{1+x}\right), \qquad (112)
\end{aligned}
$$

$$
\begin{aligned}
G(-1/y,-1,0;x) &= \int_0^x \frac{dx'}{x'+\frac{1}{y}}\int_0^{x'}\frac{dx''}{x''+1}\,\ln x'' \qquad (113)\\
&= -\zeta(3)+\left[\zeta(2)-\frac{1}{2}\ln^2(1-y)+\frac{1}{6}\ln^2(y)\right]\ln(y)\\
&\quad-\left[\ln(x)\ln(1+x)-\frac{1}{3}\ln^2(1+x)+\frac{1}{2}\ln(1+x)\ln(y)\right.\\
&\quad\left.-\ln(1-y)\ln(y)\right]\ln(1+x)+\left[2\ln(x)\ln(1+x)-\ln^2(1+x)\right.\\
&\quad-\frac{1}{2}\ln(x)\ln(1+xy)+\ln(1+x)\ln(1+xy)\\
&\quad\left.-\frac{1}{3}\ln^2(1+xy)\right]\ln(1+xy)+\left[\ln(x)+\ln(1-y)\right]\mathrm{Li}_2(-xy)\\
&\quad-\left[\ln(x)+\ln(1-y)-\ln(1+xy)\right]\mathrm{Li}_2(-x)
\end{aligned}
$$

$$
\begin{aligned}
&-\Big[\ln(x)-\ln(1+x)+\ln(1-y)\Big]\mathrm{Li}_2\left(\frac{x(1-y)}{1+x}\right)\\
&-\Big[\ln(1+x)-\ln(1-y)\Big]\mathrm{Li}_2\left(-\frac{1-y}{(1+x)y}\right)\\
&+\Big[\ln(1+x)-\ln(1+xy)\Big]\Big[\mathrm{Li}_2\left(\frac{(1+x)y}{1+xy}\right)+\mathrm{Li}_2\left(\frac{1+xy}{1+x}\right)\Big]\\
&+\mathrm{Li}_3(-x)+\mathrm{Li}_3\left(\frac{x(1-y)}{1+x}\right)-\mathrm{Li}_3\left(-\frac{1-y}{(1+x)y}\right)-\mathrm{Li}_3(-xy)\\
&-\mathrm{Li}_3\left(\frac{(1+x)y}{1+xy}\right)+\mathrm{Li}_3\left(\frac{1+xy}{1+x}\right)+\mathrm{S}_{1,2}(y)+\mathrm{S}_{1,2}(-xy). \qquad (114)
\end{aligned}
$$

D Limiting cases

In order to be able to impose the initial conditions for the solutions of the differential equations for the MIs, we need the expressions of the 2dHPLs in some particular point of one of the variables. In our case we needed the following values of 2dHPLs at $x = 1$ in terms of HPLs of argument y:

$$
\begin{aligned}
G(-y;1) &= H(-1;y)-H(0;y), && (115)\\
G(-1/y;1) &= H(-1;y), && (116)\\
G(-y,0;1) &= -\zeta(2)+H(0,-1;y)-H(0,0;y), && (117)\\
G(-1/y,0;1) &= -H(0,-1;y), && (118)\\
G(-y,0,0;1) &= \zeta(2)H(0;y)-H(0,0,-1;y)+H(0,0,0;y), && (119)\\
G(-1/y,0,0;1) &= H(0,0,-1;y), && (120)\\
G(-y,-1,0;1) &= -\frac{3}{2}\zeta(3)+\frac{1}{2}\zeta(2)[H(1;y)-H(-1;y)] && \\
&\quad -H(1,0,-1;y)+H(1,0,0;y), && (121)\\
G(-1/y,-1,0;1) &= -\frac{1}{2}\zeta(2)[H(1;y)+H(-1;y)]+H(0,0,-1;y) && \\
&\quad +H(1,0,-1;y)\,. && (122)
\end{aligned}
$$

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