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Renormalons: technical introduction

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Abstract

The large-β_0 limit of QCD is discussed, with the emphasize on simple technical methods of calculating various quantities at the order $1/\beta_0$. Many examples, mainly from heavy quark physics, are considered. Some QCD results based on renormalization group (and not restricted to the large-β_0 limit) are also discussed.

1 Large-β_0 Limit

It is well known that perturbative series do not converge. They are asymptotic series, i.e., the difference between the exact result and its approximation up to the order α_s^L, divided by α_s^L, tends to 0 in the limit $\alpha_s \to 0$. Large-order behaviour of various perturbative series attracted considerable attention during recent years. Most of the results obtained so far are model-dependent: they are derived in the large-β_0 limit, i.e., at $n_f \to -\infty$. There are some hints that the situation in the real QCD may be not too different from this limit, but this cannot be proved. However, a few results are rigorous consequences of QCD; they are based on the renormalization group. Many more applications are discussed in the excellent review [1], where additional references can be found.

Let's consider a perturbative quantity A such that the tree diagram for it contains no gluon propagators. We can always normalize the tree value of A to be 1. Then the perturbative series for the bare quantity A_0 has the form

$$A_0 = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a'_{Ln} n_f^n \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^L , \qquad (1.1)$$

where L is the number of loops. This series can be rewritten in terms of $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ instead of n_f:

$$A_0 = 1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_{Ln} \beta_0^n \left(\frac{g_0^2}{(4\pi)^{d/2}} \right)^L . \tag{1.2}$$

Now we are going to consider β_0 as a large parameter such that $\beta_0 \alpha_s \sim 1$, and consider only a few terms in the expansion in $1/\beta_0 \sim \alpha_s$:

$$A_0 = 1 + \frac{1}{\beta_0} f \left(\frac{\beta_0 g_0^2}{(4\pi)^{d/2}} \right) + \mathcal{O}\left(\frac{1}{\beta_0^2} \right) . \tag{1.3}$$

This regime is called large-β_0 limit, and can only hold in QCD with $n_f \to -\infty$. Note that it has nothing in common with the large-N_c limit, because we cannot control powers of N_c in the coefficients a_{Ln}.

There is some empirical evidence [2] that the two-loop coefficients $a_{21}\beta_0 + a_{20}$ for many quantities are well approximated by $a_{21}\beta_0$. It is easy to find a'_{21} from the diagram with the quark-loop insertion into the gluon propagator in the one-loop correction. Then we can estimate the full two-loop coefficient as $a_{21}\beta_0 = a'_{21}\left(n_f - \frac{11}{4}C_A/T_F\right)$. This is called naive nonabelianization [2]. Of course, there is no guarantee that this will hold at higher orders. We can only hope that higher perturbative corrections are mainly due to the running of α_s; in this respect, gluonic contributions behave as $-33/2$ flavours, and QCD with $n_f = 3$ or 4 flavours is not too different from QCD with $-\infty$ flavours.

It is easy to find the coefficients $a_{L,L-1}$ of the highest degree β_0^{L-1} at L loops. They are determined by the coefficients $a'_{L,L-1}$ of n_f^{L-1}, i.e., by inserting $L-1$ quark loops into the gluon propagator in the one-loop correction. We shall assume for now that there is only one gluon propagator and no three-gluon vertices at one loop. The bare gluon propagator with $L-1$ quark loops inserted is

$$\begin{aligned} D_{\mu\nu}^{(L-1)}(p) &= -\frac{1}{(-p^2)^{1+(L-1)\varepsilon}} \left(g_{\mu\nu} + \frac{p_\mu p_\nu}{-p^2} \right) \left(-\frac{4}{3} T_F n_f \frac{g_0^2}{(4\pi)^{d/2}} \frac{D(\varepsilon)}{\varepsilon} e^{-\gamma\varepsilon} \right)^{L-1} , \\ D(\varepsilon) &= 6 e^{\gamma\varepsilon} \Gamma(1+\varepsilon) B(2-\varepsilon, 2-\varepsilon) = 1 + \frac{5}{3}\varepsilon + \cdots \end{aligned} \tag{1.4}$$

It looks like the free propagator

$$D_{\mu\nu}(p) = -\frac{1}{-p^2} \left(g_{\mu\nu} + (1 - a_0) \frac{p_\mu p_\nu}{-p^2} \right)$$

in the Landau gauge $a_0 = 0$, with a shifted power of $-p^2$ (and an extra constant factor). Let the one-loop contribution to A_0 be $(a_1 + a'_1 a_0)\, g_0^2/(4\pi)^{d/2}$. If we calculate the one-loop contribution in the Landau gauge a_1 with the denominator of the gluon propagator equal to $(-p^2)^n$ instead of just $-p^2$ and call it $a_1(n)$, then for all $L > 1$

$$a_{L,L-1} = \left(\frac{D(\varepsilon)}{\varepsilon} e^{-\gamma\varepsilon} \right)^{L-1} a_1\bigl(1 + (L-1)\varepsilon\bigr) . \tag{1.5}$$

Only the one-loop contribution a_{10} contains the additional gauge-dependent term $a_1' a_0$.

The large-β_0 limit, as formulated above, does not correspond to summation of any subset of diagrams. If we include not only quark loops, but also gluon and ghost ones, then $-\frac{4}{3}T_F n_f$ in (1.4) is replaced by

$$\beta_0 - \frac{C_A}{3}\left\{8\varepsilon + \frac{3-2\varepsilon}{2(1-\varepsilon)}(a_0+3)\left[1-\frac{\varepsilon}{2}(a_0+3)\right]\right\}\,. \tag{1.6}$$

Summing these diagrams yields a gauge-dependent result. This gauge dependence is compensated (for a gauge-invariant A_0) by other diagrams, which have more complicated topologies than a simple chain, and are impossible to sum. In the gauge $a_0 = -3$, one-loop running of α_s is produced by one-loop insertions in the gluon propagator only, without vertex contributions. Summation of chains of one-loop insertions into the gluon propagator in this gauge is equivalent to the large-β_0 limit.

In the large-β_0 limit, $\beta_1 \sim \beta_0$, $\beta_2 \sim \beta_0^2$, etc. Therefore, β-function is equal to

$$\beta = \frac{\beta_0 \alpha_s}{4\pi} \tag{1.7}$$

(this term is of order 1) plus $\mathcal{O}(1/\beta_0)$ corrections. At the leading order in $1/\beta_0$, the renormalization-group equation

$$\frac{d\log Z_\alpha}{d\beta} = -\frac{\beta}{\varepsilon+\beta} \tag{1.8}$$

can be explicitly integrated:

$$Z_\alpha = \frac{1}{1+\beta/\varepsilon}\,. \tag{1.9}$$

To the leading order in $1/\beta_0$,

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0 \log\frac{\mu}{\Lambda_{\overline{\rm MS}}}}\,. \tag{1.10}$$

The perturbative series (1.2) can be rewritten (in the Landau gauge) via the renormalized quantities:

$$A_0 = 1 + \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{F(\varepsilon,L\varepsilon)}{L}\left(\frac{\beta}{\varepsilon+\beta}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{1.11}$$

where

$$F(\varepsilon,u) = u e^{\gamma\varepsilon} a_1 (1+u-\varepsilon)\mu^{2u} D(\varepsilon)^{u/\varepsilon-1}\,. \tag{1.12}$$

If $a \neq 0$, the term $a_1' a\beta/\beta_0 + \mathcal{O}(1/\beta_0^2)$ should be added (the difference between a_0 and a is $\mathcal{O}(1/\beta_0)$).

We can expand (1.11) in the renormalized α_s, or in β (1.7), using

$$\left(\frac{\beta}{\varepsilon+\beta}\right)^L = \left(\frac{\beta}{\varepsilon}\right)^L \left[1 - L\frac{\beta}{\varepsilon} + \frac{(L)_2}{2}\left(\frac{\beta}{\varepsilon}\right)^2 - \frac{(L)_3}{3!}\left(\frac{\beta}{\varepsilon}\right)^3 + \cdots\right]$$

(here $(x)_n = x(x+1)\cdots(x+n-1) = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol). In the applications we shall consider, $F(\varepsilon, u)$ is regular at the origin:

$$F(\varepsilon, u) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_{nm}\varepsilon^n u^m\,, \tag{1.13}$$

though I know no general proof of this fact. Substituting these expansions to (1.11), we obtain a quadruple sum expressing A_0 via the renormalized quantities.

The bare quantity $A_0 = ZA$, where both Z and A have the form $1+\mathcal{O}(1/\beta_0)$. Therefore, we can find $Z-1$ with the $1/\beta_0$ accuracy just by retaining all terms with negative powers of ε in this quadruple sum. The renormalized $A-1$, with the $1/\beta_0$ accuracy, is given by terms with ε^0. It is enough to find Z_1, the coefficient of $1/\varepsilon$ in Z, in order to have the anomalous dimension

$$\gamma = -2\frac{dZ_1}{d\log\beta}\,.$$

Collecting terms with ε^{-1} in the quadruple sum for A_0, we obtain for $\beta_0 Z_1$

$$\begin{aligned}
&\beta F_{00} - \beta^2(F_{10}+F_{01}) + \beta^3(F_{20}+F_{11}+F_{02}) - \beta^4(F_{30}+F_{21}+F_{12}+F_{03}) + \cdots\\
&+ \tfrac{1}{2}\beta^2(F_{10}+2F_{01}) - \beta^3(F_{20}+2F_{11}+4F_{02}) + \tfrac{3}{2}\beta^4(F_{30}+2F_{21}+4F_{12}+8F_{03}) + \cdots\\
&+ \tfrac{1}{3}\beta^3(F_{20}+3F_{11}+9F_{02}) - \beta^4(F_{30}+3F_{21}+9F_{12}+27F_{03}) + \cdots\\
&+ \tfrac{1}{4}\beta^4(F_{30}+4F_{21}+16F_{12}+64F_{03}) + \cdots\\
&+ \cdots\\
&= \beta F_{00} - \frac{\beta^2}{2}F_{10} + \frac{\beta^3}{3}F_{20} - \frac{\beta^4}{4}F_{30} + \cdots
\end{aligned}$$

Therefore, the anomalous dimension is [3]

$$\gamma = -2\frac{\beta}{\beta_0}F(-\beta, 0) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,. \tag{1.14}$$

Collecting terms with ε^0 in the quadruple sum for A_0, we obtain for $\beta_0(A-1)$

$$\begin{aligned}
&\beta(F_{10}+F_{01}) - \beta^2(F_{20}+F_{11}+F_{02}) + \beta^3(F_{30}+F_{21}+F_{12}+F_{03})\\
&\qquad - \beta^4(F_{40}+F_{31}+F_{22}+F_{13}+F_{04}) + \cdots\\
&+ \tfrac{1}{2}\beta^2(F_{20}+2F_{11}+4F_{02}) - \beta^3(F_{30}+2F_{21}+4F_{12}+8F_{03})\\
&\qquad + \tfrac{3}{2}\beta^4(F_{40}+2F_{31}+4F_{22}+8F_{13}+16F_{04}) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ \tfrac{1}{3}\beta^3(F_{30} + 3F_{21} + 9F_{12} + 27F_{03}) - \beta^4(F_{40} + 3F_{31} + 9F_{22} + 27F_{13} + 81F_{04}) + \cdots \\
&+ \tfrac{1}{4}\beta^4(F_{40} + 4F_{31} + 16F_{22} + 64F_{13} + 256F_{04}) + \cdots \\
&+ \cdots \\
&= \beta F_{10} - \frac{\beta^2}{2}F_{20} + \frac{\beta^3}{3}F_{30} - \frac{\beta^4}{4}F_{40} + \cdots \\
&\quad + \beta F_{01} + \beta^2 F_{02} + 2\beta^3 F_{03} + 6\beta^4 F_{04} + \cdots
\end{aligned}
$$

Therefore, the renormalized quantity is [4]

$$
A(\mu) = 1 + \frac{1}{\beta_0}\int_{-\beta}^{0} d\varepsilon \frac{F(\varepsilon,0) - F(0,0)}{\varepsilon} + \frac{1}{\beta_0}\int_{0}^{\infty} du\, e^{-u/\beta}\frac{F(0,u) - F(0,0)}{u} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right), \quad (1.15)
$$

where $\beta = \beta_0\alpha_s(\mu)/(4\pi)$.

The renormalization group equation

$$
\frac{d\log A(\mu)}{d\log\alpha_s} = \frac{\gamma(\alpha_s)}{2\beta(\alpha_s)}
$$

can be conveniently solved as

$$
A(\mu) = \hat{A}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\frac{\gamma_0}{2\beta_0}} K_\gamma(\alpha_s(\mu))\,, \quad (1.16)
$$

where the function

$$
K_\gamma(\alpha_s) = \exp\int_0^{\alpha_s}\left(\frac{\gamma(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0}{2\beta_0}\right)\frac{d\alpha_s}{\alpha_s} = 1 + \frac{\gamma_0}{2\beta_0}\left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0}\right)\frac{\alpha_s}{4\pi} + \cdots \quad (1.17)
$$

satisfying

$$
K_0(\alpha_s) = 1\,, \quad K_{-\gamma}(\alpha_s) = K_\gamma^{-1}(\alpha_s)\,, \quad K_{\gamma_1+\gamma_2}(\alpha_s) = K_{\gamma_1}(\alpha_s)K_{\gamma_2}(\alpha_s)
$$

has been introduced, and

$$
\hat{A} = A(\mu_0)K_{-\gamma}(\alpha_s(\mu_0))\,.
$$

At the first order in $1/\beta_0$, we obtain from (1.14)

$$
K_\gamma(\alpha_s) = 1 + \frac{1}{\beta_0}\int_{-\beta(\alpha_s)}^{0} d\varepsilon\frac{F(\varepsilon,0) - F(0,0)}{\varepsilon}\,.
$$

Therefore,

$$
\hat{A} = 1 + \frac{1}{\beta_0}\int_0^\infty du\, e^{-u/\beta(\alpha_s(\mu_0))}\left.\frac{F(0,u) - F(0,0)}{u}\right|_{\mu_0} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,.
$$

Let's suppose that $m \gg \Lambda_{\overline{\rm MS}}$ is the characteristic hard scale in the quantity A. Then $F(\varepsilon,u)$ contains the factor $(\mu/m)^{2u}$. When taking the limit $\varepsilon \to 0$, the factor $D(\varepsilon)^{u/\varepsilon-1}$ in (1.12) becomes $\exp\left(\frac{5}{3}u\right)$. Therefore,

$$F(0,u) = \left(\frac{e^{5/6}\mu}{m}\right)^{2u} F(u)\,, \quad F(u) = u a_1 (1+u) m^{2u}\,. \tag{1.18}$$

It is most convenient to use

$$\mu_0 = e^{-5/6} m \tag{1.19}$$

in the definition of $\hat{A}$. In the rest of this Chapter, β will mean $\beta_0 \alpha_s(\mu_0)/(4\pi)$. This renormalization-group invariant is

$$\hat{A} = 1 + \frac{1}{\beta_0} \int\limits_0^\infty du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{1.20}$$

where

$$S(u) = \frac{F(u) - F(0)}{u}\,. \tag{1.21}$$

Here,

$$e^{-u/\beta} = \left(\frac{e^{5/6}\Lambda_{\overline{\rm MS}}}{m}\right)^{2u}\,. \tag{1.22}$$

If we substitute the expansion

$$S(u) = \sum_{L=1}^\infty s_L u^{L-1}$$

into the Laplace integral (1.20), we obtain the renormalized perturbative series

$$\hat{A} = 1 + \frac{1}{\beta_0} \sum_{L=1}^\infty c_L \beta^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{1.23}$$

$$c_L = (L-1)!\, s_L = \left(\frac{d}{du}\right)^{L-1} S(u)\bigg|_{u=0}\,. \tag{1.24}$$

Therefore, $S(u)$ can be obtained from $\hat{A}$ (1.23) by

$$S(u) = \sum_{L=1}^\infty \frac{c_L u^{L-1}}{(L-1)!}\,, \tag{1.25}$$

which is called Borel transform.

We see that the function $F(\varepsilon,u)$ (1.12) contains all the necessary information about the quantity A at the $1/\beta_0$ order. The anomalous dimension (1.14) is determined by $F(\varepsilon,0)$, and the renormalization-group invariant $\hat{A}$ (1.20) (which gives $A(\mu)$ (1.16)) – by $F(0,u)$. These formulae are written in the Landau gauge $a=0$; if $a \ne 0$, additional one-loop terms from the longitudinal part of the gluon propagator should be added.

2 Renormalons

The Laplace integral (1.20) is not well-defined if the Borel image $S(u)$ has singularities on the integration path – the positive half-axis $u > 0$. At the first order in $1/\beta_0$, $S(u)$ typically has simple poles. If

$$S(u) = \frac{r}{u_0 - u} + \cdots \tag{2.1}$$

where dots mean terms regular at $u = u_0$, and $u_0 > 0$, then the integral (1.20) is not well-defined near u_0. One way to make sense of this integral is to use its principal value: to make a hole $[u_0 - \delta, u_0 + \delta]$ and take the limit $\delta \to 0$. However, if we make, e.g., a hole $[u_0 - \delta, u_0 + 2\delta]$ instead, we'll get a result which differs from the principal value by the residue of the integrand times $\log 2$. Therefore, the sum of the perturbative series (1.20) contains an intrinsic ambiguity of the order of this residue. It is equal to

$$\Delta\hat{A} = \frac{r e^{-u_0/\beta}}{\beta_0} = \frac{r}{\beta_0}\left(\frac{e^{5/6}\Lambda_{\overline{\text{MS}}}}{m}\right)^{2u_0} . \tag{2.2}$$

These renormalon ambiguities are commensurate with $1/m$ power corrections – contributions of matrix elements of higher-dimensional operators to the quantity A. The full result for the physical quantity A must be unambiguous. Therefore, if one changes prescription for handling the integral across the renormalon singularity at $u = u_0$, one has to change the values of the dimension-$2u_0$ matrix elements accordingly. This shows that renormalons can only happen at integer and half-integer values of u, corresponding to dimensionalities of allowed power corrections. The largest ambiguity is associated with the renormalon closest to the origin.

The renormalon pole (2.1) yields the contribution to the coefficients c_L of the renormalized perturbative series (1.23) equal to

$$c_L = r\frac{(L-1)!}{u_0^L} \tag{2.3}$$

(see (1.24)). The series (2.1) is, clearly, divergent. Using the Stirling formula for the factorial, we can see that the terms of this series behave as

$$c_L\beta^L \sim r\left(\frac{\beta L}{e u_0}\right)^L$$

at large L. The best one can do with such a series is to sum it until its minimum term, and to assign it an ambiguity of the order of this minimum term. The minimum happens at $L \approx u_0/\beta$ loops, and the magnitude of the minimum term is given by (2.2). This is another way to look at this renormalon ambiguity. The fastest-growing contribution to c_L comes from the renormalon most close to the origin.

Note that renormalons at $u_0 < 0$ give sign-alternating factorially-growing coefficients (2.3). For such series, the integral (1.20) provides an unambiguous definition of summation called Borel sum.

Renormalon singularities can result from either UV or IR divergences of the one-loop integral. Suppose that it behaves at $k \to \infty$ as $\int d^4k/(-k^2)^{n_{\mathrm{UV}}}$, so that the degree of its UV divergence (at $d = 4$) is $\nu_{\mathrm{UV}} = 4 - 2n_{\mathrm{UV}}$. When we insert the renormalon chain, the power changes: $n_{\mathrm{UV}} \to n_{\mathrm{UV}} + (L-1)\varepsilon = n_{\mathrm{UV}} + u$ if $\varepsilon = 0$ (which is the case when calculating $S(u)$). This integral can have an UV divergence only at $u \leq 2 - n_{\mathrm{UV}} = \nu_{\mathrm{UV}}/2$. Therefore, UV renormalons can be situated at $\nu_{\mathrm{UV}}/2$ and to the left. Only quantities A with power-like UV divergences at one loop have UV renormalons at positive u. The divergence at $u = 0$ is the usual UV divergence of the one-loop integral, which is eliminated by renormalization; renormalized quantities have no UV renormalon at $u = 0$.

Similarly, if the one-loop integral behaves as $\int d^4k/(-k^2)^{n_{\mathrm{IR}}}$ at $k \to 0$ (where k is the virtual gluon momentum), so that the degree of its IR divergence is $\nu_{\mathrm{IR}} = 2n_{\mathrm{IR}} - 4$, $S(u)$ can have an IR divergence only at $u \geq 2 - n = -\nu_{\mathrm{IR}}/2$, and IR renormalons can be situated at $-\nu_{\mathrm{IR}}/2$ and integer and half-integer points to the right from it. Quantities described by off-shell diagrams have $n_{\mathrm{IR}} = 1$, and their IR renormalons are at $u = 1$ and to the right.

We can get a better understanding of the physical meaning of renormalons if we rewrite (1.20) in the form [5]

$$\hat{A} = 1 + \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) \frac{\alpha_s(\sqrt{\tau}\mu_0)}{4\pi} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) . \tag{2.4}$$

This looks like the one-loop correction, but with the running α_s under the integral sign. The function $w(\tau)$ has the meaning of the distribution function in gluon virtualities in the one-loop diagram; it is normalized to the coefficient of $\alpha_s/(4\pi)$ in the one-loop correction. Inside the $1/\beta_0$ term in (2.4), we may use the leading-order formula for running of α_s:

$$\alpha_s(\sqrt{\tau}\mu_0) = \frac{\alpha_s(\mu_0)}{1 + \beta \log \tau} = \alpha_s(\mu_0) \sum_{n=0}^\infty (-\beta \log \tau)^n .$$

Substituting this expansion into (2.4), we see, that this representation holds if $w(\tau)$ is related to the perturbative series coefficients c_L by

$$c_L = \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) (-\log \tau)^{L-1} . \tag{2.5}$$

Therefore, $S(u)$ (1.25) becomes

$$S(u) = \int\limits_0^\infty \frac{d\tau}{\tau} w(\tau) \tau^{-u} . \tag{2.6}$$

In other words, $S(u)$ is the Mellin transform of $w(\tau)$. Therefore, the distribution function $w(\tau)$ is given by the inverse Mellin transform:

$$w(\tau) = \frac{1}{2\pi i} \int\limits_{u_0 - i\infty}^{u_0 + i\infty} du\, S(u) \tau^u , \tag{2.7}$$

where u_0 should lie in the gap between IR and UV renormalons.

At $\tau < 1$ we can close the integration contour to the right. If $S(u)$ has IR renormalons $r_i/(u_i - u)$, then

$$w(\tau) = \sum_{\text{IR}} r_i \tau^{u_i} \,. \tag{2.8}$$

The leading term at small τ is given by the leftmost IR renormalon. If our quantity A is IR-finite at one loop, all $u_i > 0$, and $w(\tau) \to 0$ at $\tau \to 0$. Similarly, at $\tau > 1$ we can close the contour to the left. If the UV renormalons are $r_i/(u - u_i)$, then

$$w(\tau) = \sum_{\text{UV}} r_i \tau^{u_i} \,. \tag{2.9}$$

The leading term at large τ is given by the rightmost UV renormalon. If A is UV-finite at one loop, all $u_i < 0$, and $w(\tau) \to 0$ at $\tau \to \infty$.

All virtualities (including small ones) contribute to (2.4). Behaviour of the distribution function $w(\tau)$ in the small-virtuality region $\tau \to 0$ is determined by the IR renormalon most close to the origin. The integral (2.4) is ill-defined, just like the original integral (1.20). The one-loop α_s (1.10) becomes infinite at $\tau = (e^{5/6}\Lambda_{\overline{\text{MS}}}/m)^2$ (Landau pole), and we integrate across this pole. This happens at small τ; substituting the asymptotics (2.8) of the distribution function at small virtualities, we see that the residue at this pole, given by the IR renormalon nearest to the origin, is again equal to (2.2).

3 Light Quarks

First, we shall discuss the massless quark propagator at the order $1/\beta_0$. The one-loop expression for the quark self-energy $\Sigma(p)$ in the Landau gauge with the gluon denominator raised to the power $n = 1 + (L-1)\varepsilon$ is

$$a_1(n) = i\frac{C_F}{-p^2} \int \frac{d^d k}{\pi^{d/2}} \frac{\frac{1}{4}\operatorname{Tr} \not{p}\gamma^\mu(\not{k}+\not{p})\gamma^\nu}{[-(k+p)^2]\,(-k^2)^n} \left(g_{\mu\nu} + \frac{k_\mu k_\nu}{-k^2} \right) .$$

Using the one-loop integrals

$$\begin{gathered}\int \frac{d^d k}{[-(k+p)^2]^{n_1}[-k^2]^{n_2}} = i\pi^{d/2}(-p^2)^{d/2-n_1-n_2} G(n_1,n_2)\,,\\ G(n_1,n_2) = \frac{\Gamma(-d/2+n_1+n_2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}\,,\end{gathered} \tag{3.1}$$

we can easily find the function $F(\varepsilon, u)$ (1.12). Such functions for all off-shell massless quantities have the same Γ-function structure resulting from (3.1) with $n_2 = 1 + u - \varepsilon$:

$$F(\varepsilon,u) = \left(\frac{\mu^2}{-p^2}\right)^u e^{\gamma\varepsilon} \frac{\Gamma(1+u)\Gamma(1-u)\Gamma(2-\varepsilon)}{\Gamma(2+u-\varepsilon)\Gamma(3-u-\varepsilon)} D(\varepsilon)^{u/\varepsilon-1} N(\varepsilon,u)\,. \tag{3.2}$$

The first Γ-function in the numerator, with the positive sign in front of u, comes from the first Γ-function in the numerator of (3.1), with the negative sign in front of d, and its poles are UV divergences. The second Γ-function in the numerator, with the negative sign in front of u, comes from the second Γ-function in the numerator of (3.1), with the positive sign in front of d, and its poles are IR divergences. For $\Sigma(p)$, we obtain

$$N(\varepsilon, u) = -C_F(3 - 2\varepsilon)(u - \varepsilon)\,. \tag{3.3}$$

At one loop ($L = u/\varepsilon = 1$), the Landau-gauge self-energy vanishes; at $L = 2$, the β_0-term in the two-loop result is reproduced.

The massless-quark propagator $S(p)$ with the $1/\beta_0$ accuracy in the Landau gauge is equal to $1/\not{p}$ times (1.11), where $F(\varepsilon, u)$ is given by (3.2), (3.3). Terms with negative powers of ε in its expression via renormalized quantities form the quark-field renormalization constant Z_q. The anomalous dimension is given by (1.14):

$$\gamma = -\frac{\beta}{3\beta_0} \frac{N(-\beta, 0)}{B(2+\beta, 2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,.$$

In the general covariant gauge, the one-loop term proportional to a should be added:

$$\begin{aligned}
\gamma_q &= C_F \frac{\alpha_s}{4\pi} \left[2a + \frac{\beta\left(1 + \frac{2}{3}\beta\right)}{B(2+\beta, 2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} \right] + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \\
&= C_F \frac{\alpha_s}{4\pi} \left[2a + 3\beta\left(1 + \tfrac{5}{6}\beta - \tfrac{35}{36}\beta^2 + \cdots\right) \right]\,.
\end{aligned} \tag{3.4}$$

This perturbative series for γ_q has the radius of convergence equal to the distance from the origin to the nearest singularity, which is situated at $\beta = -5/2$; in other words, it converges at $|\beta| < 5/2$. It reproduces the leading-β_0 terms in the 2-, 3-, 4-loop results [6, 7].

The renormalized expression for $\not{p}S(p)$ is given by (1.15). If we factor out its μ-dependence as in (1.16), then the corresponding renormalization-group invariant is given by (1.20) with

$$S(u) = \frac{1}{u}\left[\frac{N(0, u)}{(1+u)(1-u)(2-u)} - \frac{N(0, 0)}{2} \right] = -\frac{3C_F}{(1+u)(1-u)(2-u)} \tag{3.5}$$

(here $\sqrt{-p^2}$ plays the role of m). The pole at $u = -1$ comes from the first Γ-function in the numerator of (3.2), and is an UV renormalon; those at $u = 1$, 2 come from the second Γ-function, and are IR renormalons (Fig. 1a). We can also see this from the power counting (Sect. 2). The light-quark self-energy seems to have a linear UV divergence. However, the leading term of the integrand at $k \to \infty$, $\not{k}/(k^2)^2$, yields 0 after integration, due to the Lorentz invariance. The actual UV divergence is logarithmic: $\nu_{\mathrm{UV}} = 0$, and UV renormalons can only be at $u \le 0$. The UV divergence at $u = 0$ is removed by renormalization, and UV renormalons are at $u < 0$. The index of the IR divergence of the self-energy, like that of any off-shell quantity, is $\nu_{\mathrm{IR}} = -2$, and IR renormalons are at $u \ge 1$.

Power corrections to the light-quark propagator form an expansion in $1/(-p^2)$, therefore, IR renormalons can only appear at positive integer values of u. For gauge-invariant quantities, the first power correction contains the gluon condensate $<G^2>$ of dimension 4, and the first IR renormalon is at $u = 2$. The quark propagator is not gauge-invariant, and the renormalon at $u = 1$ is allowed. The virtuality distribution function (2.7) is

$$w(\tau) = -3C_F \times \begin{cases} \frac{1}{2}\tau - \frac{1}{3}\tau^2 \,, & \tau < 1 \\ \frac{1}{6}\tau^{-1} \,, & \tau > 1 \end{cases}$$

(Fig. 1b).

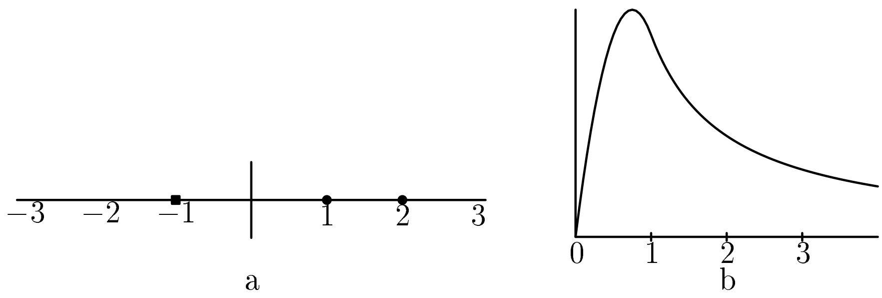


Figure 1: UV renormalons (black squares) and IR renormalons (black circles) in the light-quark self-energy (a); the virtuality distribution function (b)

Now we shall discuss light-quark currents

$$j_n(\mu) = Z_{jn}^{-1}(\mu)\bar{q}_0'\Gamma q_0 \,, \quad \Gamma = \gamma^{[\mu_1} \cdots \gamma^{\mu_n]} \,.$$

Let's calculate their vertex function $\Gamma(p, 0)$ up to one loop (Fig. 2). It is convenient to rewrite Γ as

$$\Gamma = \Gamma_+ + \Gamma_- \,, \quad \Gamma_\pm = \frac{1}{2}\left(\Gamma \pm \frac{\not{p}\Gamma\not{p}}{p^2}\right) \,,$$

where $\not{p}\Gamma_\sigma = \sigma\Gamma_\sigma\not{p}$, $\sigma = \pm 1$. Then

$$\gamma_\mu \Gamma_\sigma \gamma^\mu = 2\sigma h(d)\Gamma_\sigma \,, \tag{3.6}$$

where for n antisymmetrized γ-matrices

$$h(d) = \eta\left(n - \frac{d}{2}\right) \,, \quad \eta = (-1)^{n+1}\sigma \,. \tag{3.7}$$

The vertex function $\Gamma(p, 0)$ for a Dirac matrix $\Gamma_\pm$ is $\Gamma_\pm \cdot \Gamma(p^2)$, where the scalar function $\Gamma(p^2)$ can be calculated via $h(d)$, once and for all Dirac matrices (see [2]).

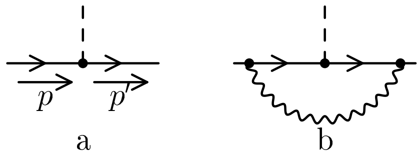


Figure 2: Proper vertex $\Gamma(p,p')$ of a bilinear quark QCD current

Calculating the vertex function $\Gamma(p^2)$ in the Landau gauge with the denominator of the gluon propagator raised to the power $n = 1 + (L-1)\varepsilon$, we obtain (1.11), (3.2) with

$$N(\varepsilon,u) = -C_F\left[2 - u - \varepsilon + 2h(u-h)\right] . \tag{3.8}$$

For the longitudinal vector current ($h = 1 - d/2$), the result can be obtained from by differentiating $\Sigma(p^2)$ (Ward identity). The anomalous dimension of the current is

$$\gamma_{jn} = \frac{d\log Z_{\Gamma n}}{d\log\mu} + \gamma_q \,,$$

where the derivative of $Z_{\Gamma n}$ is given by (1.14). We arrive at [2]

$$\begin{aligned}\gamma_{jn} &= \frac{2}{3}C_F\frac{\alpha_s}{4\pi}\frac{(n-1)(3-n+2\beta)}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\\ &= -2C_F\frac{\alpha_s}{4\pi}(n-1)\left[n - 3 + \frac{n-15}{6}\beta - \frac{13n-35}{12}\beta^2 + \cdots\right] .\end{aligned} \tag{3.9}$$

Naturally, it vanishes at $n = 1$. This perturbative series converges at $|\beta| < 5/2$. It reproduces the leading powers of β_0 in the two- and three-loop results [2, 8] In particular, the mass anomalous dimension is

$$\gamma_m = -\gamma_{j0} = 2C_F\frac{\alpha_s}{4\pi}\frac{1+\frac{2}{3}\beta}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \tag{3.10}$$

(it is known at four loops [9, 10]).

There is a general belief that one may use a naively anticommutating γ_5^{AC} in open quark lines without encountering contradictions, see [11]. The pseudoscalar currents with γ_5^{AC}, $j_{\mathrm{AC}}(\mu) = Z_{P,\mathrm{AC}}^{-1}(\mu)\bar{q}_0'\gamma_5^{\mathrm{AC}}q$, and with the 't Hooft–Veltman γ_5^{HV}, $j_{\mathrm{HV}}(\mu) = Z_{P,\mathrm{HV}}^{-1}(\mu)\bar{q}_0'\gamma_5^{\mathrm{HV}}q$ are related to each other by a finite renormalization

$$j_{\mathrm{AC}}(\mu) = Z_P(\alpha_s(\mu))j_{\mathrm{HV}}(\mu)\,, \quad Z_P(\alpha_s) = 1 + z_{\mathrm{P}1}\frac{\alpha_s}{4\pi} + z_{\mathrm{P}2}\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \tag{3.11}$$

Similarly, the axial currents are related by

$$j^\mu_{\mathrm{AC}}(\mu) = Z_A(\alpha_s(\mu))j^\mu_{\mathrm{HV}}(\mu)\,. \tag{3.12}$$

The finite renormalization constants $Z_{P,A}(\alpha_s)$ can be obtained from the currents' anomalous dimensions. Multiplying the Dirac matrix Γ in the current by γ_5^{AC} does not change the anomalous dimension; multiplying by γ_5^{HV} means $n \to 4-n$. Differentiating (3.11) and (3.12), we have

$$\begin{aligned}
\frac{d\log Z_{\mathrm{P}}(\alpha_s)}{d\log\alpha_s} &= \frac{\gamma_{j0}(\alpha_s)-\gamma_{j4}(\alpha_s)}{2\beta(\alpha_s)}\,,\\
\frac{d\log Z_{\mathrm{A}}(\alpha_s)}{d\log\alpha_s} &= \frac{\gamma_{j1}(\alpha_s)-\gamma_{j3}(\alpha_s)}{2\beta(\alpha_s)} \quad \text{where } \gamma_{j1}=0\,.
\end{aligned} \tag{3.13}$$

Therefore,

$$Z_P(\alpha_s) = K_{\gamma_{j0}-\gamma_{j4}}(\alpha_s)\,,\quad Z_A(\alpha_s) = K_{\gamma_{j1}-\gamma_{j3}}(\alpha_s) \tag{3.14}$$

(see (1.17)). For the tensor current, multiplying $\sigma^{\mu\nu}$ by γ_5^{HV} is merely a space-time transformation, e.g., $\gamma_5^{\mathrm{HV}}\sigma^{01} = -i\sigma^{23}$, and thus it does not change the anomalous dimension. Therefore, the constant relating the currents $\bar{q}'\gamma_5^{\mathrm{AC}}\sigma^{\mu\nu}q$ and $\bar{q}'\gamma_5^{\mathrm{HV}}\sigma^{\mu\nu}q$ is

$$Z_T(\alpha_s) = 1\,. \tag{3.15}$$

In the large-β_0 limit, we obtain from (3.13), (3.9)

$$\begin{aligned}
Z_A &= 1 - \frac{4}{3}\frac{C_F}{\beta_0}\int\limits_0^\beta \frac{d\beta}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)}\\
&= 1 - 4C_F\frac{\alpha_s}{4\pi}\left[1+\frac{1}{12}\beta-\frac{13}{36}\beta^2+\cdots\right]\,,\\
Z_{\mathrm{P}} &= Z_{\mathrm{A}}^2\,.
\end{aligned} \tag{3.16}$$

This reproduces the leading powers of β_0 in the three-loop results [11].

4 Heavy Quark in HQET

Now we turn to Heavy Quark Effective Theory (HQET, see, e.g., [12, 13, 14]) and discuss the heavy-quark propagator. The one-loop expression for the self-energy $\widetilde{\Sigma}(\omega)/\omega$ (Fig. 3) in the Landau gauge with the gluon denominator raised to the power $n = 1+(L-1)\varepsilon$ is

$$a_1(n) = \frac{iC_F}{\omega^2}\int\frac{d^dk}{\pi^{d/2}}\frac{\omega}{k\cdot v+\omega}\frac{v^\mu v^\nu}{(-k^2)^n}\left(g_{\mu\nu}+\frac{k_\mu k_\nu}{-k^2}\right)\,.$$

Using the one-loop HQET integrals

$$\begin{aligned}
&\int\frac{d^dk}{(-k^2)^{n_2}}\left(\frac{\omega}{k\cdot v+\omega}\right)^{n_1} = i\pi^{d/2}(-2\omega)^{d-2n_2}I(n_1,n_2)\,,\\
&I(n_1,n_2) = \frac{\Gamma(-d+n_1+2n_2)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}\,,
\end{aligned} \tag{4.1}$$

we can easily find the function $F(\varepsilon,u)$ (1.12). Such functions for all off-shell HQET quantities have the same Γ-function structure resulting from (4.1) with $n_2 = 1 + u - \varepsilon$:

$$F(\varepsilon,u) = \left(\frac{\mu}{-2\omega}\right)^{2u} e^{\gamma\varepsilon} \frac{u\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u-\varepsilon)} D(\varepsilon)^{u/\varepsilon-1} N(\varepsilon,u)\,. \tag{4.2}$$

The first Γ-function in the numerator, with the positive sign in front of u, comes from the first Γ-function in the numerator of (4.1), with the negative sign in front of d, and its poles are UV divergences. The second Γ-function in the numerator, with the negative sign in front of u, comes from the second Γ-function in the numerator of (4.1), with the positive sign in front of d, and its poles are IR divergences. For $\widetilde{\Sigma}(\omega)/\omega$, we obtain

$$N(\varepsilon,u) = -2C_F(3-2\varepsilon)\,. \tag{4.3}$$

If $a_0 \neq 0$, the one-loop term proportional to a_0 should be added to $\widetilde{\Sigma}(\omega)$. This formula reproduces the largest powers of β_0 in the known results [15] at $L = u/\varepsilon = 1$ and $L = 2$.

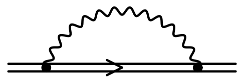


Figure 3: One-loop heavy quark self-energy in HQET

The heavy-quark propagator $\widetilde{S}(\omega) = \left[\omega - \widetilde{\Sigma}(\omega)\right]^{-1}$ with the $1/\beta_0$ accuracy in the Landau gauge is equal to $1/\omega$ times (1.11), where $F(\varepsilon,u)$ is given by (4.2), (4.3). Terms with negative powers of ε in its expression via renormalized quantities form $\widetilde{Z}_Q$. The anomalous dimension is given by (1.14):

$$\widetilde{\gamma} = \frac{\beta}{6\beta_0} \frac{N(-\beta,0)}{B(2+\beta,2+\beta)\Gamma(2+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,.$$

In the general covariant gauge, the one-loop term proportional to a should be added:

$$\begin{aligned}\widetilde{\gamma}_Q &= C_F\frac{\alpha_s}{4\pi}\left[2a - \frac{1+\frac{2}{3}\beta}{B(2+\beta,2+\beta)\Gamma(2+\beta)\Gamma(1-\beta)}\right] + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\\ &= C_F\frac{\alpha_s}{4\pi}\left[2(a-3) - 8\beta + \tfrac{10}{3}\beta^2 + \cdots\right]\,.\end{aligned} \tag{4.4}$$

This perturbative series converges at $|\beta| < 5/2$. It reproduces the leading-β_0 terms in the two- and three-loop results [15, 16, 17].

The renormalized expression for $\omega\widetilde{S}(\omega)$ is given by (1.15). If we factor out its μ-dependence as in (1.16), then the corresponding renormalization-group invariant is given by (1.20) with [18]

$$S(u) = \frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u)} N(0,u) + \frac{N(0,0)}{2u} = -6C_F\left[\frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u)} + \frac{1}{2u}\right] \tag{4.5}$$

(here -2ω plays the role of m). The first Γ-function, with the positive sign in front of u, produces UV renormalons, while the second one, with the negative sign, produces IR renormalons (Fig. 4a). We can understand this from the power counting (Sect. 2). The heavy-quark self-energy has a linear UV divergence which is not nullified by the Lorentz invariance: $\nu_{\rm UV} = 1$. This is the same divergence as that of the Coulomb energy of a point charge in classical electrodynamics. Therefore, UV renormalons are situated at $u \le 1/2$. The index of the IR divergence of the self-energy, like that of any off-shell quantity, is $\nu_{\rm IR} = -2$, and IR renormalons are at $u \ge 1$.

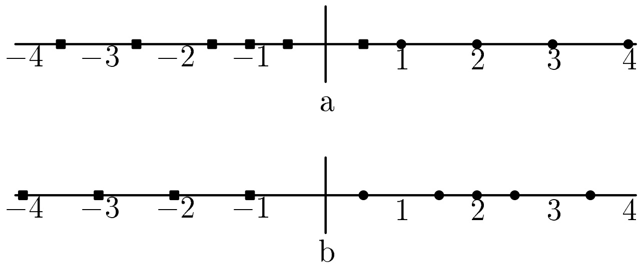


Figure 4: Renormalons in the off-shell HQET self-energy (a) and on-shell heavy-quark self-energy (b)

Here we encounter a radically new situation: an UV renormalon at $u > 0$. It leads to the ambiguity $\Delta\widetilde{\Sigma}(\omega)/\omega = (r/\beta_0)e^{5/5}\Lambda_{\overline{\rm MS}}/(-2\omega)$ where $r = 4C_F$ is the residue of $S(u)$ at $u = 1/2$. If we change the prescription for handling the pole at $u = 1/2$, we have to change the zero-energy level of HQET. Therefore, the HQET meson energy has an ambiguity $\Delta\bar{\Lambda} = \Delta\widetilde{\Sigma}(\omega)$ of order $\Lambda_{\overline{\rm MS}}/\beta_0$ [18] (see also [19])

$$\Delta\bar{\Lambda} = -2C_F e^{5/6}\frac{\Lambda_{\overline{\rm MS}}}{\beta_0}\,. \tag{4.6}$$

The structure of the leading UV renormalon at $u = 1/2$ can be investigated beyond the large-β_0 limit [20]. This approach is based on the renormalization group [21, 22]. The renormalization-group invariant corresponding to $\omega\widetilde{S}(\omega)$ is now written as

$$1 + \frac{1}{\beta_0}\int\limits_0^\infty du\, S(u)\, \exp\left[-\frac{4\pi}{\beta_0\alpha_s(\mu_0)}u\right] \tag{4.7}$$

instead of (1.20), where the exact α_s is used in the exponent, $\mu_0 = -2\omega e^{-5/6}$, and $\mathcal{O}(1/\beta_0^2)$ is absent. The singularity of $S(u)$ at $u = 1/2$ becomes a branching point

$$S(u) = \frac{r}{\left(\frac{1}{2} - u\right)^{1+a}} + \cdots\,,$$

with the cut from $1/2$ to $+\infty$, instead of the simple pole. The renormalon ambiguity of $\widetilde{\Sigma}(\omega)/\omega$ is defined, as before, as the difference of the integrals (4.7) below and above the real axis divided by $2\pi i$:

$$
\begin{aligned}
\Delta\bar{\Lambda} &= \frac{r\omega}{\beta_0}\frac{1}{2\pi i}\int\limits_C \frac{du}{\left(\frac{1}{2}-u\right)^{1+a}} \exp\left[-\frac{4\pi}{\beta_0\alpha_s(\mu_0)}u\right] \\
&= \frac{r}{2\beta_0\Gamma(1+a)}(-2\omega)\exp\left[-\frac{2\pi}{\beta_0\alpha_s(\mu_0)}\right]\left(\frac{\beta_0\alpha_s(\mu_0)}{4\pi}\right)^{-a}
\end{aligned}
\tag{4.8}
$$

(Fig. 5; we have used $\Gamma(-a)\Gamma(1+a) = -\pi/\sin\pi a$). But this must be just some number times $\Lambda_{\overline{\mathrm{MS}}}$, and cannot depend on ω!

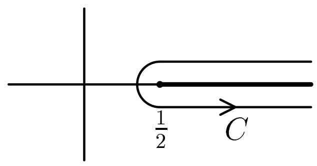


Figure 5: Integration contour

We have to use a formula for $\alpha_s(\mu)$ more precise than the one-loop one (1.10). The renormalization-group equation is solved by separation of variables:

$$
\begin{aligned}
&\frac{2\pi}{\beta_0}\int\frac{d\alpha_s}{\alpha_s^2}\left[1-\frac{\beta_1}{\beta_0}\frac{\alpha_s}{4\pi}+\mathcal{O}(\alpha_s^2)\right] = -\int d\log\mu\,, \\
&\frac{2\pi}{\beta_0\alpha_s(\mu)}+\frac{\beta_1}{2\beta_0^2}\log\frac{\alpha_s(\mu)}{4\pi}+\mathcal{O}(\alpha_s) = \log\frac{\mu}{\Lambda_{\overline{\mathrm{MS}}}}\,,
\end{aligned}
$$

and hence

$$
\Lambda_{\overline{\mathrm{MS}}} = \mu\exp\left[-\frac{2\pi}{\beta_0\alpha_s(\mu)}\right]\left(\frac{\alpha_s(\mu)}{4\pi}\right)^{-\frac{\beta_1}{2\beta_0^2}}\left[1+\mathcal{O}(\alpha_s)\right]\,. \tag{4.9}
$$

The UV renormalon ambiguity $\Delta\bar{\Lambda}$ must be equal to $\Lambda_{\overline{\mathrm{MS}}}$ times some number:

$$
\Delta\bar{\Lambda} = N_0\Delta_0\,,\quad \Delta_0 = -2C_F e^{5/6}\frac{\Lambda_{\overline{\mathrm{MS}}}}{\beta_0}\,. \tag{4.10}
$$

The normalization factor N_0 is only known in the large-β_0 limit:

$$
N_0 = 1+\mathcal{O}(1/\beta_0)\,;
$$

in general, it is just some unknown number of order 1. Comparing (4.8) with (4.10), we conclude that at $u\to 1/2$

$$
S(u) = -\frac{4C_F N_0'}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}}\left[1+\mathcal{O}\left(\tfrac{1}{2}-u\right)\right]\,, \tag{4.11}
$$

where

$$N_0' = N_0 \Gamma\left(1 + \frac{\beta_1}{2\beta_0^2}\right) \beta_0^{\frac{\beta_1}{2\beta_0^2}} . \tag{4.12}$$

The result for the power is exact; the normalization cannot be found within this approach.

The self-energy with a kinetic-energy insertion $\widetilde{\Sigma}_k$ (Fig. 6) can be also easily calculated in the large-β_0 limit. In the Landau gauge, raising the gluon denominator to the power $n = 1 + (L-1)\varepsilon$:

$$\widetilde{\Sigma}_k(\omega) = iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{v^\mu \left[2(k\cdot v+\omega)k_\perp^\nu - k_\perp^2 v^\nu\right]}{(-k^2)^n (k\cdot v+\omega)^2} \left[g_{\mu\nu} + \frac{k_\mu k_\nu}{-k^2}\right] ,$$

we obtain (4.2) with

$$N(\varepsilon, u) = 2C_F(3-2\varepsilon)^2 \omega^2 , \tag{4.13}$$

and hence

$$\Delta\widetilde{\Sigma}_k(\omega) = -3\omega\Delta\bar{\Lambda} . \tag{4.14}$$

This leads to the UV renormalon ambiguity of the heavy-quark field renormalization constant [23]

$$\Delta\widetilde{Z}_Q = -\frac{3}{2}\frac{\Delta\bar{\Lambda}}{m} . \tag{4.15}$$

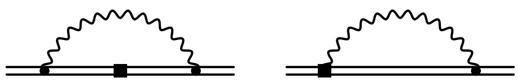


Figure 6: One-loop diagrams for $\widetilde{\Sigma}_k$

Let's now discuss the heavy–light quark current in HQET. If the light quark is massless, we may take $\frac{1}{4}\,\mathrm{Tr}$ of γ-matrices on the light-quark line of any diagram for $\widetilde{\Gamma}(\omega, 0)$. All diagrams with insertions to the gluon propagator of the one-loop diagram (Fig. 7b), as well as this one-loop diagram itself in the Landau gauge, vanish due to transversality of the gluon propagator. Therefore, to the first order in $1/\beta_0$, $\widetilde{\Gamma}(\omega,0) = 1$, and $\widetilde{\gamma}_j = \frac{1}{2}(\widetilde{\gamma}_Q + \gamma_q)$ in the Landau gauge. This anomalous dimension is gauge-invariant, and [2]

$$\begin{aligned}\widetilde{\gamma}_j &= -C_F \frac{\alpha_s}{4\pi} \frac{1+\frac{2}{3}\beta}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \\ &= -3C_F\frac{\alpha_s}{4\pi}\left[1 + \tfrac{5}{6}\beta - \tfrac{35}{36}\beta^2 + \cdots\right] ,\end{aligned} \tag{4.16}$$

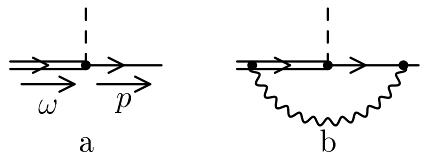


Figure 7: Proper vertex $\widetilde{\Gamma}(\omega, p)$ of a heavy–light HQET current

from (3.4) and (4.4). This perturbative series converges at $\beta_0|\alpha_s| < 4\pi$. It reproduces the leading-β_0 terms in the two- and three-loop results [15, 24, 17]. Note that $\widetilde{\gamma}_j = \frac{1}{2}\gamma_{j0}$ (3.9) at the first order in $1/\beta_0$.

Finally, we discuss the heavy–heavy current

$$\widetilde{J} = \widetilde{Z}_J^{-1}(\cosh\vartheta)\widetilde{J}_0\,, \quad \widetilde{J}_0 = \widetilde{Q}^{+}_{v'0}\widetilde{Q}_{v0}\,, \quad \cosh\vartheta = v\cdot v'$$

in HQET. At one loop (Fig. 8), we use the Fourier transform of the Landau-gauge gluon propagator with the denominator raised to the power n,

$$\frac{i}{2^{2n-1}\pi^{d/2}}\,\frac{\Gamma(d/2-n)}{\Gamma(n+1)}\,\frac{(2n-1)x^2 g_{\mu\nu} + (d-2n)x_\mu x_\nu}{(-x^2+i0)^{d/2-n+1}}\,.$$

to obtain the coordinate-space vertex

$$\begin{aligned}
&\widetilde{\Gamma}_1(t,t';\cosh\vartheta) = -C_F\frac{g_0^2}{2^{2n+1}\pi^{d/2}}\frac{\Gamma(d/2-n)}{\Gamma(n+1)}\theta(t)\theta(t')\\
&\times\frac{(2n-1)x^2\cosh\vartheta + (d-2n)(t+t'\cosh\vartheta)(t'+t\cosh\vartheta)}{(-x^2)^{d/2}}\,,\\
&x^2 = t^2 + t'^2 + 2tt'\cosh\vartheta\,.
\end{aligned}$$

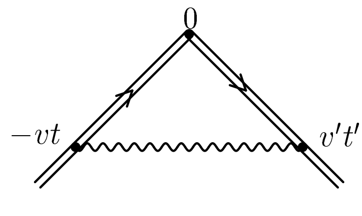


Figure 8: One-loop heavy–heavy vertex

The momentum-space vertex function is expressed via the coordinate-space one as

$$\widetilde{\Gamma}(\omega,\omega';\cosh\vartheta) = \int dt\,dt'\,e^{i\omega t + i\omega' t'}\widetilde{\Gamma}(t,t';\cosh\vartheta)\,.$$

Ultraviolet divergences of $\widetilde{\Gamma}(\omega,\omega';\cosh\vartheta)$ do not depend on the residual energies ω, ω', and we may nullify them. An infrared cutoff is then necessary to avoid IR $1/\varepsilon$ terms. Proceeding to the variables

$$t=\tau\frac{1+\xi}{2}\,,\quad t'=\tau\frac{1-\xi}{2}\,,$$

we obtain the coefficient $a_1(n)$

$$a_1(n)=-C_F 2^{d-2n-2}\frac{\Gamma(d/2-n)}{\Gamma(n+1)}\int\limits_0^T\frac{d\tau}{\tau^{d-2n-1}}\times$$
$$\int\limits_{-1}^{+1}d\xi\,\frac{(2n-1)\cosh\vartheta(c^2-s^2\xi^2)+(d-2n)(c^4-s^4\xi^2)}{(-c^2+s^2\xi^2)^{d/2-n+1}}\,,$$

where $c=\cosh\frac{\vartheta}{2}$, $s=\sinh\frac{\vartheta}{2}$, and the upper limit T provides an infrared cutoff. Changing the integration variable $\xi=\tanh\psi/\tanh(\vartheta/2)$, we obtain

$$\begin{aligned}a_1(n)=C_F\frac{\Gamma(d/2-n-1)}{\Gamma(n+1)}\left(\frac{i}{2}T\cosh\frac{\vartheta}{2}\right)^{2n+2-d}&\int\limits_{-\vartheta/2}^{+\vartheta/2}\frac{d\psi}{\cosh^{2n+2-d}\psi}\\ \times\left[\left(\frac{d}{2}+n-1\right)\coth\vartheta+\frac{d/2-n}{\sinh\vartheta}\cosh 2\psi\right]&\end{aligned}$$

(it becomes real in the Euclidean space $T\to-iT_E$). Therefore (1.12),

$$\begin{aligned}F(\varepsilon,u)=&-C_F\frac{\Gamma(1-u)}{\Gamma(2+u-\varepsilon)}e^{\gamma\varepsilon}D(\varepsilon)^{u/\varepsilon-1}\left(\frac{i}{2}\mu T\cosh\frac{\vartheta}{2}\right)^{2u}\\ &\times\int\limits_{-\vartheta/2}^{+\vartheta/2}\frac{d\psi}{\cosh^{2u}\psi}\left[(2+u-2\varepsilon)\coth\vartheta+\frac{1-u}{\sinh\vartheta}\sinh 2\psi\right]\,.\end{aligned}$$

The anomalous dimension corresponding to $\widetilde{Z}_\Gamma$ is (1.14)

$$\widetilde{\gamma}_\Gamma=\frac{1}{3}C_F\frac{\alpha_s}{4\pi}\frac{2(1+\beta)\vartheta\coth\vartheta+1}{B(2+\beta,2+\beta)\Gamma(2+\beta)\Gamma(1-\beta)}\,.$$

In order to obtain $\widetilde{\gamma}_J=\widetilde{\gamma}_\Gamma+\widetilde{\gamma}_Q$, we add (4.4):

$$\begin{aligned}\widetilde{\gamma}_J&=\frac{2}{3}C_F\frac{\alpha_s}{4\pi}\frac{\vartheta\coth\vartheta-1}{B(2+\beta,2+\beta)\Gamma(1+\beta)\Gamma(1-\beta)}\\ &=4C_F\frac{\alpha_s}{4\pi}\left(1+\frac{5}{3}\beta-\frac{1}{3}\beta^2+\cdots\right)(\vartheta\coth\vartheta-1)\,.\end{aligned}\tag{4.17}$$

It vanishes at $\vartheta=0$ as expected. It reproduces the leading β_0 terms in the two-loop result [25].

5 On-shell Heavy Quark in QCD

Now, we turn to the on-shell mass and wave-function renormalization of a heavy quark in QCD at the order $1/\beta_0$. It is convenient [16] to introduce the function

$$T(t) = \frac{1}{4m}\,\mathrm{Tr}(\rlap{/}v + 1)\Sigma(mv(1+t))\,,$$

then the renormalization constants are

$$Z_m^{\mathrm{os}} = 1 - T(0)\,, \quad Z_Q^{\mathrm{os}} = \left[1 - T'(0)\right]^{-1}\,.$$

At one loop (Fig. 9), in the Landau gauge, with the gluon denominator raised to the power $n = 1 + (L-1)\varepsilon$, we have

$$\begin{aligned} a_1(n) &= -iC_F \int \frac{d^dk}{\pi^{d/2}} \frac{\mathrm{Tr}(\rlap{/}v+1)\gamma^\mu(\rlap{/}p + \rlap{/}k + m)\gamma^\nu}{4mD_1(t)D_2^n}\left[g_{\mu\nu} + \frac{k_\mu k_\nu}{D_2}\right] \\ &= -2iC_F \int \frac{d^dk}{\pi^{d/2}} \frac{1}{D_1(t)D_2^n}\left[1 - \frac{(d-2)D_2}{4m^2}(1-t) + \mathcal{O}(t^2)\right]\,, \end{aligned}$$

where

$$p = mv(1+t)\,, \quad D_1(t) = m^2 - (k+p)^2\,, \quad D_2 = -k^2\,,$$

Expanding $1/D_1(t)$ to the linear term in t and using the one-loop on-shell integrals

$$\begin{aligned} &\int \frac{d^dk}{\left[m^2 - (k+mv)^2\right]^{n_1}(-k^2)^{n_2}} = i\pi^{d/2} m^{d-2(n_1+n_2)} M(n_1,n_2)\,, \\ &M(n_1,n_2) = \frac{\Gamma(d-n_1-2n_2)\Gamma(-d/2+n_1+n_2)}{\Gamma(n_1)\Gamma(d-n_1-n_2)}\,, \end{aligned} \tag{5.1}$$

we find $F(\varepsilon,u)$ (1.12).

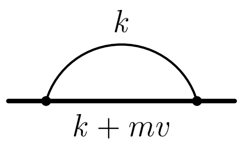


Figure 9: One-loop on-shell heavy-quark self-energy

The functions F for all on-shell quantities have a common Γ-function structure resulting from (5.1) with $n_2 = 1 + u - \varepsilon$:

$$F(\varepsilon,u) = \left(\frac{\mu}{m}\right)^{2u} e^{\gamma\varepsilon}\frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u-\varepsilon)} D(\varepsilon)^{u/\varepsilon-1} N(\varepsilon,u)\,. \tag{5.2}$$

The first Γ-function in the numerator, with the positive sign in front of u, comes from the second Γ-function in the numerator of (5.1), with the negative sign in front of d, and its poles are UV divergences. The second Γ-function in the numerator, with the negative sign in front of u, comes from the first Γ-function in the numerator of (5.1), with the positive sign in front of d, and its poles are IR divergences. For $T(t)$, we obtain [2]

$$N(\varepsilon, u) = 2C_F(3 - 2\varepsilon)(1 - u)\bigl[1 - (1 + u - \varepsilon)t\bigr] + \mathcal{O}(t^2)\,. \tag{5.3}$$

The on-shell mass renormalization constant $Z_m^{\rm os} = m_0/m = 1 - T(0)$ with the $1/\beta_0$ accuracy is given by (1.11), (5.2) with $N(\varepsilon, u)$ equal to minus (5.3) at $t = 0$. Retaining only terms with negative powers of ε, we obtain the $\overline{\rm MS}$ mass renormalization constant Z_m (because $Z_m^{\rm os}$ contains no IR divergences). Using (1.14), we reproduce the mass anomalous dimension (3.10). Retaining terms with ε^0, we get $Z_m^{\rm os}/Z_m(\mu) = m(\mu)/m$ in the form (1.15). As usual, it is convenient to express $m(\mu)$ via the renormalization-group invariant $\hat{m}$ (1.16). Then the ratio [18]

$$\begin{aligned}
&\frac{m}{\hat{m}} = 1 + \frac{1}{\beta_0}\int\limits_0^\infty du\, e^{-u/\beta} S(u) + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,,\\
&S(u) = 6C_F\left[\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u) - \frac{1}{2u}\right]\,.
\end{aligned} \tag{5.4}$$

The first Γ-function, with the positive sign in front of u, produces UV renormalons, while the second one, with the negative sign, produces IR renormalons (Fig. 4b). We can understand this from the power counting (Sect. 2). The QCD quark self-energy has a logarithmic UV divergence ($\nu_{\rm UV} = 0$), and hence UV renormalons are situated at $u < 0$. The index of the IR divergence of the on-shell quark self-energy is $\nu_{\rm IR} = -1$, and IR renormalons are at $u \geq 1/2$.

The ratio (5.4) can be represented [5] in the form (2.4). At $\tau > 1$, the distribution function is given by the sum (2.9) over the UV renormalons at $u = -n$, $n = 1, 2, 3, \ldots$:

$$w(\tau) = 6C_F \sum_{n=1}^\infty \frac{(n+1)\,(2n)!}{n!\,(n+2)!}\left(-\frac{1}{\tau}\right)^n = \frac{1}{2}C_F\tau^2\left[1 - \frac{6}{\tau^2} - \left(1 - \frac{2}{\tau}\right)\sqrt{1 + \frac{4}{\tau}}\right]\,.$$

At $\tau < 1$, it is the sum (2.8) over the IR renormalons at $u = n + \frac{1}{2}$ ($n = 0, 1, 2, \ldots$) and at $u = 2$:

$$\begin{aligned}
w(\tau) &= \frac{1}{2}C_F\left[\tau^2 - 3\sum_{n=0}^\infty \frac{(2n-1)\,(2n-1)!!\,(2n-5)!!}{(2n)!}\frac{\tau^{n+\frac{1}{2}}}{(-4)^n}\right]\\
&= \frac{1}{2}C_F\left[(2-\tau)\sqrt{\tau(4+\tau)} + \tau^2\right]
\end{aligned}$$

(both of these series are easily summed using the Newton binomial expansion). Finally, we obtain (Fig. 10)

$$w(\tau) = \frac{1}{2}C_F\left[(2-\tau)\sqrt{\tau(4+\tau)} + \tau^2 - 6\theta(\tau - 1)\right]\,. \tag{5.5}$$

At $\tau \to 0$, $w(\tau) \sim \sqrt{\tau}$, and at $\tau \to \infty$, $w(\tau) \sim 1/\tau$, according to the positions of the nearest IR and UV renormalons.

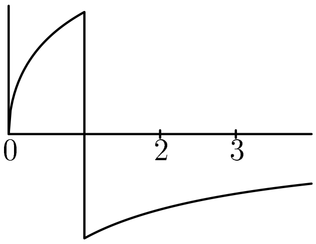


Figure 10: Virtuality distribution function

The IR renormalon ambiguity of the on-shell mass is [18], from the residue of $S(u)$ at the leading IR renormalon $u = 1/2$,

$$\Delta m = 2C_F e^{5/6} \frac{\Lambda_{\overline{\mathrm{MS}}}}{\beta_0} \,. \tag{5.6}$$

The meson mass is a measurable quantity, and must be unambiguous. In HQET, it is an expansion in $1/m$. Its leading term, m, is a short-distance quantity – a parameter of QCD. The first correction, $\bar{\Lambda}$, is a long-distance quantity, determined by the meson structure at the confinement scale. However, $\overline{\mathrm{MS}}$ regularization scheme contains no strict momentum cutoffs. As a result, the on-shell mass m also contains a contribution from large distances, where perturbation theory is ill-defined. This produces the IR renormalon ambiguity (5.6), which is suppressed by 1/m as compared to the leading term. Likewise, $\bar{\Lambda}$ contains a contribution from small distances, which leads to the UV renormalon ambiguity (4.6). They compensate each other in the physical quantity – the meson mass. In other words, in $\overline{\mathrm{MS}}$ the separation of the short- and long-distance contributions is ambiguous, though the full result is not.

This cancellation should hold beyond the large-β_0 limit. Therefore [20],

$$S(u) = \frac{2C_F N_0'}{\left(\frac{1}{2} - u\right)^{1+\frac{\beta_1}{2\beta_0^2}}} \left[1 + \mathcal{O}\left(\tfrac{1}{2} - u\right)\right] \,, \tag{5.7}$$

where the power is exact. The coefficients in the perturbative series

$$\frac{m}{\hat{m}} = 1 + \frac{1}{\beta_0} \sum_{L=1}^{\infty} c_L \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L$$

at $L \gg 1$ are, according to (1.24),

$$c_{n+1} = 2^{1+a} 2C_F N_0' (2\beta_0)^n (1+a)_n \left[1 + \mathcal{O}(1/n)\right] \,, \quad a = \frac{\beta_1}{2\beta_0^2} \,.$$

From the Stirling formula, $\Gamma(n+1+a) = n^a n![1+\mathcal{O}(1/n)]$, and we arrive at

$$c_{n+1} = 4C_F N_0\, n!\, (2\beta_0)^n (2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left[1+\mathcal{O}(1/n)\right] . \tag{5.8}$$

This result is model-independent.

Our calculation of $T(t)$ also yields $Z_Q^{\text{os}} = [1-T'(0)]^{-1}$ at the first order in $1/\beta_0$. It has the form (1.11), (5.2) with

$$N_Z(\varepsilon,u) = -2C_F(3-2\varepsilon)(1-u)(1+u-\varepsilon) \tag{5.9}$$

(see (5.3)). If we retain only negative powers of ε, we obtain $Z_q(\mu)/\widetilde{Z}_Q(\mu)$ (because $\widetilde{Z}_Q^{\text{os}} = 1$). Therefore, calculating the corresponding anomalous dimension by (1.14), we obtain

$$\gamma_q - \widetilde{\gamma}_Q = 2C_F \frac{\alpha_s}{4\pi} \frac{(1+\beta)\left(1+\frac{2}{3}\beta\right)}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) .$$

This difference is gauge-invariant at the $1/\beta_0$ level; it agrees with (3.4), (4.4). If we retain terms with ε^0, we get the finite combination $Z_Q^{\text{os}}\widetilde{Z}_Q(\mu)/Z_q(\mu)$ of the form (1.15); the corresponding renormalization-group invariant (1.20) has [28]

$$S(u) = -6C_F\left[\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u^2) - \frac{1}{2u}\right] .$$

6 Chromomagnetic Interaction

Now we shall discuss [23] the chromomagnetic interaction coefficient $C_m(\mu)$ in the HQET Lagrangian. It is defined by matching the on-shell scattering amplitudes in an external chromomagnetic fields in QCD and HQET at the linear order in the momentum transfer q. All loop diagrams in HQET contain no scale and hence vanish. The QCD amplitude at the first order in $1/\beta_0$ is given by the L-loop diagrams with $L-1$ quark loops (Fig. 11). The results have the form (5.2). The diagram of Fig. 11a is calculated in the standard way, and gives

$$N_a(\varepsilon,u) = (2C_F - C_A)(3+2u-u^2-5\varepsilon+3\varepsilon u-2\varepsilon u^2+2\varepsilon^2-2\varepsilon^2 u) .$$

We should sum the diagrams in 11b in l from 0 to $L-1$. All terms in the sum are equal, so that the summation just gives the factor $L = u/\varepsilon$:

$$N_b(\varepsilon,u) = C_A(5-2u-3\varepsilon)\frac{u}{\varepsilon} .$$

In order to calculate the diagrams in Fig. 11c, we need the triangle quark loop with the linear accuracy in q; it is a combination of one-loop propagator integrals (3.1). Again, all terms in the sum in l from 0 to $L-2$ are equal, and the summation just gives the factor $L-1$:

$$N_c(\varepsilon,u) = -C_A\frac{10-4u-28\varepsilon+9\varepsilon u+23\varepsilon^3-4\varepsilon^2 u-6\varepsilon^3}{2(1-\varepsilon)}\left(\frac{u}{\varepsilon}-1\right) .$$

Also, we should include the one-loop on-shell quark wave-function renormalization contribution (5.9) multiplied by the Born scattering amplitude (which is just 1). Finally, we arrive at [23]

$$\begin{aligned} N(\varepsilon,u) &= C_F N_F(\varepsilon,u) + C_A N_A(\varepsilon,u)\,, \\ N_F(\varepsilon,u) &= 4u(1+u-2\varepsilon u)\,, \\ N_A(\varepsilon,u) &= \frac{2-u-\varepsilon}{2(1-\varepsilon)}(2+3u-5\varepsilon-6\varepsilon u+2\varepsilon^2+4\varepsilon^2 u)\,. \end{aligned} \tag{6.1}$$

The sum is regular at the origin $\varepsilon = u = 0$, unlike separate contributions. It reproduces the leading β_0 terms in the two-loop result [26].

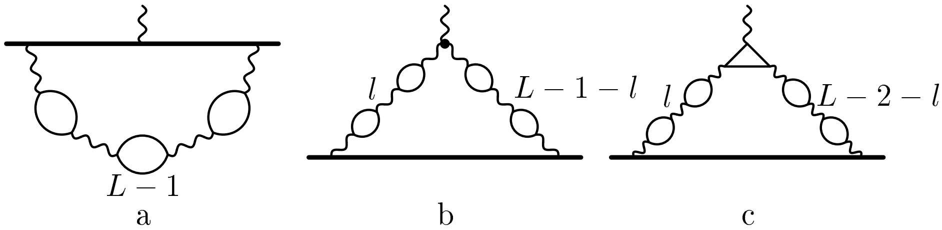


Figure 11: Quark scattering in an external gluon field

Now we can easily find the anomalous dimension $\widetilde{\gamma}_m$ and $C_m(\mu)$ with the $1/\beta_0$ accuracy. The anomalous dimension (1.14) is [23]

$$\begin{aligned} \widetilde{\gamma}_m &= C_A \frac{\alpha_s}{2\pi} \frac{\beta(1+2\beta)\Gamma(5+2\beta)}{24(1+\beta)\Gamma^3(2+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right) \\ &= C_A \frac{\alpha_s}{2\pi}\left[1 + \tfrac{13}{6}\beta - \tfrac{1}{2}\beta^2 + \cdots\right]\,. \end{aligned} \tag{6.2}$$

It reproduces the leading-β_0 term of the two-loop result [27, 26]

$$\widetilde{\gamma}_m = C_A \frac{\alpha_s}{2\pi}\left[1 + (13\beta_0 - 25C_A)\frac{\alpha_s}{24\pi} + \cdots\right]\,.$$

The perturbative series (6.2) converges at $\beta_0|\alpha_s| < 4\pi$.

The renormalization-group invariant $\hat{C}_m$ corresponding to $C_m(\mu)$ (see (1.16)) has the form (1.20) with [23]

$$S(u) = \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}\left[4u(1+u)C_F + \frac{1}{2}(2-u)(2+3u)C_A\right] - \frac{C_A}{u}\,. \tag{6.3}$$

The renormalon poles coincide with those in Fig. 4b. Taking the residue at the leading IR pole $u = 1/2$ and comparing with (4.6), we obtain

$$\Delta\hat{C}_m = -\left(1 + \frac{7}{8}\frac{C_A}{C_F}\right)\frac{\Delta\bar{\Lambda}}{m}\,. \tag{6.4}$$

In physical quantities, such as the mass splitting $m_{B^*} - m_B$, this IR renormalon ambiguity is compensated by UV renormalon ambiguities of the matrix elements in the $1/m$ correction. Detailed investigation of this cancellation allows one to find the exact nature of the singularity of $S(u)$ at $u = 1/2$: it is a branching point, a sum of three terms with different fractional powers of $\frac{1}{2} - u$, where the powers are known exactly, but the normalizations – only in the large-β_0 limit. The large-L asymptotics of the perturbative series for $\hat{C}_m$ can be found. These results have been obtained in [23]. We shall not discuss them here, because they require the use of $1/m^2$ terms in the HQET Lagrangian. A similar analysis of bilinear heavy–light currents will be presented in the next Section.

We can rewrite $\hat{C}_m$ in the form (2.4) with [23]

$$\begin{aligned} w(\tau) &= C_F w_F(\tau) + C_A w_A(\tau)\,, \\ w_F(\tau) &= 2\tau \left[\frac{2 + 4\tau + \tau^2}{\sqrt{\tau(4+\tau)}} - 2 - \tau \right], \\ w_A(\tau) &= \frac{\tau}{4} \left[\frac{14 + 5\tau}{\sqrt{\tau(4+\tau)}} - 5 \right] - \theta(\tau - 1) \end{aligned} \tag{6.5}$$

(Fig. 12; these formulae can be derived in the same way as (5.5)).

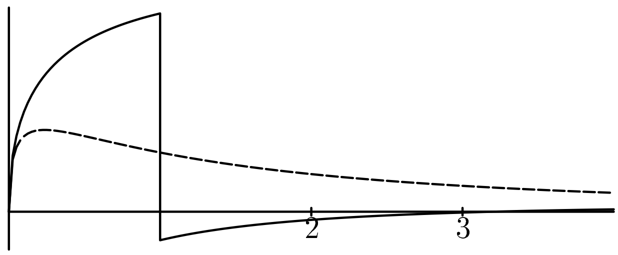


Figure 12: Distribution functions w_{F} (dashed line) and w_{A} (solid line)

7 Heavy–Light Currents

Hadronic matrix elements of QCD operators, such as quark currents j, are expanded in $1/m$

$$<j> = C<\tilde{j}> + \frac{1}{2m} \sum B_i <\tilde{O}_i> + \mathcal{O}\left(\frac{1}{m^2}\right)\,, \tag{7.1}$$

to separate short-distance contributions – the matching coefficients C, B_i, ..., and long-distance ones – HQET matrix elements $<\tilde{j}>$, $<\tilde{O}_i>$, ... The QCD matrix element $<j>$ contains no renormalon ambiguities, because the operator j has the lowest dimensionality in its channel. In schemes without strict separation of large and small momenta, such as $\overline{\text{MS}}$, this procedure artificially introduces infrared renormalon ambiguities in matching coefficients and ultraviolet renormalon ambiguities in HQET matrix elements. When calculating matching coefficients C, ..., we integrate over all loop momenta, including small ones. Therefore, they contain, in addition to the main short-distance contributions, also contributions from large distances, where the perturbation theory is ill-defined. They produce infrared renormalon singularities, which lead to ambiguities $\sim (\Lambda_{\overline{\text{MS}}}/m)^n$ in the matching coefficients C, ... Similarly, HQET matrix elements of higher-dimensional operators $<\widetilde{O}_i>$, ... contain, in addition to the main large-distance contributions, also contributions from short distances, which produce several UV renormalon singularities at positive u. They lead to ambiguities of the order $\Lambda^n_{\overline{\text{MS}}}$ times lower-dimensional matrix elements (e.g., $<\widetilde{\jmath}>$). These two kinds of renormalon ambiguities have to cancel in physical full QCD matrix elements $<j>$ (7.1) [28] (see also [29]).

Let's consider the leading QCD/HQET matching coefficient $C_\Gamma(\mu)$ for the heavy–light current with a Dirac matrix Γ having the properties

$$\rlap/v\Gamma = \sigma\Gamma\rlap/v\,, \quad \sigma = \pm 1\,,$$

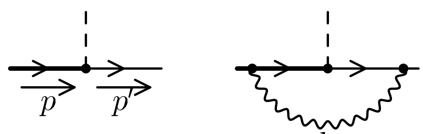
and (3.6). The QCD vertex function $\Gamma(mv,0)$ at one loop (Fig. 13), in the Landau gauge, with the gluon denominator raised to a power n, can be calculated, once and for all Dirac matrices, via $h(d)$ (3.7) [2]:

$$a_1(n) = \frac{iC_F}{2(d-1)} \int \frac{d^d k}{\pi^{d/2}} \frac{2(d-1) + (dD_2/m^2 + 4)h - 2(D_2/m^2+4)h^2}{D_1 D_2^n}\,.$$

At the first order in $1/\beta_0$, $\Gamma(mv,0)$ has the form (1.11), (5.2) with [2]

$$N(\varepsilon,u) = -C_F\left(2 - u - \varepsilon + 2uh - 2h^2\right)\,. \tag{7.2}$$

In order to obtain the renormalized matrix element $\left(Z_Q^{\text{os}}\right)^{1/2}\Gamma(mv,0)$, we should add $\frac{1}{2}N_Z(\varepsilon,u)$ (5.9).



a b

Figure 13: Proper vertex $\Gamma(p,p')$ of a heavy–light QCD current

With the considered accuracy, all loop corrections in HQET vanish. Retaining negative powers of ε in $\left(Z_Q^{\text{os}}\right)^{1/2}\Gamma(mv,0)$, we obtain $Z_j(\mu)/\widetilde{Z}_j(\mu)$. The corresponding anomalous dimension (1.14) is

$$\gamma_{jn} - \widetilde{\gamma}_j = C_F \frac{\alpha_s}{12\pi} \frac{2+\beta-2(n-2-\beta)^2 + (3+2\beta)(1+\beta)}{B(2+\beta,2+\beta)\Gamma(3+\beta)\Gamma(1-\beta)} + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\,, \tag{7.3}$$

in agreement with (3.9) and (4.16). Retaining ε^0 terms, we obtain $C_\Gamma(\mu)$ in the form (1.15). The corresponding renormalization-group invariant (1.16) has the form (1.20) with [2]

$$S(u) = -C_F \left\{ \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \left[5 - u - 3u^2 + 2u\eta(n-2) - 2(n-2)^2\right] \right.$$
$$\left. - \frac{5-2(n-2)^2}{2u} \right\}. \qquad (7.4)$$

Comparing the residue at the leading IR renormalon $u = \frac{1}{2}$ with $\Delta\bar\Lambda$ (4.6), we obtain the ambiguity of the matching coefficient [2]

$$\Delta C_\Gamma(\mu) = \frac{1}{3} \left[\frac{15}{4} + \eta(n-2) - 2(n-2)^2 \right] \frac{\Delta\bar\Lambda}{m}. \qquad (7.5)$$

Matching coefficients for the currents with γ_5^{AC} and γ_5^{HV} have identical $S(u)$ and $\hat C_\Gamma$; they only differ by $K_\gamma(\alpha_s)$ in (1.16). For notational convenience, we shall use the v rest frame. From (7.2) and [2]

$$Z_P = \frac{C_1}{C_{\gamma_5^{\mathrm{HV}}}}, \quad Z_A = \frac{C_{\gamma^0}}{C_{\gamma_5^{\mathrm{HV}}\gamma^0}} = \frac{C_{\gamma^i}}{C_{\gamma_5^{\mathrm{HV}}\gamma^i}}$$

we trivially reproduce (3.16). Taking into account [2]

$$\frac{m}{m(\mu)} = \frac{C_1(\mu)}{C_{\gamma^0}(\mu)},$$

the result (7.2) also reproduces the corresponding formula for $m/m(\mu)$, namely (5.3) with $t = 0$.

The ratio

$$\frac{f_{B^*}}{f_B} = \frac{C_{\gamma^i}}{C_{\gamma^0}}$$

is given by (1.20) with [28]

$$S(u) = 4C_F \frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u)}. \qquad (7.6)$$

It can be rewritten in the form (2.4). Summing the series (2.8), (2.9) over the residues of $S(u)$, we obtain [5]

$$w(\tau) = -\frac{2}{3} C_F \left[(1+\tau)\sqrt{\tau(4+\tau)} - \tau(3+\tau) \right] \qquad (7.7)$$

(Fig. 14).

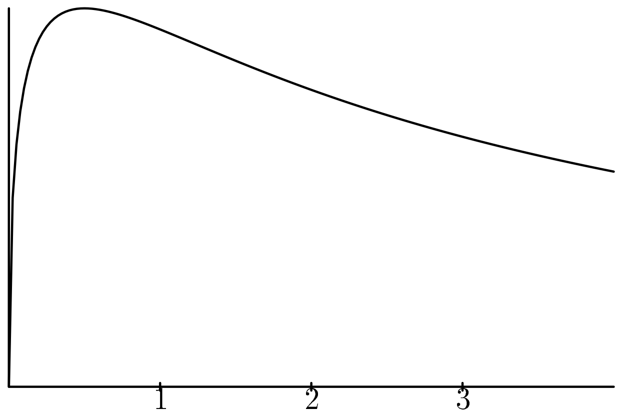


Figure 14: Virtuality distribution function for f_{B^*}/f_B

Now we shall discuss the expansion (7.1) for the matrix elements from B to vacuum of the currents with $\Gamma = \gamma_5^{\rm AC}$, $\gamma_5^{\rm AC}\gamma^0$ in more detail [30, 31]. The leading HQET current $\widetilde{\jmath} = \bar{q}\gamma_5^{\rm AC}\widetilde{Q}$ has the matrix element

$$<0|\widetilde{\jmath}(\mu)|B> = -i\sqrt{m_B}F(\mu)\,. \tag{7.8}$$

There are 4 dimension-4 HQET operators: 2 local ones and 2 bilocal ones. The local operators are full derivatives, and their matrix elements are expressed via $\bar{\Lambda}$ times (7.8). The bilocal operators are

$$\widetilde{O}_{jk} = i\int dx\ \mathrm{T}\left\{\widetilde{\jmath}(0),\widetilde{O}_k(x)\right\}\,,\quad \widetilde{O}_{jm} = i\int dx\ \mathrm{T}\left\{\widetilde{\jmath}(0),\widetilde{O}_m(x)\right\}\,, \tag{7.9}$$

where $\widetilde{O}_k$, $\widetilde{O}_m$ are the kinetic operator and the chromomagnetic one in the HQET lagrangian. Their matrix elements are

$$<0|\widetilde{O}_{jk}(\mu)|B> = -i\sqrt{m_B}F(\mu)G_k(\mu)\,,\quad <0|\widetilde{O}_{jm}(\mu)|B> = -i\sqrt{m_B}F(\mu)G_m(\mu)\,. \tag{7.10}$$

In the leading logarithmic approximation (LLA),

$$G_k(\mu) = \hat{G}_k - \bar{\Lambda}\frac{\widetilde{\gamma}_0^k}{2\beta_0}\log\frac{\alpha_s(\mu)}{4\pi}\,,\quad G_m(\mu) = \hat{G}_m\left(\frac{\alpha_s(\mu)}{4\pi}\right)^{\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} + \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\bar{\Lambda}\,, \tag{7.11}$$

where $\hat{G}_{k,m}$ do not depend on μ, and $\widetilde{\gamma}^{k,m}$ are mixing anomalous dimensions of $\widetilde{O}_{jk,jm}$ with local operators, see [31]. The renormalization-group invariant QCD matrix elements of the pseudoscalar current and the axial one are in LLA

$$\begin{aligned}\left\{\begin{array}{c}\hat{f}_B^P\\ f_B\end{array}\right\} = \left(\frac{\alpha_s(m)}{4\pi}\right)^{\frac{\widetilde{\gamma}_{j0}}{2\beta_0}}\frac{\hat{C}_\Gamma\hat{F}}{\sqrt{m_B}}\Bigg\{1 &+ \\ + \frac{1}{2m}\left[\left(-\frac{\widetilde{\gamma}_0^k}{2\beta_0}\log\frac{\alpha_s(m)}{4\pi} \pm 1 + \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right)\bar{\Lambda} + \hat{G}_k + \hat{G}_m\left(\frac{\alpha_s(m)}{4\pi}\right)^{\frac{\widetilde{\gamma}_{m0}}{2\beta_0}}\right]&\Bigg\}\,.\end{aligned} \tag{7.12}$$

The next-to-leading corrections are also known [32, 31].

We are interested in UV renormalon ambiguities of the matrix elements of $\widetilde{O}_{jk,jm}$. By dimensional analysis, they are proportional to $\Delta\bar{\Lambda}$ times the matrix element of the lower-dimensional operator $\widetilde{\jmath}$ with the same external states. We may use quark states instead of hadron ones. Specifically, we consider transition from an off-shell heavy quark with residual energy $\omega < 0$ to a light quark with zero momentum, this is enough to ensures the absence of IR divergences. For $\widetilde{O}_{jk}$, all loop corrections to the vertex function vanish. The kinetic-energy vertices contain no Dirac matrices, and we may take $\frac{1}{4}\,\mathrm{Tr}$ on the light-quark line; this yields k^α at the vertex, and the gluon propagator with insertions is transverse. There is one more contribution: the matrix element of $\widetilde{\jmath}$ should be multiplied by the heavy-quark wave-function renormalization $\widetilde{Z}_Q^{1/2}$, which has an UV renormalon ambiguity (4.15). Therefore [28],

$$\Delta G_{\mathrm{k}}(\mu) = -\frac{3}{2}\Delta\bar{\Lambda} \tag{7.13}$$

(this ambiguity is μ-independent at the first order in $1/\beta_0$).

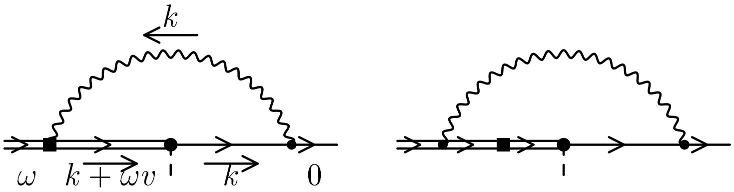


Figure 15: Matrix element of $\widetilde{O}_{jk}$

For $\widetilde{O}_{jm}$, a straightforward calculation of the diagram in Fig. 16 gives the bare matrix element of the usual form (1.11), (4.2) with

$$N(\varepsilon, u) = -6C_F C_m^0 \frac{\omega}{m} \,. \tag{7.14}$$

The renormalization-group invariant matrix element has the form (1.20) with $\mu_0 = -2\omega e^{-5/6}$ and

$$S(u) = -6C_F C_m(\mu_0)\frac{\omega}{m}\left(\frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u)} + \frac{1}{2u}\right) \,. \tag{7.15}$$

Taking the residue at the pole $u = 1/2$, we find the UV renormalon ambiguity $C_m(\mu_0)\Delta\bar{\Lambda}/m$ times the matrix element of $\widetilde{\jmath}$, and we obtain [28]

$$\Delta G_m(\mu) = 2\Delta\bar{\Lambda} \tag{7.16}$$

(again, μ-independent at this order).

In the full QCD matrix elements (7.12), the IR renormalon ambiguities (7.5) of the leading matching coefficients C_Γ are compensated, at the $1/\beta_0$ order, by the UV renormalon

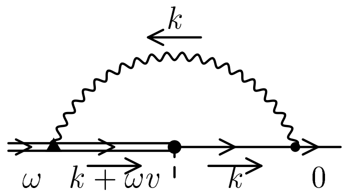


Figure 16: Matrix element of $\widetilde{O}_{jm}$

ambiguities of the subleading matrix elements $\Delta\bar{\Lambda}$ (4.6) and $\Delta G_{k,m}$ (7.13), (7.16). This cancellation must hold beyond the large-β_0 limit. The subleading matrix elements are controlled by the renormalization group. The requirement of cancellation allows one to investigate the structure of the leading IR renormalon singularity of C_Γ [31].

In the large-β_0 limit,

$$\Delta\hat{G}_k = -\frac{3}{2}\Delta\bar{\Lambda}\,, \quad \Delta\hat{G}_m = \left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right)\Delta\bar{\Lambda}\,,$$

see (7.13), (7.16), (7.11). In general, they must be equal to $\Lambda_{\overline{\mathrm{MS}}}$ times some numbers:

$$\Delta\hat{G}_k = -\frac{3}{2}N_1\Delta_0\,, \quad \Delta\hat{G}_m = N_2\left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right)\Delta_0 \tag{7.17}$$

(see (4.10)). The normalization factors $N_{1,2}$ are only known in the large-β_0 limit:

$$N_i = 1 + \mathcal{O}(1/\beta_0)\,;$$

in general, they are just some unknown numbers of order 1. Using (4.9), we can represent the UV renormalon ambiguities of the $1/m$ corrections in (7.12) as $\exp[-2\pi/(\beta_0\alpha_s(\mu_0))]$ times a sum of terms with different fractional powers of $\alpha_s(\mu_0)/(4\pi)$. It is convenient to replace $\log[\alpha_s(\mu_0)/(4\pi)] \to [(\alpha_s(\mu_0)/(4\pi))^\delta - 1]/\delta$, and take the limit $\delta \to 0$ at the end of calculation.

In order to cancel this ambiguity, we should have the branching point

$$S_\Gamma(u) = \sum_i \frac{r_i}{\left(\frac{1}{2} - u\right)^{1+a_i}} \tag{7.18}$$

instead of a simple pole (7.4). Then (see (4.8))

$$\Delta\hat{C}_\Gamma = \frac{1}{\beta_0}\exp\left[-\frac{2\pi}{\beta_0\alpha_s(\mu_0)}\right]\sum_i \frac{r_i}{\Gamma(1+a_i)}\left(\frac{\beta_0\alpha_s(\mu_0)}{4\pi}\right)^{-a_i}\,. \tag{7.19}$$

The requirement of cancellation of the ambiguities in (7.12) gives for $\Gamma = 1$, γ^0

$$\begin{aligned} S_\Gamma(u) = \frac{C_F}{\left(\frac{1}{2} - u\right)^{1+\frac{\beta_1}{2\beta_0^2}}} \Bigg\{ &\left[-\frac{\widetilde{\gamma}_0^k}{2\beta_0}\left(\log\frac{\frac{1}{2}-u}{\beta_0} - \psi\left(1 + \frac{\beta_1}{2\beta_0^2}\right)\right) \pm 1 + \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right] N_0' \\ &- \frac{3}{2}N_1' + \left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2' \left(\tfrac{1}{2} - u\right)^{\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \Bigg\}\,. \end{aligned} \tag{7.20}$$

The similarly requirement gives for $\Gamma = \gamma^i$, $\gamma^i\gamma^0$

$$S_\Gamma(u) = \frac{C_F}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}} \left\{ \left[-\frac{\widetilde{\gamma}_0^k}{2\beta_0} \left(\log\frac{\frac{1}{2}-u}{\beta_0} - \psi\left(1+\frac{\beta_1}{2\beta_0^2}\right)\right) + \frac{1}{3}\left(\pm 1 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right)\right] N_0' \right.$$
$$\left. - \frac{3}{2}N_1' - \frac{1}{3}\left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2' \left(\tfrac{1}{2}-u\right)^{\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \right\}. \qquad (7.21)$$

Here

$$N_1' = N_1 \Gamma\left(1+\frac{\beta_1}{2\beta_0^2}\right) \beta_0^{\frac{\beta_1}{2\beta_0^2}}, \quad N_2' = N_2 \Gamma\left(1+\frac{\beta_1}{2\beta_0^2} - \frac{\widetilde{\gamma}_{m0}}{2\beta_0}\right) \beta_0^{\frac{\beta_1}{2\beta_0^2}-\frac{\widetilde{\gamma}_{m0}}{2\beta_0}}$$

(see (4.12)). Corrections $\mathcal{O}\left(\frac{1}{2}-u\right)$ were calculated in [31]. In the large-β_0 limit, the simple pole behaviour with (7.5) is reproduced.

The asymptotics of the perturbative coefficients c_L^Γ at $L \gg 1$ is determined by the renormalon singularity closest to the origin. Similarly to (5.8), we obtain, for $\Gamma = 1$, γ^0,

$$c_{n+1}^\Gamma = 2C_F\, n!\, (2\beta_0)^n\, (2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left[\left(\frac{\widetilde{\gamma}_0^k}{2\beta_0} \log 2\beta_0 n \pm 1 + \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_0 \right.$$
$$\left. - \frac{3}{2}N_1 + \left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2 (2\beta_0 n)^{-\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \right], \qquad (7.22)$$

and for $\Gamma = \gamma^i$, $\gamma^i\gamma^0$,

$$c_{n+1}^\Gamma = 2C_F\, n!\, (2\beta_0)^n\, (2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left[\left(\frac{\widetilde{\gamma}_0^k}{2\beta_0} \log 2\beta_0 n + \frac{1}{3}\left(\pm 1 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right)\right) N_0 \right.$$
$$\left. - \frac{3}{2}N_1 - \frac{1}{3}\left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2 (2\beta_0 n)^{-\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \right]. \qquad (7.23)$$

Corrections $\mathcal{O}(1/n)$ were calculated in [31].

For the ratio f_{B^*}/f_B, the Borel image of the perturbative series is

$$S(u) = \frac{4}{3}\frac{C_F}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}} \left[\left(1 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_0' - \left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2' \left(\tfrac{1}{2}-u\right)^{\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \right], \qquad (7.24)$$

and the asymptotics of the coefficients is

$$c_{n+1} = \frac{8}{3} C_F\, n!\, (2\beta_0)^n (2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left[\left(1 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_0 - \left(2 - \frac{\widetilde{\gamma}_0^m}{\widetilde{\gamma}_{m0}}\right) N_2 (2\beta_0 n)^{-\frac{\widetilde{\gamma}_{m0}}{2\beta_0}} \right]. \qquad (7.25)$$

8 Heavy–Heavy Currents

First we consider the leading matching coefficients for the currents $\bar{c}\Gamma b$ at $\vartheta = 0$ at $1/\beta_0$ order [33]. The one-loop vertex $\Gamma(m_b v, m_c v)$ (Fig. 17) contains integrals

$$\int \frac{d^d k}{[m_b^2 - (k + m_b v)^2]^{n_1} [m_c^2 - (k + m_c v)^2]^{n_2} (-k^2)^n} .$$

The denominators are linearly dependent. We can multiply the integrand by

$$1 = \frac{m_b [m_c^2 - (k + m_c v)^2] - m_c [m_b^2 - (k + m_b v)^2]}{(m_b - m_c)(-k^2)} ,$$

thus lowering n_1 or n_2, until one of these denominators disappear. Remaining integrals are single-mass (5.1). We have

$$\begin{aligned} a_1(n) &= C_F \frac{\Gamma(d-2n-1)\Gamma(-d/2+n+1)}{\Gamma(d-n-1)} \frac{m_b^{d-2n-1}\Phi(m_c/m_b) - m_c^{d-2n-1}\Phi(m_b/m_c)}{m_b - m_c} , \\ \Phi(r) &= \frac{d-1}{d-2n-3} r + \frac{2h^2 + (d-2n-2)h - d + n + 1}{d-n-1} . \end{aligned} \tag{8.1}$$

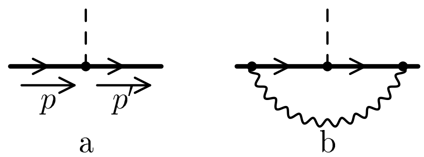


Figure 17: Proper vertex $\Gamma(p, p')$ of a heavy–heavy QCD current

Adding the on-shell wave-function renormalization (5.9) for b and c, we obtain

$$F(\varepsilon, u) = \left(\frac{\mu^2}{m_b m_c}\right)^u e^{\gamma\varepsilon} \frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u-\varepsilon)} D(\varepsilon)^{u/\varepsilon - 1} N(\varepsilon, u) , \tag{8.2}$$

$$\begin{aligned} N(\varepsilon, u) = 2C_F \Bigg[&\left((n-2)^2 - u\eta(n-2) + 2\varepsilon(n-2) - u\varepsilon\eta - \frac{4 - u^2 - 4\varepsilon - 2u\varepsilon^2}{1+2u} \right) R_1 \\ &+ (3 - 2\varepsilon)\frac{1-2u}{1+2u}(1 - u - u^2 + u\varepsilon) R_0 \Bigg] , \end{aligned} \tag{8.3}$$

where

$$R_0 = \cosh\frac{L}{2} , \quad R_1 = \frac{\sinh\frac{(1-2u)L}{2}}{\sinh\frac{L}{2}} , \quad L = \log\frac{m_b}{m_c} ,$$

for the on-shell QCD matrix element (which is equal to the matching coefficient, because all loop corrections in HQET vanish). The corresponding anomalous dimension (1.14) reproduces γ_{jn} (3.9), because $\widetilde{\gamma}_J = 0$ at $\vartheta = 0$. The function $S(u)$ (1.21) for the matching coefficient is

$$
\begin{aligned}
S(u) = C_F \biggl\{ & 2\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \biggl[\left((n-2)^2 - u\eta(n-2) - \frac{4-u^2}{1+2u} \right) R_1 \\
& + 3\frac{(1-2u)(1-u-u^2)}{1+2u} R_0 \biggr] - \frac{(n-1)(n-3)}{u} \biggr\}
\end{aligned}
\tag{8.4}
$$

with

$$\mu_0 = e^{-5/6}\sqrt{m_b m_c}$$

(see (1.19)). There is no pole at $u = 1/2$, the leading IR renormalon is at $u = 1$. Therefore, the IR renormalon ambiguity of the matching coefficients at $\vartheta = 0$ is $\sim (\Lambda_{\overline{\mathrm{MS}}}/m_{b,c})^2/\beta_0$.

For the vector and axial currents [33],

$$
\begin{aligned}
S_{\gamma^0}(u) &= 6C_F \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \frac{1-u-u^2}{1+2u} \Bigl[-R_1 + (1-2u)R_0 \Bigr] \\
&= 6C_F \left(\frac{L}{2}\coth\frac{L}{2} - 1 \right) \left(1 - \frac{3}{2}u + \cdots \right) , \\
S_{\gamma_5^{\mathrm{AC}}\boldsymbol{\gamma}}(u) &= 2C_F \frac{\Gamma(u)\Gamma(1-2u)}{(1+2u)\Gamma(3-u)} \Bigl[-(3-u+u^2)R_1 + (1-2u)(1-u-u^2)R_0 \Bigr] \\
&= C_F \left[3L\coth\frac{L}{2} - 8 - \left(\frac{5}{2}L\coth\frac{L}{2} - 6 \right) u + \cdots \right] .
\end{aligned}
\tag{8.5}
$$

The matching coefficients don't depend on μ, μ', and are given by (1.20):

$$
\begin{aligned}
H_{\gamma^0} &= 6C_F \frac{\alpha_s(\mu_0)}{4\pi} \left(\frac{L}{2}\coth\frac{L}{2} - 1 \right) \left(1 - \frac{3}{2}\beta + \cdots \right) , \\
H_{\gamma_5^{\mathrm{AC}}\boldsymbol{\gamma}} &= C_F \frac{\alpha_s(\mu_0)}{4\pi} \left[3L\coth\frac{L}{2} - 8 - \left(\frac{5}{2}L\coth\frac{L}{2} - 6 \right) \beta + \cdots \right] .
\end{aligned}
\tag{8.6}
$$

Now we consider the general case $\vartheta \neq 0$ [28]. For a generic Dirac matrix Γ satisfying

$$\gamma_\mu \Gamma \gamma^\mu = 2h(d)\Gamma ,$$

there are 4 leading HQET currents in the expansion:

$$
\begin{aligned}
& J = \sum_i H_i \widetilde{J}_i + \frac{1}{2m_b}\sum_i G_i \widetilde{O}_i + \frac{1}{2m_c}\sum_i G'_i \widetilde{O}'_i + \mathcal{O}(1/m_{b,c}^2) , \\
& J = \bar{c}\Gamma b , \quad \widetilde{J}_i = \bar{\widetilde{c}}_{v'}\Gamma_i \widetilde{b}_v , \quad \Gamma_i = \Gamma ,\ \not{v}\Gamma ,\ \Gamma\not{v}' ,\ \not{v}\Gamma\not{v}' .
\end{aligned}
\tag{8.7}
$$

The one-loop vertex $\Gamma(m_b v, m_c v')$ in the Landau gauge, with the gluon denominator raised to a power n, is

$$\Gamma_1 = iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu(\not{k} + m_c\not{v}' + m_c)\Gamma(\not{k} + m_b\not{v} + m_b)\gamma^\mu}{(-k^2)^n(-k^2 - 2m_b\, v\cdot k)(-k^2 - 2m_c\, v'\cdot k)} .$$

Using Feynman parametrization, we have

$$\begin{aligned}
&\Gamma_1 = \frac{iC_F n(n+1)}{\pi^{d/2}} \int \frac{(1-x-x')^{n-1} dx\, dx'\, d^d k'}{(a^2-k'^2)^{n+2}} \times \\
&\gamma_\mu(\not{k}' + m_c(1-x')\not{v}' - m_b x\not{v} + m_c)\Gamma(\not{k}' + m_b(1-x)\not{v} - m_c x'\not{v}' + m_b)\gamma^\mu\,, \\
&k' = k + m_b x v + m_c x' v'\,, \\
&a^2 = (m_b x v + m_c x' v')^2 = m_b^2 x^2 + m_c^2 x'^2 + 2 m_b m_c x x' \cosh\vartheta\,.
\end{aligned}$$

We calculate the loop integral and substitute $x = \xi(1+z)/2$, $x' = \xi(1-z)/2$,

$$a^2 = m_b m_c \xi^2 a_+ a_-\,, \quad a_\pm = \cosh\frac{L\pm\vartheta}{2} + z\sinh\frac{L\pm\vartheta}{2}\,.$$

The ξ integration is trivial, and we obtain H_i having the form (1.11), without the leading 1 for $H_{2,3,4}$; the functions $F_i(\varepsilon,u)$ have the form (8.2) with

$$\begin{aligned}
N_1(\varepsilon,u) = C_F &\left\{ \int\limits_{-1}^{+1} \frac{dz}{(a_+a_-)^{1+u}} \left[a_+a_- h^2(1-2u) - \frac{(1-z^2)h}{4} u(1-2u) \right.\right. \\
&\left. + (1+u-\varepsilon)(2-u-\varepsilon)\cosh\vartheta + (\cosh L + z\sinh L) u(2-u-\varepsilon) \right] \\
&\left. - (3-2\varepsilon)(1-u)(1+u-\varepsilon)(r^u + r^{-u}) \right\}, \\
N_2(\varepsilon,u) = &- C_F \frac{u}{2} \int\limits_{-1}^{+1} \frac{dz}{(a_+a_-)^{1+u}} \left[\frac{e^L(1+z)^2 h}{2}(1-2u) + (1-z)(2-u-\varepsilon) \right], \\
N_3(\varepsilon,u) = &- C_F \frac{u}{2} \int\limits_{-1}^{+1} \frac{dz}{(a_+a_-)^{1+u}} \left[\frac{e^{-L}(1-z)^2 h}{2}(1-2u) + (1+z)(2-u-\varepsilon) \right], \\
N_4(\varepsilon,u) = &- C_F \frac{hu(1-2u)}{4} \int\limits_{-1}^{+1} \frac{dz\,(1-z^2)}{(a_+a_-)^{1+u}}\,.
\end{aligned} \tag{8.8}$$

At $\vartheta = 0$, the integrals can be easily calculated, and $N_1+N_2+N_3+N_4$ reproduces (8.3). The anomalous dimension (1.14) corresponding to (8.8) is $\gamma_{jn} - \widetilde{\gamma}_J$, see (3.9), (4.17). The functions (1.21)

$$S_i(u) = \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} N_i(0,u) - \frac{N_i(0,0)}{2u}$$

have the leading IR renormalon pole at $u = 1/2$, thus producing the ambiguities (2.2)

$$\Delta H_i = -\frac{N_i(0,1/2)}{3C_F} \frac{\Delta\bar{\Lambda}}{\sqrt{m_b m_c}}$$

in the matching coefficients. It is easy to calculate $N_i(0,1/2)$ using the integrals

$$\int\limits_{-1}^{+1} \frac{dz}{(a_+a_-)^{3/2}} = \frac{4\cosh\frac{L}{2}}{\cosh\vartheta+1}\,, \quad \int\limits_{-1}^{+1} \frac{z\,dz}{(a_+a_-)^{3/2}} = -\frac{4\sinh\frac{L}{2}}{\cosh\vartheta+1}\,.$$

We obtain [28]

$$\begin{aligned} &\Delta H_1 = \left(\frac{1}{\cosh\vartheta+1} - \frac{3}{4}\right)\left(\frac{1}{m_c}+\frac{1}{m_b}\right)\Delta\bar\Lambda\,,\\ &\Delta H_2 = \frac{1}{\cosh\vartheta+1}\,\frac{\Delta\bar\Lambda}{2m_c}\,, \quad \Delta H_3 = \frac{1}{\cosh\vartheta+1}\,\frac{\Delta\bar\Lambda}{2m_b}\,, \quad \Delta H_4 = 0\,. \end{aligned} \tag{8.9}$$

They do not depend on the Dirac matrix Γ in the current. As expected, $\Delta H_1+\Delta H_2+\Delta H_3$ vanishes at $\vartheta=0$.

In matrix elements of QCD currents (8.7), these IR renormalon ambiguities in the leading matching coefficients H_i must be compensated by UV renormalon ambiguities in matrix elements of the subleading operators $\widetilde{O}_i$, $\widetilde{O}'_i$. There are two kinds of subleading operators – local and bilocal. First we consider local operators, whose coefficients are completely fixed by reparametrization invariance [34]:

$$\begin{aligned} \sum_{\text{local}} G_i\widetilde{O}_i = \bar{\tilde{c}}_{v'}\Big[&(H_1\,\Gamma + H_2\,\rlap{/}v\Gamma + H_3\,\Gamma\rlap{/}v' + H_4\,\rlap{/}v\Gamma\rlap{/}v')\,i\rlap{/}D + 2i\overleftarrow{\rlap{/}D}\,(H_2\,\Gamma + H_4\,\Gamma\rlap{/}v')\\ &+ 2\,(H_1'\,\Gamma + H_2'\,\rlap{/}v\Gamma + H_3'\,\Gamma\rlap{/}v' + H_4'\,\rlap{/}v\Gamma\rlap{/}v')\,iv'\cdot D\Big]\tilde{b}_v\,, \end{aligned} \tag{8.10}$$

where H_i' are derivatives in the argument $\cosh\vartheta$, and similarly for $\sum G_i'\widetilde{O}_i'$. These local operators contain either D_μ or $\overleftarrow{D}_\mu$. Let's decompose these derivatives into components in the (v,v') plane and those orthogonal to this plane. Projection of D_μ onto the longitudinal plane is

$$D_\mu \to \frac{(v_\mu\,v'\cdot D + v'_\mu\,v\cdot D)\cosh\vartheta - v_\mu\,v\cdot D - v'_\mu\,v'\cdot D}{\sinh^2\vartheta}\,,$$

and similarly for $\overleftarrow{D}_\mu$. All operators with longitudinal derivatives can be rewritten, using equations of motion, as full derivatives of the leading currents $\widetilde{J}_i$. When we are interested in matrix elements from a ground-state meson into a ground-state meson, we may replace

$$i\partial_\mu\widetilde{J}_i \to \bar\Lambda(v-v')_\mu\widetilde{J}_i\,.$$

In this case, projecting onto the longitudinal plane means

$$iD_\mu \to \bar\Lambda\frac{v_\mu\cosh\vartheta - v'_\mu}{\cosh\vartheta+1}\,, \quad -i\overleftarrow{D}_\mu \to \bar\Lambda\frac{v'_\mu\cosh\vartheta - v_\mu}{\cosh\vartheta+1}\,.$$

The longitudinal part of the local $1/m_{b,c}$ contribution (8.10) to the QCD matrix element $<J>$ is easily derived by this substitution. It, clearly, has an UV renormalon ambiguity proportional to $\Delta\bar{\Lambda}$. Matrix elements of operators with transverse derivatives cannot be written as matrix elements of the leading currents $\widetilde{J}_i$ times some scalar factors, they require new independent form factors. Therefore, they contain no UV renormalon ambiguities, which should have the form $\Delta\bar{\Lambda}$ times lower-dimensional matrix elements of $\widetilde{J}_i$. The above derivation is exact (not only valid in the large-β_0 limit). At the first order in $1/\beta_0$, we may replace $H_1 \to 1$, $H_{2,3,4} \to 0$, $H_i' \to 0$. The contribution of the local subleading operators to the ambiguity of $<J>$ in this approximation is [28]

$$\left(1 - \frac{1}{\cosh\vartheta + 1}\right)\left(\frac{1}{m_c} + \frac{1}{m_b}\right)\frac{\Delta\bar{\Lambda}}{2}<\widetilde{J}_1> - \frac{1}{\cosh\vartheta + 1}\left(\frac{\Delta\bar{\Lambda}}{2m_c}<\widetilde{J}_2> + \frac{\Delta\bar{\Lambda}}{2m_b}<\widetilde{J}_3>\right) . \quad (8.11)$$

Now we turn to bilocal subleading operators, and consider the operator

$$i\int dx \ \mathrm{T}\left\{\widetilde{J}_1(0), \widetilde{O}_{kc}(x)\right\}$$

with the insertion of the c-quark kinetic energy. It appears in the expansion (8.7) with the coefficient H_1. The one-loop vertex (Fig. 18) with the gluon denominator raised to the power n is

$$a_1(n) = i\frac{C_F}{2m_c}\int\frac{d^dk}{\pi^{d/2}}\left(k_\perp^\mu - \frac{k_\perp^2 v'^\mu}{k\cdot v' + \omega'}\right)\frac{v^\nu}{(k\cdot v + \omega)(k\cdot v' + \omega')(-k^2)^n}\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{-k^2}\right) ,$$

where $k_\perp = k - (k\cdot v')\,v'$. We are interested in the UV renormalon at $u = 1/2$; therefore, to make subsequent formulae shorter, we shall calculate $F(u)$ (1.18) instead of the full function $F(\varepsilon, u)$, and omit terms regular at $u = 1/2$. We also set $\omega' = \omega$ for simplicity, and obtain

$$\begin{aligned} F(u) = & -i\frac{C_F}{2m_c}u(-2\omega)^{2u}\left[\int\frac{d^4k}{\pi^2}\frac{1}{(-k\cdot v' - \omega)(-k^2)^{1+u}}\right. \\ & \left. + \int\frac{d^4k}{\pi^2}\frac{\cosh\vartheta}{(-k\cdot v - \omega)(-k\cdot v' - \omega)^2(-k^2)^u} + \cdots\right] , \end{aligned}$$

where dots mean integrals without linear UV divergences at $u = 0$ (and hence having no UV renormalon singularity at $u = 1/2$), and -2ω plays the role of m in the definition (1.18).

The first integral is trivial (see (4.1)):

$$-\frac{i}{\pi^2}(-2\omega)^{-1+2u}\int\frac{d^4k}{(-k\cdot v' - \omega)(-k^2)^{1+u}} = 2\frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(1+u)} .$$

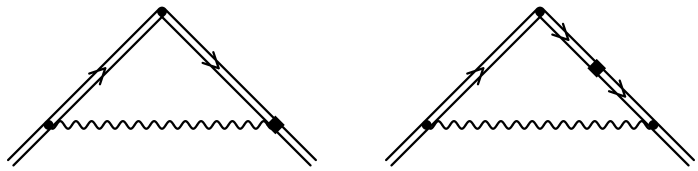


Figure 18: Kinetic-energy insertions into the c-quark line

For the second one, we use the HQET Feynman parametrization [15]:

$$\begin{aligned} I = & -\frac{i}{\pi^2}(-2\omega)^{-1+2u}\int\frac{d^4k}{(-k\cdot v-\omega)(-k\cdot v'-\omega)^2(-k^2)^u} \\ = & -8\frac{i}{\pi^2}(-2\omega)^{-1+2u}\frac{\Gamma(3+u)}{\Gamma(u)}\int\frac{y'\,dy\,dy'\,d^4k}{\left[-k^2-2yv\cdot k-2y'v'\cdot k-2\omega(y+y')\right]^{3+u}} \\ = & \, 8u(-2\omega)^{-1+2u}\int\frac{y'\,dy\,dy'}{\left[y^2+y'^2+2yy'\cosh\vartheta-2\omega(y+y')\right]^{1+u}}\,. \end{aligned}$$

The substitution $y=(-2\omega)\xi(1-z)/2$, $y'=(-2\omega)\xi(1+z)/2$ gives

$$I=2u\int\frac{\xi^{1-u}\,d\xi\,(1+z)\,dz}{\left[\left(\cosh^2\frac{\vartheta}{2}-z^2\sinh^2\frac{\vartheta}{2}\right)\xi+1\right]^{1+u}}\,.$$

Then the substitution $\left(\cosh^2\frac{\vartheta}{2}-z^2\sinh^2\frac{\vartheta}{2}\right)\xi=\eta$ leads to the factored form

$$I=2u\int_0^\infty\frac{\eta^{1-u}d\eta}{(\eta+1)^{1+u}}\int_{-1}^{+1}\frac{dz}{\left[\cosh^2\frac{\vartheta}{2}-z^2\sinh^2\frac{\vartheta}{2}\right]^{2-u}}\,,$$

where

$$\int_0^\infty\frac{\eta^{1-u}d\eta}{(\eta+1)^{1+u}}=\frac{\Gamma(-1+2u)\Gamma(2-u)}{\Gamma(1+u)}\,.$$

Collecting all contributions, we obtain (1.21)

$$\begin{aligned} S(u) = & \, C_F\frac{-2\omega}{m_c}\frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(1+u)}\times \\ & \left[1+u(1-u)\cosh\vartheta\int_{-1}^{+1}\frac{dz}{\left[\cosh^2\frac{\vartheta}{2}-z^2\sinh^2\frac{\vartheta}{2}\right]^{2-u}}\right]+\cdots \end{aligned} \tag{8.12}$$

where dots mean terms regular at $u=1/2$.

The residue at the pole $u = 1/2$ can be obtained using

$$\int_{-1}^{+1} \frac{dz}{\left[\cosh^2 \frac{\vartheta}{2} - z^2 \sinh^2 \frac{\vartheta}{2}\right]^{3/2}} = \frac{1}{\cosh\vartheta + 1}\,.$$

The corresponding UV renormalon ambiguity is given by (2.2) with -2ω instead of m. Adding also the external-line renormalization effect $\Delta\widetilde{Z}_c/2$ (4.15), we obtain

$$\left(\frac{1}{2} - \frac{1}{\cosh\vartheta + 1}\right) \frac{\Delta\bar{\Lambda}}{2m_c}\,.$$

The contribution of the bilocal operator with the b-quark kinetic energy contains m_b instead of m_c. Therefore, contribution of all bilocal operators with kinetic-energy insertions to the ambiguity of $<J>$ is [28]

$$\left(\frac{1}{2} - \frac{1}{\cosh\vartheta + 1}\right) \left(\frac{1}{m_c} + \frac{1}{m_b}\right) \frac{\Delta\bar{\Lambda}}{2} <\widetilde{J}_1>\,. \tag{8.13}$$

Matrix elements of bilocal operators with a c or b-quark chromomagnetic insertion cannot be represented as matrix elements of the leading currents $\widetilde{J}_i$ times scalar factors – they require new independent form factors. Therefore, they have no UV renormalon ambiguities, which should be equal to $\Delta\bar{\Lambda}$ times lower-dimensional matrix elements.

Summing the ambiguity $\sum \Delta H_i <\widetilde{J}_i>$ (8.9) of the QCD matrix element $<J>$ due to the IR renormalon in the leading matching coefficients H_i at $u = 1/2$, the ambiguity (8.11) due to the UV renormalon in the local subleading operators at $u = 1/2$, and the contribution of the bilocal subleading operators (8.13), we see that they cancel, at the first order in $1/\beta_0$, for any Dirac structure Γ of the current J [28].

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