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The Low-Energy Structure of Little Higgs Models

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The mechanism of electroweak symmetry breaking in Little Higgs Models is analyzed in an effective field theory approach. This enables us to identify observable effects irrespective of the specific structure and content of the heavy degrees of freedom. We parameterize these effects in a common operator basis and present the complete set of anomalous contributions to gauge-boson, Higgs, and fermion couplings. If the hypercharge assignments of the model retain their standard form, electroweak precision data are affected only via the S and T parameters and by contact interactions. As a proof of principle, we apply this formalism to the minimal model and consider the current constraints on the parameter space. Finally, we show how the interplay of measurements at LHC and a Linear Collider could reveal the structure of these models.

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I. INTRODUCTION

In the Standard Model (SM) of electroweak interactions no symmetry protects the Higgs boson mass from large radiative corrections. Various scenarios have been developed which address this problem by embedding the SM in a richer structure. Recently, a new class of models has been found (the Little Higgs models [1–3]) where the Higgs doublet is part of a multiplet of pseudo-Goldstone bosons. The Goldstone-boson multiplet is associated to the spontaneous breaking of a global symmetry at a scale Λ which is placed in the multi-TeV range, considerably higher than the electroweak scale v. Thus, Λ acts as a cutoff which separates the weakly-interacting low-energy range from a possible strongly-interacting sector at higher energies. The large value of Λ would explain the fact that no sign of such new dynamics has yet been detected in the low-energy observables which are presently accessible to us.

Since the Higgs bosons have interactions with gauge bosons and massive fermions, the global symmetry can only be approximate, and the symmetry-breaking scale Λ cannot be arbitrarily high. Denoting the characteristic scale of the Goldstone multiplet (the analog of the pion decay constant) by F, there are order-of-magnitude relations

$$v \sim F/4\pi \sim \Lambda/16\pi^2 \qquad (1)$$

which should be satisfied if large fine-tuning of the parameters is excluded.

In the energy range between F and Λ, Little Higgs models are weakly-interacting models which contain, apart from the SM particles, extra vector bosons, scalars, and fermions. Their spectrum and interactions are arranged in such a way that the symmetries force at least one of the pseudo-Goldstone bosons, the Higgs particle, to be light: $m_H = O(v)$. The other new particles have masses of order F (up to several TeV) and therefore have not yet been observed directly in present experiments.

Nevertheless, indirect effects of the virtual exchange of heavy particles affect the interactions of SM particles. This fact has been used for computing the shifts to low-energy observables and thus to constrain Little Higgs models from precision data [4–8]. In the present paper, we derive the complete low-energy effective Lagrangian, using standard techniques of the effective-theory approach. The Littlest Higgs model of Ref. [2] has all essential features and is thus well suited as a concrete example, which we adopt throughout the derivation. For each sector of the model, we also indicate the possible modifications that appear in the general case.

After integrating out all heavy particles, the information on the specific model is encoded in the values of coefficients of dimension-six operators. This not just allows for a very simple picture of the low-energy constraints, but gives us the opportunity to present the complete pattern of anomalous couplings in the gauge-boson, top-quark and Higgs sectors, which will be probed at future colliders. In the final section we make use of those results and develop a strategy for reconstructing a complete model from data at a Linear Collider in combination with LHC.

II. INTEGRATING OUT HEAVY FIELDS

To get a picture of the low-energy trace of a heavy particle, one may set up the theory in a path-integral formalism. For instance, for two interacting scalars Φ, φ,

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where φ is massless, the generating functional of Green functions reads

$$\mathcal{Z}[j,J] = \int \mathcal{D}\Phi\,\mathcal{D}\varphi \exp\Big[i\int dx\Big(\tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}M^2\Phi^2 - \lambda\varphi^2\Phi - \ldots + J\Phi + j\varphi\Big)\Big], \quad (2)$$

where the dots indicate additional terms in the scalar potential. The low-energy effective theory which is applicable at energy scales $E \ll M$, is obtained by setting the source J to zero (since Φ does not appear as an asymptotic state) and formally integrating out the heavy field(s). In the example, this is achieved by completing the square, such that

$$\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}M^2\Phi^2 - \lambda\varphi^2\Phi = -\frac{1}{2}\Phi'(M^2+\partial^2)\Phi' + \frac{\lambda^2}{2M^2}\varphi^2\left(1+\frac{\partial^2}{M^2}\right)^{-1}\varphi^2, \quad (3)$$

where

$$\Phi' = \Phi + \frac{\lambda}{M^2}\left(1+\frac{\partial^2}{M^2}\right)^{-1}\varphi^2, \quad (4)$$

and evaluating the integral over Φ', which results in a trivial factor. The residual effective Lagrangian for φ contains virtual Φ exchange encoded as $(1+\partial^2/M^2)^{-1}$, which has to be expanded in powers of $1/M^2$ and truncated at finite order.

This method accounts for all tree-level effects, where terms higher than quadratic in Φ are treated perturbatively as operator insertions. In particular, book-keeping is simple even if Φ gets a vacuum expectation value or if there is nontrivial mixing between heavy and light fields. At loop level, there are UV-divergent corrections to the coefficients in the effective Lagrangian which can systematically be determined by appropriate matching conditions. In more general theories, there are also one-loop terms which stem from the expansion of the Jacobi determinant, if it depends on further light fields (e.g., gauge fields).

In the context of the minimal SM, an experimental precision at the per mil to percent level is consistent with a truncation of the expansion at order $1/M^2$, if M is in the TeV range. Thus, the appropriate effective Lagrangian of Little Higgs models is given by the SM, possibly extended by extra light Higgs multiplets, and augmented by a small set of dimension-six operators [9–11]. Generically, the squared ratio v^2/M^2 of the electroweak and the new-physics scales is of the same order as $1/16\pi^2$, the prefactor of loop corrections. In the present context, this rough equality is dictated by naturalness. Furthermore, while loop corrections involving only SM particles are important, loop corrections involving heavy fields are suppressed by additional powers of v^2/M^2.

The exception to this rule are loop corrections which are quadratically divergent. The quadratic divergence of the SM Higgs mass is ameliorated to a logarithmic one by the matching conditions of Little Higgs models, such that it is proportional to $M^2/(4\pi)^2 \sim v^2$. However, for all operators of the form $(h^\dagger h)^n$ with $n>1$, there is an uncancelled quadratic divergence which is cut off only by the unknown UV completion of the theory. The result is the well-known Coleman-Weinberg potential [12], which has UV-sensitive coefficients of order one at the matching scale.

The generic suppression of radiative corrections which involve new particles is partly reduced by logarithmic enhancement if some masses become exceptionally light or heavy. Such enhanced loop corrections can have a detectable effect on observables which are very precisely measured [5, 13]. However, we should keep in mind that there also unknown contributions from physics beyond the UV cutoff Λ. These can be encoded in dimension-eight operators and in corrections to the coefficients of dimension-six operators, and their effect may be enhanced by the presence of new strong interactions in the high-energy range. Such terms are parameterically of comparable magnitude as the loop effects due to the new heavy particles [14]. Thus, if we do not want to be specific about the UV completion, we can restrict our calculation of the low-energy effects in Little Higgs models to the coefficients of dimension-six operators at tree-level.

A. The Model

In all Little Higgs models, the SM gauge group $SU(2)\times U(1)$ is extended in a nontrivial way, so there are new heavy vector bosons in the spectrum which cancel the leading cutoff dependence in the Higgs self-energy. After breaking of the high-energy symmetry, the heavy states arrange themselves as massive multiplets of $SU(2)\times U(1)$.

Looking for low-energy effects, the interesting cases are triplets and singlets of $SU(2)$ with zero hypercharge. These vector-boson multiplets can directly couple to the SM fermions and mix with the SM vector bosons at leading order. After integrating out all heavy fields, they induce dimension-six operators in the low-energy effective theory. In some models [3], the extended gauge symmetry yields additional exotic multiplets of heavy vector bosons (e.g., $SU(2)$ doublets). Such states may be detected by direct observation at high-energy colliders. However, their virtual effects at tree level involve operators of dimension eight and higher only. As argued before, such terms are small and compete with the low-energy trace of the unknown UV completion, so we can consistently neglect them.

The Littlest Higgs model [2] contains exactly one extra triplet and one extra singlet of heavy vector bosons. The SM gauge group is the result of the simultaneous spontaneous breaking

$$SU(2)_1 \times SU(2)_2 \to SU(2) \quad (5)$$

$$U(1)_1 \times U(1)_2 \to U(1). \quad (6)$$

This setup is easily generalized to more complicated models where multiple vector boson triplets and singlets may exist. In particular, some or all extra group factors could be part of a larger simple group. In that case, there are relations among the gauge couplings which restrict the allowed values of the mixing angles. Here, we will treat all gauge couplings as independent parameters.

We denote the $SU(2)$ and $U(1)$ gauge fields by $A_i^{a,\mu}$ and B_i^μ $(i = 1,2)$, respectively. In the Lagrangian, the triplet and singlet parts are coupled by the Goldstone-boson interactions:

$$\mathcal{L} = \mathcal{L}_0^{(3)} + \mathcal{L}_0^{(1)} + \mathcal{L}_0^G. \tag{7}$$

The gauge-field Lagrangians are

$$\mathcal{L}_0^{(3)} = -\sum_i \frac{1}{2g_i^2} \operatorname{Tr} \mathbf{A}_{i,\mu\nu} \mathbf{A}_i^{\mu\nu} - 2 \operatorname{tr} A_1^\mu J_\mu^{(3)}, \tag{8}$$

$$\mathcal{L}_0^{(1)} = -\sum_i \frac{1}{2g_i'^2} \operatorname{Tr} \mathbf{B}_{i,\mu\nu} \mathbf{B}_i^{\mu\nu} - \sum_i B_i^\mu J_{i,\mu}^{(1)}. \tag{9}$$

Here, we define the matrix-valued field strengths as

$$\mathbf{A}_i^{\mu\nu} = \partial^\mu \mathbf{A}_i^\nu - \partial^\nu \mathbf{A}_i^\mu + i[\mathbf{A}_i^\mu, \mathbf{A}_i^\nu], \tag{10}$$

$$\mathbf{B}_i^{\mu\nu} = \partial^\mu \mathbf{B}_i^\nu - \partial^\nu \mathbf{B}_i^\mu, \tag{11}$$

with $\mathbf{A}_i^\mu = A_i^{\mu,a} T_i^a$ and $\mathbf{B}_i^\mu = B_i^\mu Y_i$ $(i = 1,2)$.

The vector bosons couple to the fermionic triplet and singlet currents $J_\mu^{(3)} = J_\mu^{(3),a} \frac{\tau^a}{2}$ and $J_{i,\mu}^{(1)}$ $(i = 1,2)$, respectively. The triplet current is the usual left-handed isospin current. This current can interact with one gauge field only. By contrast, each singlet vector field may couple to its own fermion current, which has the general form

$$\begin{aligned} J_i^{(1),\mu} &= y_{L,i} \bar{L}_L \gamma^\mu L_L + y_{\nu,i} \bar{\nu}_R \gamma^\mu \nu_R + y_{\ell,i} \bar{\ell}_R \gamma^\mu \ell_R \\ &\quad + y_{Q,i} \bar{Q}_L \gamma^\mu Q_L + y_{u,i} \bar{u}_R \gamma^\mu u_R + y_{d,i} \bar{d}_R \gamma^\mu d_R \end{aligned} \tag{12}$$

with *a priori* arbitrary $U(1)$ charges $y_{f,i}$ [6, 7]. The usual hypercharges are obtained as the sum,

$$y_f = \sum_i y_{f,i} \tag{13}$$

for each fermion field f. In order to avoid flavor-changing neutral currents, we may assume that all $U(1)$ charges are generation-independent. Note that we cannot draw any conclusions from the requirement of anomaly cancellation, since the UV completion of the model may provide additional fermions which add to the anomalies, but do not mix into the low-energy spectrum.

Some models [3] predict $SU(2)$ triplet gauge bosons A_R^μ which couple to triplet currents made of right-handed SM fermions. Since the chirality structure of the observed charged currents is known to be left-handed to a good approximation (at least, for the first two generations), we have to assume that these $SU(2)$ bosons are orthogonal to the left-handed $SU(2)$ sector. Thus, the triplet Lagrangian is unaffected up to the order we are interested in. Nevertheless, the neutral component of a A_R^μ triplet can mix with the hypercharge boson, so we should treat it as a B^μ boson which generates an extra $U(1)$ symmetry. We just have to keep in mind that at low energies, charged A_R^μ exchange induces an extra right-handed four-fermion contact interaction.

Turning to the Goldstone sector, let us first discuss the particular realization of the Littlest Higgs model [2], where the gauge group is embedded in a global $SU(5)$ group, broken down to $SO(5)$. The representation is usually written in terms of 5×5 matrices, where the $SU(2)$ generators $T_{1,2}^a$ $(a = 1,2,3)$ and $U(1)$ generators $Y_{1,2}$ are given by

$$T_1^a = \frac{1}{2} \begin{pmatrix} \tau^a & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad \text{and} \quad T_2^a = \frac{1}{2} \begin{pmatrix} 0 & & \\ & 0 & \\ & & -\tau^{a*} \end{pmatrix} \tag{14}$$

and

$$Y_1 = \operatorname{diag}(3,3,-2,-2,-2)/10, \tag{15}$$

$$Y_2 = \operatorname{diag}(2,2,2,-3,-3)/10, \tag{16}$$

respectively.

The Goldstone Lagrangian describes spontaneous symmetry breaking at the scale F, which is expected in the TeV range. Here, it is parameterized by a complex symmetric 5×5 matrix. Using the fields H (light Higgs), $w^\pm, z$ (SM Goldstones), and $\Phi^{\pm\pm}, \Phi^\pm, \Phi_0, \Phi_1$ (heavy scalars) as building blocks,

$$h = \begin{pmatrix} w^+ \\ \frac{1}{\sqrt{2}}(v + H + iz) \end{pmatrix}, \quad \phi = \begin{pmatrix} \sqrt{2}\,\Phi^{++} & \Phi^+ \\ \Phi^+ & \Phi_0 + i\Phi_1 \end{pmatrix}. \tag{17}$$

the matrix is defined as

$$\Xi = \left(\exp \tfrac{2i}{F}\Pi\right) \Xi_0, \quad \text{where} \quad \Xi_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad \Pi = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & h & \phi \\ h^\dagger & 0 & h^T \\ \phi^\dagger & h^* & 0 \end{pmatrix}. \tag{18}$$

The covariant derivative is given by

$$\begin{aligned} D^\mu \Xi &= \partial^\mu \Xi \\ &\quad + i \sum_{k=1,2} \left[(\mathbf{A}_k^\mu \Xi + \Xi (\mathbf{A}_k^\mu)^T) + (\mathbf{B}_k^\mu \Xi + \Xi (\mathbf{B}_k^\mu)^T) \right], \end{aligned} \tag{19}$$

such that the Goldstone Lagrangian reads

$$\mathcal{L}_0^G = \frac{F^2}{8} \operatorname{Tr}(D_\mu \Xi)(D^\mu \Xi)^*. \tag{20}$$

The generalization to more complicated models is straightforward. There may be multiple light scalars h

and heavy scalars ϕ. In the light sector, apart from extra Higgs doublets, there may be singlets, triplets, etc. The ρ parameter constraints make it unlikely that any component of a higher multiplet has a sizable vacuum expectation value, so such extra scalars have little impact on phenomenology [31]. In the heavy sector, we are most interested in triplets with hypercharge 2 or 0 and singlets with hypercharge 0. These can couple to Higgs doublets via

$$h^\dagger \phi_2 h^*, \quad h^\dagger \phi_0 h, \quad h^\dagger h \sigma_0, \tag{21}$$

and will induce dimension-six operators after being integrated out. While the Littlest Higgs model contains a single ϕ_2 multiplet, other models realize the ϕ_0 and σ_0 cases.

In the fermion sector of Little Higgs models, the top quark mass is generated by mixing the known top quark t with new vector-like heavy quarks. This interaction has the additional properties of cancelling the quadratic cutoff dependence of the Higgs mass and generating electroweak symmetry breaking by driving the Higgs mass squared parameter negative. The simplest setup involves just one heavy vector-like fermion T which is a $SU(2)$ singlet. Many models predict a more complicated multiplet structure. In some cases, all fermions have heavy partners which make them fit into multiplets of an enlarged gauge symmetry. However, the basic principles of constructing the Yukawa sector [1] are common to all models.

In the Littlest Higgs model, the heavy-fermion Lagrangian is built from the chiral fields

$$Q_R\colon\ b_R,\ t_R,\ T_R, \quad \text{and} \quad Q_L\colon\ q_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix},\ T_L, \tag{22}$$

namely

$$\mathcal{L}_t = \sum_Q \bar{Q} i \not{D} Q + \mathcal{L}_Y - \lambda_2 F(\bar{T}_L T_R + \text{h.c.}). \tag{23}$$

The Yukawa interaction $\mathcal{L}_Y$ combines q_L and T_L in a common $SU(3)$ multiplet. This implements the Little-Higgs symmetry structure in the fermion sector, such that the leading cutoff dependence due to top-quark loops is cancelled. We define a 3×3 matrix $\hat{\chi}_L^{ij} = \epsilon^{ijk}\chi_L^k$, where

$$\chi_L = \begin{pmatrix} i\tilde{q}_L \\ T_L \end{pmatrix} \quad \text{with} \quad \tilde{q}_L = i\tau^2 q_L = i\tau^2 \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \tag{24}$$

and promote this to a 5×5 matrix by padding zeroes:

$$\hat{\chi}_L = \begin{pmatrix} i\tau^2 T_L & iq_L & 0 \\ -iq_L^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

With these definitions, the Yukawa interaction is given by

$$\mathcal{L}_Y = \lambda_1 F \bar{t}_R \operatorname{Tr}\left[\Xi^*(iT_2^2)\Xi^* \hat{\chi}_L\right] + \text{h.c.}, \tag{26}$$

where T_2^2 is the generator defined in (14).

The masses of the light leptons and quarks can be generated by similar interactions [32], where in those cases naturalness does not require the presence of further heavy states if the corresponding quadratic divergences are cut off at the scale Λ [1]. An interesting property of the Littlest Higgs model is the possibility to write lepton-number violating interactions like

$$\mathcal{L}_N = -g_N F (\bar{L}^c)^T \Xi L, \quad \text{where} \quad L = \begin{pmatrix} \tilde{\ell}_L \\ 0 \\ 0 \end{pmatrix}$$
$$\text{and} \quad \tilde{\ell}_L = i\tau^2 \ell_L = i\tau^2 \ell_L \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \tag{27}$$

which are invariant under the full gauge symmetry. After electroweak symmetry breaking, such operators generate Majorana masses for left-handed neutrinos of order $g_N v^2/F$. Since F is not large enough to account for the huge suppression of the observed neutrino masses, the coefficient g_N must itself be small. For instance, it could be proportional to some power of F/Λ', where Λ' is a high scale where lepton number is broken.

In the general case, the construction of Yukawa interactions proceeds along similar lines. In at least one term, a component of the top quark is combined with the new state(s) T in a common multiplet of the enlarged global symmetry, while there is another interaction that generates a T mass term. This structure is consistent with the Little-Higgs symmetry and thus allows for a sizable top-quark Yukawa coupling without generating unwanted terms in the one-loop scalar potential [1–3].

B. Heavy Vector Fields

Introducing the physical heavy vector bosons X_μ, Y_μ and the SM gauge fields W_μ, B_μ, we express the gauge fields of the Littlest Higgs model as

$$A_1^\mu = W^\mu + g_X c^2 X^\mu, \quad B_1^\mu = B^\mu + g_Y c'^2 Y^\mu, \tag{28}$$
$$A_2^\mu = W^\mu - g_X s^2 X^\mu, \quad B_2^\mu = B^\mu - g_Y s'^2 Y^\mu, \tag{29}$$

where

$$c = \frac{g_1}{\sqrt{g_1^2+g_2^2}}, \qquad s = \frac{g_2}{\sqrt{g_1^2+g_2^2}},$$
$$g_X = \frac{g}{cs}, \qquad g = \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}, \tag{30}$$

and, analogously,

$$c' = \frac{g_1'}{\sqrt{g_1'^2+g_2'^2}}, \qquad s' = \frac{g_2'}{\sqrt{g_1'^2+g_2'^2}},$$
$$g_Y = \frac{g'}{c's'}, \qquad g' = \frac{g_1' g_2'}{\sqrt{g_1'^2+g_2'^2}}, \tag{31}$$

and rewrite the gauge terms in the Lagrangian:

$$
\begin{aligned}
\mathcal{L}_0^{(3)} &= -\frac{1}{2g^2}\,\mathrm{tr}\, W_{\mu\nu}W^{\mu\nu} - 2\,\mathrm{tr}\, W^\mu J_\mu^{(3)} \\
&\quad - 2g_X c^2\,\mathrm{tr}\, X^\mu J_\mu^{(3)} - \frac{1}{2}\,\mathrm{tr}\, X_{\mu\nu}X^{\mu\nu}, \qquad (32)\\
\mathcal{L}_0^{(1)} &= -\frac{1}{4g'^2}B_{\mu\nu}B^{\mu\nu} - B^\mu J_{Y,\mu}^{(1)} \\
&\quad - g_Y Y^\mu J_\mu^{(1)} - \frac{1}{4}Y_{\mu\nu}Y^{\mu\nu}. \qquad (33)
\end{aligned}
$$

For the matter fields X_μ and Y_μ, the field strengths are $X^{\mu\nu} = D^\mu X^\nu - D^\nu X^\mu$ (with the covariant derivative in the adjoint representation) and $Y^{\mu\nu} = \partial^\mu Y^\nu - \partial^\nu Y^\mu$, while the SM gauge field strengths have their standard form, $W^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu + i[W^\mu, W^\nu]$ and $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$.

In general, the singlet currents $J_Y^{(1)}$ and $J^{(1)}$ are linearly independent. We express the original currents $J_{1,2}^{(1)}$ as

$$
\begin{aligned}
J_1^\mu &= (1-a)J_Y^\mu + J_N^\mu, \qquad (34a)\\
J_2^\mu &= aJ_Y^\mu - J_N^\mu, \qquad (34b)
\end{aligned}
$$

where J_Y is the canonical hypercharge current. J_N describes the terms which deviate from the canonical hypercharge assignments. Note that there is some ambiguity in defining J_N, since we can subtract an arbitrary multiple of J_Y. This is accounted for by the parameter a. The current which is coupled to the heavy vector boson Y in (33) is then given by

$$
J_\mu^{(1)} = (c'^2 - a)J_{Y,\mu}^{(1)} + J_{N,\mu}^{(1)}. \qquad (35)
$$

Furthermore, we introduce the Higgs current

$$
V_\mu = i\left[h(D_\mu h)^\dagger - (D_\mu h)h^\dagger\right], \qquad (36)
$$

which may be decomposed into its singlet and triplet parts:

$$
V_\mu^{(1)} = \mathrm{tr}\, V_\mu, \quad V_\mu^{(3)} = V_\mu - \frac{1}{2}\,\mathrm{tr}\, V_\mu. \qquad (37)
$$

For later use we also define field strength tensors,

$$
V_{\mu\nu}^{(3)} = D_\mu V_\nu^{(3)} - D_\nu V_\mu^{(3)} \quad \text{and} \quad V_{\mu\nu}^{(1)} = \partial_\mu V_\nu^{(1)} - \partial_\nu V_\mu^{(1)}, \qquad (38)
$$

where

$$
D_\mu V_\nu^{(3)} \equiv \partial_\mu V_\nu^{(3)} + i[W_\mu, V_\nu^{(3)}]. \qquad (39)
$$

With these definitions, the Goldstone Lagrangian (20) can be expanded to yield

$$
\begin{aligned}
\mathcal{L}_0^G = {} & M_X^2\,\mathrm{tr}\, X_\mu X^\mu + g_X\frac{c^2-s^2}{2}\,\mathrm{tr}[X^\mu V_\mu^{(3)}] \\
& + \frac{1}{2}M_Y^2 Y_\mu Y^\mu + g_Y\frac{c'^2-s'^2}{4}Y^\mu V_\mu^{(1)} \\
& + \frac{1}{2}\,\mathrm{tr}(D_\mu\phi)^\dagger(D^\mu\phi) + (D_\mu h)^\dagger(D^\mu h) \\
& - \frac{1}{6F^2}\,\mathrm{tr}\left[V_\mu^{(3)}V^{(3),\mu}\right] + \dots, \qquad (40)
\end{aligned}
$$

where the omitted terms are higher-dimension interactions which are irrelevant for our discussion. The vector-boson masses are given by

$$
M_X = g_X F/2 \quad \text{and} \quad M_Y = g_Y F/(2\sqrt{5}), \qquad (41)
$$

respectively. After electroweak symmetry breaking, the physical masses of the X and Y bosons get corrections of order v^2/F, but this is irrelevant for our discussion.

Following the lines of the beginning of this section, the X and Y vector fields are integrated out by completing the square in the Lagrangian. This is achieved by the redefinitions

$$
X'_\mu = X_\mu - \frac{g_X c^2}{M_X^2}J_\mu^{(3)} + g_X\frac{c^2-s^2}{4M_X^2}V_\mu^{(3)} \qquad (42)
$$

$$
Y'_\mu = Y_\mu - \frac{g_Y}{M_Y^2}J_\mu^{(1)} + g_Y\frac{c'^2-s'^2}{4M_Y^2}V_\mu^{(1)} \qquad (43)
$$

and leads to the low-energy effective Lagrangian

$$
\mathcal{L} = \mathcal{L}^{(3)} + \mathcal{L}^{(1)}, \qquad (44)
$$

where the triplet and singlet parts are given by

$$
\begin{aligned}
\mathcal{L}^{(3)} = {} & -\frac{1}{2g^2}\,\mathrm{tr}[W_{\mu\nu}W^{\mu\nu}] - 2\,\mathrm{tr}[W^\mu J_\mu^{(3)}] \\
& + f_{JJ}^{(3)}\,\mathrm{tr}[J^{(3),\mu}J_\mu^{(3)}] + f_{VV}^{(3)}\,\mathrm{tr}[V^{(3),\mu}V_\mu] \qquad (45)\\
& + f_{VJ}^{(3)}\,\mathrm{tr}[V^{(3),\mu}J_\mu^{(3)}] \\
\mathcal{L}^{(1)} = {} & -\frac{1}{4g'^2}B_{\mu\nu}B^{\mu\nu} - B^\mu J_{Y,\mu}^{(1)} \\
& + f_{JJ}^{(1)}J^{(1),\mu}J_\mu^{(1)} + f_{VV}^{(1)}V^{(1),\mu}V_\mu^{(1)} \qquad (46)\\
& + f_{VJ}^{(1)}V^{(1),\mu}J_{Y,\mu}^{(1)} + f_{VN}^{(1)}V^{(1),\mu}J_{N,\mu}^{(1)},
\end{aligned}
$$

respectively. The Littlest Higgs values of the coefficients are

$$
\begin{aligned}
f_{JJ}^{(3)} &= -\frac{4c^4}{F^2}, \qquad (47a)\\
f_{VV}^{(3)} &= -\frac{1}{6F^2}\left(1 + \frac{3}{2}(c^2-s^2)^2\right), \qquad (47b)\\
f_{VJ}^{(3)} &= \frac{2c^2(c^2-s^2)}{F^2}, \qquad (47c)\\
f_{JJ}^{(1)} &= -\frac{10}{F^2}, \qquad (47d)\\
f_{VV}^{(1)} &= -\frac{5}{8F^2}(c'^2-s'^2)^2, \qquad (47e)\\
f_{VJ}^{(1)} &= \frac{5(c'^2-a)(c'^2-s'^2)}{F^2}, \qquad (47f)\\
f_{VN}^{(1)} &= \frac{5(c'^2-s'^2)}{F^2}. \qquad (47g)
\end{aligned}
$$

The overall structure of the effective Lagrangian (44–46) is generic to Little Higgs models. In extended models, there are extra $U(1)$ gauge symmetries which are associated to multiple linearly independent currents J_N. (In the original version of the Littlest Higgs model [2], the extra singlet current J_N and the parameter a are both zero.) If there are multiple Higgs doublets in the light spectrum, we can construct multiple currents $V_{i,\mu}$. One linear combination of these is the Noether current of the electroweak symmetries and plays the role of V_μ in the Littlest Higgs model, while the others provide extra interactions which induce anomalous couplings in the multi-doublet Higgs sector. In the present paper, we restrict ourselves to the discussion of a single Higgs doublet and leave the multi-doublet case as a straightforward extension.

Otherwise, the information about the specific model is encoded in the values of the operator coefficients. In particular, the factor $\sqrt{5}$ in the Y mass (41) corresponds to factors of 5 in the singlet coefficients (47d–47g). In models with a different $U(1)$ embedding, this prefactor will take a different value. The constant term in $f_{VV}^{(3)}$ is a consequence of the nonlinear Goldstone boson representation. The analogous constant term in $f_{VV}^{(1)}$ happens to be zero in the Littlest Higgs model. The terms which involve mixing angles depend on the corresponding vector boson spectrum. For instance, there is a variant of the Littlest Higgs model where the extra $U(1)$ symmetry is ungauged [6, 7, 16]. In this model, the singlet coefficients vanish identically.

C. Heavy Scalars and the Higgs Boson

In the expansion of the Goldstone Lagrangian (40), the kinetic energy of the heavy scalar multiplet ϕ

$$\mathcal{L}_0^\phi = \frac{1}{2}\,\mathrm{tr}(D_\mu\phi)^\dagger(D^\mu\phi) \tag{48}$$

produces an extra contribution to the effective Lagrangian when ϕ has been integrated out. This effective interaction has to be combined with the other terms in the Coleman-Weinberg potential of the scalar fields, which is generated at one-loop order.

In the Littlest Higgs model, the potential involves the Higgs doublet h and the triplet ϕ. The coefficients are generated by gauge-boson and fermion exchange and are therefore proportional to the gauge and Yukawa couplings:

$$\begin{aligned}\mathcal{L}_0^{CW} &= -\frac{1}{2}M_\phi^2\,\mathrm{tr}[\phi\phi^\dagger] + \mu^2(h^\dagger h) - \lambda_4(h^\dagger h)^2 \\ &\quad - \mathrm{i}\lambda_{2\phi}\left(h^\dagger\phi h^* - h^T\phi^\dagger h\right) - \lambda_{2\phi\phi}\,\mathrm{tr}[(\phi\phi^\dagger)(hh^\dagger)] \\ &\quad - \mathrm{i}\lambda_{4\phi}(h^\dagger h)\left(h^\dagger\phi h^* - h^T\phi^\dagger h\right) - \lambda_6(h^\dagger h)^3\end{aligned} \tag{49}$$

with the ϕ mass parameter

$$M_\phi^2 = -F^2\,\left[(g_1^2 + g_2^2 + g_1'^2 + g_2'^2)k + \lambda_1^2 k'\right] \tag{50}$$

and the coupling constants

$$\lambda_{2\phi} = \frac{F}{2\sqrt{2}}\left[(g_1^2 + g_1'^2 - g_2^2 - g_2'^2)k - \lambda_1^2 k'\right] \tag{51a}$$

$$\lambda_4 = M_\phi^2/4F^2 \tag{51b}$$

$$\lambda_{2\phi\phi} = -2M_\phi^2/3F^2 \tag{51c}$$

$$\lambda_{4\phi} = -\lambda_{2\phi}/F^2 \tag{51d}$$

$$\lambda_6 = -M_\phi^2/6F^4 \tag{51e}$$

which are sensitive to the UV completion of the theory via the dimensionless parameters k and k'. To get the correct pattern of electroweak symmetry breaking, the ϕ mass squared M_ϕ^2 must be positive. This implies the relation

$$\left(\frac{e^2}{s_w^2 s^2 c^2} + \frac{e^2}{c_w^2 s'^2 c'^2}\right)k + \frac{\lambda_t^2}{c_t^2}k' < 0 \tag{52}$$

which the unknown coefficients k and k' have to satisfy.

The Higgs mass parameter μ^2 is given to leading-logarithmic one-loop order by

$$\begin{aligned}\mu^2 = -\frac{3}{64\pi^2}&\left[3g^2M_X^2\log\frac{\Lambda^2}{M_X^2} + g'^2M_Y^2\log\frac{\Lambda^2}{M_Y^2}\right] \\ &- \frac{\lambda}{16\pi^2}M_\phi^2\log\frac{\Lambda^2}{M_\phi^2} + \frac{3\lambda_t^2}{8\pi^2}M_T^2\log\frac{\Lambda^2}{M_T^2},\end{aligned} \tag{53}$$

but there are constant one-loop and two-loop corrections to this quantity with prefactors of the order $F^2/16\pi^2 \sim \Lambda^2/(4\pi)^4$ which are not necessarily negligible.

To get all terms that we will need later, we integrate out the heavy scalar using the redefinition

$$\begin{aligned}\phi' = \phi - \frac{2\mathrm{i}\lambda_{2\phi}}{M_\phi^2}&\left(1 + \frac{D^2}{M_\phi^2} + \frac{2\lambda_{2\phi\phi}}{M_\phi^2}hh^\dagger\right)^{-1} \\ &\times\left(1 + \frac{\lambda_{4\phi}}{\lambda_{2\phi}}h^\dagger h\right)hh^T.\end{aligned} \tag{54}$$

Expanding the resulting effective Lagrangian up to second order, we obtain

$$\begin{aligned}\mathcal{L}^\phi = \frac{2\lambda_{2\phi}^2}{M_\phi^2}&\left((h^\dagger h)^2 + 2\left(\frac{\lambda_{4\phi}}{\lambda_{2\phi}} - \frac{\lambda_{2\phi\phi}}{M_\phi^2}\right)(h^\dagger h)^3\right. \\ &\left. + \frac{1}{M_\phi^2}\,\mathrm{tr}\,D_\mu(h^*h^\dagger)D^\mu(hh^T) + \ldots\right).\end{aligned} \tag{55}$$

The first term in this expression modifies the coefficient λ_4,

$$\lambda_4^{\text{eff}} = \frac{M_\phi^2}{4F^2} - \frac{2\lambda_{2\phi}^2}{M_\phi^2}. \tag{56}$$

Hence, up to corrections of order v^4/F^2, the Higgs mass is given by

$$\begin{aligned} m_H^2 &= 2\lambda_4^{\rm eff} v^2 \\ &= -2v^2 \left(\frac{e^2}{s_w^2 c^2} + \frac{e^2}{c_w^2 c'^2}\right) k \\ &\times \frac{\left(\frac{e^2}{s_w^2 s^2} + \frac{e^2}{c_w^2 s'^2}\right) k + \frac{\lambda_t^2}{c_t^2} k'}{\left(\frac{e^2}{s_w^2 s^2 c^2} + \frac{e^2}{c_w^2 s'^2 c'^2}\right) k + \frac{\lambda_t^2}{c_t^2} k'}. \end{aligned} \tag{57}$$

The μ mass parameter is related to this by $\mu^2 = m_H^2/2$. For electroweak symmetry breaking to occur, μ^2 has to be positive, so the relation

$$\frac{\lambda_{2\phi}^2}{M_\phi^4} < \frac{1}{8F^2} \tag{58}$$

must be satisfied [4, 6].

The other terms in (55) are dimension-six operators, which may be rewritten as

$$\begin{aligned} \mathcal{L}_6^\phi = -\frac{4\lambda_{2\phi}^2}{3F^2 M_\phi^2}(h^\dagger h)^3 + \frac{4\lambda_{2\phi}^2}{M_\phi^4} \big[(h^\dagger h)\left((D_\mu h)^\dagger (D^\mu h)\right) \\ + \left((D_\mu h)^\dagger h\right)\left(h^\dagger (D^\mu h)\right)\big]. \end{aligned} \tag{59}$$

Again, this particular expression is specific to the Littlest Higgs model. However, in more general models the structure is similar: The effective Higgs potential contains h^4 and h^6 terms, while the exchange of heavy scalars between light Higgs bosons generates derivative interactions. The quantum numbers of the heavy scalar determine the structure of these terms, i.e., the square bracket in (59). Introducing the operators

$$\mathcal{O}_{VV}^{(3)} = \operatorname{tr} V^{(3),\mu} V_\mu^{(3)}, \tag{60}$$

$$\mathcal{O}_{hh} = (h^\dagger h)\left((D_\mu h)^\dagger (D^\mu h)\right), \tag{61}$$

$$\mathcal{O}_{h,1} = \left((D_\mu h)^\dagger h\right)\left(h^\dagger (D^\mu h)\right), \tag{62}$$

from integrating out triplets with hypercharge 2 (ϕ_2), hypercharge 0 (ϕ_0), or singlets σ_0, we obtain interactions of the form

$$\phi_2 : \mathcal{O}_{hh} + \mathcal{O}_{h,1}, \tag{63}$$

$$\phi_0 : -\mathcal{O}_{VV}^{(3)} + 3\mathcal{O}_{hh} - \mathcal{O}_{h,1}, \tag{64}$$

$$\sigma_0 : -\mathcal{O}_{VV}^{(3)} + 2\mathcal{O}_{hh}, \tag{65}$$

respectively. In the Littlest Higgs model, only ϕ_2 is present. These derivative interactions combine with the triplet and singlet Higgs current interactions we have encountered when integrating out the vector fields.

D. Heavy Fermions and the Top Quark

Finally, we derive the low-energy effective Lagrangian in the fermion sector. We expand the Yukawa term of the Littlest Higgs model $\mathcal{L}_Y$ (26) to order $1/F^2$:

$$\begin{aligned} \mathcal{L}_Y = &-\lambda_1 F\left(1 - \frac{1}{F^2}h^\dagger h\right)\bar t_R T_L \\ &+ \lambda_1\sqrt{2}\left(1 - \frac{2}{3F^2}h^\dagger h\right)\bar t_R h^T \tilde q_L \\ &- \frac{i\lambda_1}{F}\bar t_R h^\dagger \phi \tilde q_L + \ldots + \text{h.c.} \end{aligned} \tag{66}$$

Combining this with the T mass term,

$$\mathcal{L}_T = -\lambda_2 F \bar T_R T_L + \text{h.c.}, \tag{67}$$

we diagonalize the two top-like states to leading order by the rotation

$$t_R \to c_t t_R + s_t T_R, \tag{68}$$

$$T_R \to -s_t t_R + c_t T_R, \tag{69}$$

where the mixing angle is given by

$$s_t = \frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \qquad c_t = \frac{\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}. \tag{70}$$

We may first integrate out the heavy scalar ϕ in the expression (66). Using the leading term of (54), this is equivalent to the replacement

$$\phi \to \frac{2\mathrm{i}\lambda_{2\phi}}{M_\phi^2} h h^T. \tag{71}$$

In the rotated basis, the Yukawa terms take the form

$$\begin{aligned} \mathcal{L}_Y + \mathcal{L}_T = &-\frac{\lambda_t F}{c_t s_t}\bar T_R T_L + \lambda_t\sqrt{2}\,\frac{s_t}{c_t}\bar T_R h^T \tilde q_L \\ &+ \frac{\lambda_t}{F} h^\dagger h \bar t_R T_L \\ &+ \lambda_t\sqrt{2}\left(1 - \frac{\beta}{F^2}h^\dagger h\right)\bar t_R h^T \tilde q_L + \text{h.c.}, \end{aligned} \tag{72}$$

where

$$\lambda_t = \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \quad \text{and} \quad \beta = \frac{2}{3} - \frac{\sqrt{2}\lambda_{2\phi}F}{M_\phi^2}. \tag{73}$$

To get the low-energy effective Lagrangian, we combine the chiral states T_L and T_R to a Dirac field T with mass

$$M_T = \lambda_t F / c_t s_t + O(v^2/F). \tag{74}$$

Completing the square in the Lagrangian,

$$T' = T + \lambda_t (i\not{D} - M_T)^{-1}\left(\sqrt{2}\,\frac{s_t}{c_t}h^T \tilde q_L + \frac{1}{F}h^\dagger h t_R\right), \tag{75}$$

we can integrate out T'. We expand the result up to the order $1/F^2$ and obtain

$$\begin{aligned} \mathcal{L}_f^{\rm eff} = &\sum_{Q = q_L, t_R, b_R} \bar Q (i\not{D}) Q + \frac{2s_t^4}{F^2}\bar{\tilde q}_L h^* (i\not{D}) h^T \tilde q_L \\ &+ \lambda_t\sqrt{2}\left(1 - \frac{\beta - s_t^2}{F^2}h^\dagger h\right)\left(\bar t_R h^T \tilde q_L + \text{h.c.}\right). \end{aligned} \tag{76}$$

Using the operator definitions

$$\mathcal{O}_{Vq} = \bar{\tilde{q}}_L \not{V}^T \tilde{q}_L, \quad (77)$$

$$\mathcal{O}_{Vt} = \bar{t}_R \not{V}^{(1)} t_R, \quad (78)$$

$$\mathcal{O}_{hq} = h^\dagger h \left(\bar{t}_R h^T \tilde{q}_L + \text{h.c.}\right), \quad (79)$$

this can be rewritten in the form

$$\mathcal{L}_f^{\text{eff}} = \sum_{Q=b,t} \bar{Q}(i\not{D})Q + \lambda_t\sqrt{2}\left(\bar{t}_R h^T \tilde{q}_L + \text{h.c.}\right) + f_{Vq}\mathcal{O}_{Vq} + f_{Vt}\mathcal{O}_{Vt} + f_{hq}\mathcal{O}_{hq}, \quad (80)$$

where the coefficients in the Littlest Higgs model are

$$f_{Vq} = -\frac{s_t^4}{F^2}, \quad f_{Vt} = 0, \quad f_{hq} = \frac{\sqrt{2}\lambda_t}{F^2}(c_t^2 s_t^2 - \beta), \quad (81)$$

and β is the coefficient resulting from scalar interactions given in (73).

In the effective Lagrangian (80), all reference to the specific model is encoded in the values of the coefficients f_{Vq}, f_{Vt}, and f_{hq}. Since in the Littlest Higgs model there is no mixing of the left-handed fields, the anomalous coupling f_{Vt} of the right-handed top quark vanishes. In general, this need not be the case. Furthermore, other quarks and leptons may also mix with heavy partners. Such mixings are constrained by the absence of flavor-changing neutral currents. We will not consider this complication in the present paper.

The Lagrangian (80) gives rise to the top mass

$$m_t = \lambda_t v + \frac{f_{hq}}{2\sqrt{2}} v^3. \quad (82)$$

The small correction to the canonical value $\lambda_t v$ is detectable only if λ_1 and λ_2 are determined directly, i.e., by measuring production and decay of the heavy T at the percent level. This accuracy is not likely to become feasible in the near future [16]. From the viewpoint of the low-energy effective theory, it is more appropriate to take m_t as an input parameter and absorb the correction in the mass term. Thus, we rewrite (80) as

$$\mathcal{L}_f^{\text{eff}} = \sum_{Q=b,t} \bar{Q}(i\not{D})Q + \frac{m_t}{v}\sqrt{2}\left(\bar{t}_R h^T \tilde{q}_L + \text{h.c.}\right) + f_{Vq}\mathcal{O}_{Vq} + f_{Vt}\mathcal{O}_{Vt} + f_{hq}\mathcal{O}'_{hq}, \quad (83)$$

where in the redefined operator

$$\mathcal{O}'_{hq} = \left(h^\dagger h - v^2/2\right)\left(\bar{t}_R h^T \tilde{q}_L + \text{h.c.}\right) \quad (84)$$

the contribution to the top mass is removed.

III. EQUATIONS OF MOTION

The effective Lagrangian consisting of (45, 46) and (55) is not yet well suited for discussing physical observables. The reason is the presence of couplings V-J between the Higgs and fermion currents, which after spontaneous symmetry breaking induce anomalous couplings of the W and Z bosons to fermions. This is not a problem, but since gauge-boson interactions with fermions define the gauge couplings of the SM, it is convenient to eliminate the corrections by appropriate field redefinitions, i.e., by applying the equations of motion. The result will be more transparent, and the coefficients in the effective Lagrangian can be more easily related to measurable quantities.

It is natural to separate triplet and singlet terms in this procedure. Here, the triplet terms conserve the approximate custodial $SU(2)_c$ symmetry of the SM, while the singlet terms (which, incidentally, are more model-dependent) induce $SU(2)_c$ violation and thus contribute to the ρ parameter.

A. Custodial-$SU(2)$ Conserving Terms

In (45), the total contribution linear in the fermionic triplet current $J_\mu^{(3)}$ is given by

$$\mathcal{L}_J^{(3)} = -2\,\text{tr}\left[\left(W^\mu - \frac{1}{2} f_{VJ}^{(3)} V^{(3),\mu}\right) J_\mu^{(3)}\right]. \quad (85)$$

A (nonlinear) redefinition of W_μ eliminates the extra term in (85). This is equivalent to an application of the equations of motion, which for the W field read

$$0 = \frac{\delta\mathcal{L}}{\delta W^\mu} = -\frac{2}{g^2} D^\nu W_{\mu\nu} + V_\mu^{(3)} - 2J_\mu^{(3)} + \ldots \quad (86)$$

The omitted terms are of dimension five and higher and thus irrelevant for our discussion. To eliminate the $V_\mu J^\mu$ term, we add the operator

$$\begin{aligned} 0 &= \frac{1}{2} f_{VJ}^{(3)}\,\text{tr}[V_\mu^{(3)} \frac{\delta\mathcal{L}}{\delta W_\mu}] \\ &= -\frac{1}{g^2} f_{VJ}^{(3)}\,\text{tr}[V_\mu^{(3)} D_\nu W^{\mu\nu}] \\ &\quad + \frac{1}{2} f_{VJ}^{(3)}\,\text{tr}[V^{(3),\mu} V_\mu^{(3)}] - f_{VJ}^{(3)}\,\text{tr}[V^{(3),\mu} J_\mu^{(3)}] \end{aligned} \quad (87)$$

to the effective Lagrangian, such that the V-J coupling vanishes in the result. Applying partial integration to the first operator on the right-hand side of (87) and combining the additional terms with (45), we obtain

$$\mathcal{L}^{(3)} = -\frac{1}{2g^2}\,\text{tr}[W_{\mu\nu}W^{\mu\nu}] - 2\,\text{tr}[W^\mu J_\mu^{(3)}] + f_{JJ}^{(3)}\mathcal{O}_{JJ}^{(3)} + f_{VW}\mathcal{O}_{VW} + f_{VV}^{(3)}\mathcal{O}_{VV}^{(3)}, \quad (88)$$

where the dimension-six operators are defined as

$$\mathcal{O}_{JJ}^{(3)} = \text{tr}\, J^{(3),\mu} J_\mu^{(3)}, \quad (89)$$

$$\mathcal{O}_{VW} = \text{tr}\, V_{\mu\nu}^{(3)} W^{\mu\nu}, \quad (90)$$

$$\mathcal{O}_{VV}^{(3)} = \text{tr}\, V^{(3),\mu} V_\mu^{(3)}. \quad (91)$$

The coefficients in (88) are, in the Littlest Higgs model,

$$f_{JJ}^{(3)} = -\frac{4c^4}{F^2}, \tag{92}$$

$$f_{VW} = -\frac{c^2(c^2-s^2)}{g^2F^2}, \tag{93}$$

$$f_{VV}^{(3)} = \frac{(1+2c^2)(c^2-s^2)}{4F^2} - \frac{1}{6F^2}. \tag{94}$$

To convert this result into a more useful form, we expand the derivative acting on V and rewrite it in terms of the basis introduced in [10]:

$$\mathcal{O}_{VW} = -4\mathcal{O}_W - 2\mathcal{O}_{BW} - 2\mathcal{O}_{WW}, \tag{95}$$

where

$$\mathcal{O}_W = i(D_\mu h)^\dagger W^{\mu\nu}(D_\nu h), \tag{96}$$

$$\mathcal{O}_{BW} = -\frac{1}{2}B_{\mu\nu}h^\dagger W^{\mu\nu}h, \tag{97}$$

$$\mathcal{O}_{WW} = -\frac{1}{2}(h^\dagger h)\,\mathrm{tr}\,W_{\mu\nu}W^{\mu\nu}. \tag{98}$$

The last operator renormalizes the kinetic energy of the vector bosons. Noting that the gauge coupling g (in our convention) appears in the dimension-four Lagrangian only as the prefactor of the W kinetic energy, we can add a term proportional to $\mathrm{tr}\,W_{\mu\nu}W^{\mu\nu}$ to the Lagrangian and completely absorb its effect into a redefinition of g, a shift which is unobservable in the effective theory. (The same argument has been applied to the top quark mass above.) Hence, we can replace (95) by

$$\mathcal{O}_{VW} = -4\mathcal{O}_W - 2\mathcal{O}_{BW} - 2\mathcal{O}'_{WW}, \tag{99}$$

where now

$$\mathcal{O}'_{WW} = -\frac{1}{2}(h^\dagger h - v^2/2)\,\mathrm{tr}\,W_{\mu\nu}W^{\mu\nu}. \tag{100}$$

The second operator on the right-hand side of (87) should also be investigated:

$$\mathcal{O}_{VV}^{(3)} = \mathrm{tr}\,V_\mu^{(3)}V^\mu = \mathrm{tr}\,V_\mu V^\mu - \frac{1}{2}\,\mathrm{tr}\,V_\mu\,\mathrm{tr}\,V^\mu. \tag{101}$$

This can be rewritten as [33]

$$\mathcal{O}_{VV}^{(3)} = 3\mathcal{O}_{hh} + \frac{1}{2}(h^\dagger h)((D^2h)^\dagger h + h^\dagger(D^2h)), \tag{102}$$

where $\mathcal{O}_{hh}$ is defined in (61). Similar to the treatment of $\mathcal{O}_{WW}$, we add a term proportional to $(D_\mu h)^\dagger(D^\mu h)$ to the Lagrangian and absorb it in the Higgs kinetic energy, while on the other hand we replace $\mathcal{O}_{hh}$ by

$$\mathcal{O}'_{hh} = (h^\dagger h - v^2/2)\left((D_\mu h)^\dagger(D^\mu h)\right). \tag{103}$$

This implies a redefinition of the physical value of v. From the SM Lagrangian, the Higgs part of which reads

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} = {} & (D_\mu h)^\dagger(D^\mu h) + \mu^2(h^\dagger h) - \lambda(h^\dagger h)^2 \\ & - (h^\dagger J_S + J_S^\dagger h),\end{aligned} \tag{104}$$

we read off the equation of motion for h,

$$D^2h = \mu^2 h - 2\lambda(h^\dagger h)h - J_S, \tag{105}$$

where J_S is the scalar current of the massive fermions which couples to the Higgs field. For the quartic coupling λ, we should take the effective coupling λ_4^{eff} (56). Thus, we can express $\mathcal{O}_{VV}^{(3)}$ as

$$\mathcal{O}_{VV}^{(3)} = 3\mathcal{O}'_{hh} - 6\lambda_4^{\text{eff}}\mathcal{O}'_{h,3} - \frac{1}{2}\mathcal{O}'_{J_S}, \tag{106}$$

where the additional operators are

$$\mathcal{O}'_{h,3} = \frac{1}{3}(h^\dagger h - v^2/2)^3, \tag{107}$$

$$\mathcal{O}'_{J_S} = (h^\dagger h - v^2/2)(h^\dagger J_S + J_S^\dagger h). \tag{108}$$

Again, we have absorbed terms proportional to v^2 in the definition of μ^2, λ, and the physical fermion masses. Additional contributions to $\mathcal{O}'_{hh}$ and $\mathcal{O}_{h,3}$ come from the terms in (59) which encode heavy-scalar exchange.

B. Custodial-$SU(2)$ Violating Terms

From integrating out the heavy hypercharge boson, we have obtained an interaction of the form

$$\mathcal{L}_J^{(1)} = -\left(B^\mu - f_{VJ}^{(1)}V^{(1),\mu}\right)J_{Y,\mu}^{(1)} + f_{VN}^{(1)}V^{(1),\mu}J_{N,\mu}^{(1)}. \tag{109}$$

Analogous to the triplet case, the coupling of $V^{(1)}$ with the hypercharge current $J_Y^{(1)}$ can be eliminated from the effective Lagrangian by the equations of motion. However, if the model provides a $U(1)$ current $J_N^{(1)}$ which is linearly independent from the hypercharge current, the resulting extra term in (109) cannot be removed in this way.

Nevertheless, we proceed as before and add the term

$$\begin{aligned}0 &= f_{VJ}^{(1)}V_\mu^{(1)}\frac{\delta\mathcal{L}}{\delta B_\mu} \\ &= f_{VJ}^{(1)}V_\mu^{(1)}\left(-\frac{1}{g'^2}\partial^\nu B_{\mu\nu} + \frac{1}{2}V_\mu^{(1)} - J_{Y,\mu}^{(1)}\right),\end{aligned} \tag{110}$$

such that the result reads

$$\begin{aligned}\mathcal{L}^{(1)} = {} & -\frac{1}{4g'^2}B_{\mu\nu}B^{\mu\nu} - B^\mu J_{Y,\mu}^{(1)} + f_{JJ}^{(1)}\mathcal{O}_{JJ}^{(1)} \\ & + f_{VB}\mathcal{O}_{VB} + f_{VV}^{(1)}\mathcal{O}_{VV}^{(1)} + f_{VN}^{(1)}\mathcal{O}_{VN}^{(1)}.\end{aligned} \tag{111}$$

Here, the operators are defined as

$$\mathcal{O}_{JJ}^{(1)} = J^{(1),\mu}J_\mu^{(1)}, \tag{112}$$

$$\mathcal{O}_{VB} = V_{\mu\nu}^{(1)}B^{\mu\nu}, \tag{113}$$

$$\mathcal{O}_{VV}^{(1)} = V^{(1),\mu}V_\mu^{(1)}, \tag{114}$$

$$\mathcal{O}_{VN}^{(1)} = V^{(1),\mu}J_{N,\mu}^{(1)}. \tag{115}$$

In the Littlest Higgs model, the operator coefficients are

$$f_{JJ}^{(1)} = -\frac{10}{F^2}, \tag{116}$$

$$f_{VB} = -\frac{5(c'^2 - a)(c'^2 - s'^2)}{2g'^2F^2}, \tag{117}$$

$$f_{VV}^{(1)} = \frac{5(1 + 2c'^2 - 4a)(c'^2 - s'^2)}{8F^2}, \tag{118}$$

$$f_{VN}^{(1)} = \frac{5(c'^2 - s'^2)}{F^2}. \tag{119}$$

Switching to a more familiar basis, we expand the operators as follows:

$$\mathcal{O}_{VB} = -8\mathcal{O}_B - 4\mathcal{O}_{BW} - 4\mathcal{O}'_{BB}, \tag{120}$$

$$\mathcal{O}_{VV}^{(1)} = 2\mathcal{O}'_{hh} - 12\lambda_4^{\text{eff}}\mathcal{O}'_{h,3} - \mathcal{O}'_{J_S} + 4\mathcal{O}_{h,1} \tag{121}$$

where the new terms are

$$\mathcal{O}_B = \frac{i}{2}(D_\mu h)^\dagger (D_\nu h) B^{\mu\nu}, \tag{122}$$

$$\mathcal{O}'_{BB} = -\frac{1}{4}(h^\dagger h - v^2/2) B_{\mu\nu} B^{\mu\nu}, \tag{123}$$

$$\mathcal{O}'_{h,1} = \left((D_\mu h)^\dagger h\right)\left(h^\dagger (D^\mu h)\right) - (v^2/2)(D_\mu h^\dagger)(D^\mu h). \tag{124}$$

Analogous to $\mathcal{O}'_{hh}$, in the definition of the operator $\mathcal{O}'_{h,1}$ the contribution which would modify the Higgs kinetic energy in unitary gauge has been subtracted and absorbed in the definition of v. Finally, we note that from (59) we get an additional contribution to the coefficient of the $SU(2)_c$-violating operator $\mathcal{O}_{h,1}$.

IV. PRECISION OBSERVABLES

In the previous sections, we have derived the low-energy effective Lagrangian of a Little Higgs model, which is applicable in the energy range below the lowest-lying new particle in the spectrum. Collecting all terms, we list the complete result in the appendix as Eq. (A1).

Below we will discuss the contributions to the electroweak precision observables which follow from this expression. Anomalous vector-boson and Higgs couplings are the subject of the next section.

A. Oblique Corrections

The two operators $\mathcal{O}_{BW}$ and $\mathcal{O}_{h,1}$ influence the gauge-boson two-point functions. These corrections are usually expressed in terms of the S, T, U parameters [10, 17, 18]. In our context, there is no dimension-six operator which corresponds to U, so ΔU is zero. The other two parameters get contributions from the exchange of heavy particles:

Expanding the operator

$$\mathcal{O}_{BW} = -\frac{1}{2} B_{\mu\nu} h^\dagger W^{\mu\nu} h \tag{125}$$

in terms of physical fields, we have to modify the rotation of neutral fields by the correction present in (A1)

$$W^3 = \frac{e}{s_w}\left[c_w\left(1 + M_Z^2(f_{VW} + 2f_{VB})\right) Z + s_w A\right], \tag{126}$$

$$B = \frac{e}{c_w}\left[-s_w\left(1 + M_Z^2(f_{VW} + 2f_{VB})\right) Z + c_w A\right], \tag{127}$$

in order to get the correct kinetic energies of Z and A in the effective Lagrangian. Correspondingly, the gauge couplings g and g' are given by

$$g = \frac{e}{s_w}(1 + M_W^2(f_{VW} + 2f_{VB})), \tag{128}$$

$$g' = \frac{e}{c_w}\left(1 + [M_Z^2 - M_W^2](f_{VW} + 2f_{VB})\right), \tag{129}$$

if expressed in terms of e and s_w, c_w.

Here, e is the ordinary electromagnetic coupling. (In practice, we have to account for a nontrivial scale dependence in this quantity, but this effect is universal and independent of our discussion.) For the definition of the weak mixing angle s_w, we first consider the special case where $J_N = 0$, i.e., the hypercharge vector bosons couple only to the standard hypercharge current. This covers, in particular, the original Littlest Higgs model where the fermions are gauged only under one $U(1)$ group. Then, the sine of the weak mixing angle s_w is measured directly in Z decays, since in our framework both the vector and the axial vector coupling receive the same correction,

$$\Delta v_f/v_f = \Delta a_f/a_f = M_Z^2(f_{VW} + 2f_{VB}), \tag{130}$$

such that the ratio v_f/a_f is unaffected.

As a result, $\mathcal{O}_{BW}$ contributes to the S parameter. In our case, we have

$$\Delta S = 8\pi v^2(f_{VW} + 2f_{VB}) \tag{131}$$

$$= -8\pi c^2(c^2 - s^2)\frac{v^2}{g^2F^2} - 40\pi(c'^2 - a)(c'^2 - s'^2)\frac{v^2}{g'^2F^2}. \tag{132}$$

The second equation gives the value in the Littlest Higgs model.

Turning to the $SU(2)_c$-violating sector, the operator $\mathcal{O}'_{h,1}$ (124) yields a correction to the W mass (but not the Z mass):

$$\Delta M_W^2/M_W^2 = -\frac{v^2}{2} f_{h,1}. \tag{133}$$

This is equivalent to a contribution to the T parameter:

$$\alpha\Delta T = -\frac{v^2}{2} f_{h,1} \tag{134}$$

$$= -\frac{5}{4}(1 + 2c'^2 - 4a)(c'^2 - s'^2)\frac{v^2}{F^2} - \frac{2v^2\lambda_{2\phi}^2}{M_\phi^4}, \tag{135}$$

where again the second line applies to the Littlest Higgs model only.

Collecting all contributions, the physical vector masses get shifted as follows:

$$M_W^2 = \left(\frac{ev}{2s_w}\right)^2 (1+x), \qquad M_Z^2 = \left(\frac{ev}{2s_w c_w}\right)^2 (1+y), \tag{136}$$

where

$$x = \alpha\left(\frac{\Delta S}{4s_w^2} + \Delta T\right) = 2M_W^2(f_{VW} + 2f_{VB}) - \frac{v^2}{2} f_{h,1}, \tag{137}$$

$$y = \alpha\left(\frac{\Delta S}{4s_w^2 c_w^2}\right) = 2M_Z^2(f_{VW} + 2f_{VB}). \tag{138}$$

B. Non-Universal Hypercharges

If the model contains a current J_N which is linearly independent from the hypercharge current J_Y, the situation becomes more complicated. This typically happens if the fermions are charged under more than one $U(1)$ gauge group, since there is no particular reason to have the two $U(1)$ charges proportional to each other.

We may use the freedom of choosing the parameter a in (34a, 34b) to remove, for instance, the left-handed lepton contribution in J_N. Then, the unitary-gauge interactions induced by $\mathcal{O}_{VN}$ are

$$\begin{aligned}\mathcal{L}_{VN} = \frac{-2M_W^2}{gc_w} f_{VN}\,&(z_\ell \bar\ell_R \not{Z} \ell_R + z_Q \bar Q_L \not{Z} Q_L \\ &+ z_u \bar u_R \not{Z} u_R + z_d \bar u_R \not{Z} d_R) + \ldots,\end{aligned} \tag{139}$$

with some fixed parameters z_f, where the dots indicate couplings which involve the Higgs field. To verify that the terms in $\mathcal{L}_{VN}$ cannot be eliminated, we recall that the couplings to both the neutral isospin and hypercharge currents satisfy the sum rules

$$g_L^\nu + g_L^\ell = g_R^\nu + g_R^\ell, \tag{140}$$
$$g_L^u + g_L^d = g_R^u + g_R^d, \tag{141}$$
$$g_L^\nu + g_L^\ell = -3(g_L^u + g_L^d). \tag{142}$$

Any linear combination of the two currents also satisfies these sum rules. In particular, this holds for the electromagnetic current and for the current coupled to the Z-boson. Higher-dimensional bosonic operators do not affect this property.

However, by definition, the sum rules are violated by a nonvanishing J_N. Therefore, its presence can be constrained, e.g., by measuring the ratios

$$r_\ell = \frac{g_R^\nu + g_R^\ell}{g_L^\nu + g_L^\ell}, \qquad r_q = \frac{g_R^u + g_R^d}{g_L^u + g_L^d}, \qquad r_{q\ell} = -3\frac{g_L^u + g_L^d}{g_L^\nu + g_L^\ell}. \tag{143}$$

In other words, if any of these quantities deviates from unity, we know that the model contains a third linearly independent current, which in the present context is due to non-universal charge assignments for the $U(1)$ gauge groups.

In this situation, the standard two-parameter analysis of the electroweak precision observables is no longer appropriate. If the extra $U(1)$ charges (i.e., the parameters z_ℓ, z_Q, z_u, z_d) are taken as unknowns, many of the electroweak observables such as $A_{LR}, A_{FB}^b, \Gamma_Z, \Gamma_{\bar\nu\nu}$, etc. become independent of each other. The $U(1)$ charges may even depend on the fermion generation, as far as the constraints on flavor-changing neutral currents are respected. It is interesting that the numerical quality of the present electroweak fit is rather poor [20], so there might already be a hint of such new-physics contributions. On the other hand, for a specific model with fixed $U(1)$ charge assignments, it is straightforward to include the appropriate modifications in the expressions for the Z-fermion couplings. The remaining free parameters are formally equivalent to the S and T parameters which we have discussed in the preceding section. However, the numerical fit to the electroweak data has to be reconsidered in this framework [7].

C. Four-Fermion Interactions

At very low energies, the W and Z bosons are also integrated out and give way to the four-fermion interactions of the Fermi model. These interactions get corrections from the exchange of heavy vector bosons, i.e., from the operators

$$\mathcal{O}_{JJ}^{(3)} = \mathrm{tr}\, J^{(3),\mu} J_\mu^{(3)} \quad \text{and} \quad \mathcal{O}_{JJ}^{(1)} = J^{(1),\mu} J_\mu^{(1)}. \tag{144}$$

which are both present in (A1). Looking at charged-current interactions, together with the shift in the W mass this correction effectively modifies the relation of the Fermi constant and the Higgs vacuum expectation value v:

$$\sqrt{2}G_F = \frac{1}{v^2}(1+z) \quad \text{with} \quad z = -\alpha\Delta T - \frac{v^2}{4} f_{JJ}^{(3)}. \tag{145}$$

It is customary to choose G_F (as measured in muon decay) as an independent parameter of the SM. If this is complemented by M_Z and α (i.e., e), we have to account for the shifts in the vector boson masses, and define the parameters $\hat v_0$ and $\hat s_0$ by the relations

$$\hat v_0 = (\sqrt{2}G_F)^{-1/2} \quad \text{and} \quad M_Z = \frac{e\hat v_0}{2\hat s_0 \hat c_0}. \tag{146}$$

The two definitions of the weak mixing angle are thus related by

$$\begin{aligned} s_w^2 &= \hat s_0^2\left(1 + \frac{\hat c_0^2}{\hat c_0^2 - \hat s_0^2}(y+z)\right), \\ c_w^2 &= \hat c_0^2\left(1 - \frac{\hat s_0^2}{\hat c_0^2 - \hat s_0^2}(y+z)\right). \end{aligned} \tag{147}$$

D. Constraints on the Littlest Higgs Model

Electroweak precision data constrain the allowed parameter space of Little Higgs models. We have seen that in the case of universal hypercharges, up to the order v^2/F^2, all corrections to low-energy observables can be parameterized in terms of ΔS, ΔT, and two extra parameters which introduce contact interactions of the triplet and singlet currents. Even in the non-universal case, for any specific model where all hypercharge assignments are fixed we may take into account their effects in the electroweak observables explicitly. Then, in addition to S and T, the coefficient $f_{VN}^{(1)}$ is left as a free parameter, which can be constrained by the analysis of Z decays.

Contact interactions have been sought for both at hadron and lepton colliders. Since they are formally of higher order on the Z pole (the interference of signal and background vanishes on the resonance), they yield an independent set of constraints. The exact form depends on the $U(1)$ charge assignments.

For illustration, let us consider the original version of the Littlest Higgs model [2] where the situation is particularly simple, since all fermions couple to the first $U(1)$ group only ($a = 0$ and $J_N = 0$). The present exclusion limits for Z' bosons [19] can be turned into limits on the values of $f_{JJ}^{(3)}$ and $f_{JJ}^{(1)}$ (92, 116), i.e., on the ratios c^2/F and c'^2/F. Thus, for a given value of F these constraints can be evaded if c and c' are both small. This is the limit where the massive vector bosons become superheavy and simultaneously decouple from fermions. In fact, for c or c' less than about 0.1, the vector boson masses are of the same order as the cutoff Λ where the Little Higgs model breaks down as a low-energy effective theory, and new (strong) interactions may be expected.

From current experimental data, the combined limit for a Z' boson with SM-like couplings is $M_{Z'} \gtrsim 1.5$ TeV [19]. For the Littlest Higgs model, this translates roughly into

$$c^2 \lesssim F/4.5 \text{ TeV} \quad \text{and} \quad c'^2 \lesssim F/10 \text{ TeV}. \tag{148}$$

The limits for charged heavy vector bosons are somewhat weaker.

In the limit $c \sim c' \to 0$ where all contact interactions disappear, the correction to S (131) also vanishes. However, there remains a constant contribution to T [6],

$$\alpha\Delta T(c' = 0) = \frac{5v^2}{4F^2} - \frac{2v^2\lambda_{2\phi}^2}{M_\phi^4}, \tag{149}$$

where the second term depends on the parameters in the Coleman-Weinberg potential.

The first term in (149) is due to the existence of the heavy hypercharge boson Y. This positive shift in T pushes the model out of the exclusion contour in the S-T plane allowed by electroweak data for a light Higgs boson, unless F is larger than about 4 TeV. The second term, the shift due to heavy-scalar exchange, is negative. However, the bound (58) implies that the net ΔT will not be smaller than $v^2/\alpha F^2$.

When discussing Little Higgs models, it is usually assumed that the Higgs boson is light, presumably close to the lower experimental limit $m_0 \approx 115$ GeV. However, this is not necessarily true: Depending on the parameters in the Coleman-Weinberg potential (e.g., if the two contributions in the denominator of (57) almost cancel each other), the physical Higgs mass can take any value that is not in conflict with unitarity. Increasing the Higgs mass with respect to the reference value m_0, we get additional shifts in S and T. These are approximately given by [17]

$$\Delta S = \frac{1}{12\pi}\ln\frac{m_H^2}{m_0^2} \quad \text{and} \quad \Delta T = -\frac{3}{16\pi c_w^2}\ln\frac{m_H^2}{m_0^2} \tag{150}$$

(The complete one-loop formulae can be found, e.g., in [10].) As a consequence, the positive contribution to T can be partially cancelled by an increase in the Higgs mass. Note that this would also reduce the amount of fine-tuning in the model.

In the presence of oblique *and* non-oblique corrections one has to carefully define the S and T parameters. In our effective theory they are defined on the operator level and can be read off from the gauge boson masses (136), if the scale v is given. Introducing the abbreviations $g_W = e/s_w$ and $g_Z = e/(s_w c_w)$ for the couplings of W and Z to fermions (e and s_w defined at the Z pole), using (128, 130, 136–138) we can make up dimensionless ratios

$$\frac{M_W^2}{c_w^2 M_Z^2} = 1 - \frac{\alpha}{4c_w^2}\Delta S + \alpha\Delta T, \tag{151}$$

$$\frac{\Gamma_W^2}{M_W^2} \propto g_W^4(1 + 2\frac{\alpha}{4s_w^2}\Delta S), \tag{152}$$

$$\frac{\Gamma_Z^2}{M_Z^2} \propto g_Z^4(1 + 2\frac{\alpha}{4s_w^2 c_w^2}\Delta S), \tag{153}$$

where v drops out. Here, $\Gamma_{W/Z}$ stand for either the total width or for a partial decay width of the corresponding vector boson. The prefactors in (152, 153) are known functions which at leading order just depend on s_w, while higher-order corrections, in a consistent approximation, add incoherently to the new-physics contribution considered here. Accepting the fact that Γ_W is not sufficiently well measured to be relevant here, we nevertheless can extract S and T from (151) and (153) alone, i.e., exclusively from Z- and W-pole data.

This is not the conventional way of extracting the oblique parameters [16, 20], where e and s_w are used as above, but the low-energy observable G_F is included as a dimensionful quantity which sets the scale v. Due to the presence of non-oblique new physics, the relation between G_F and v is modified by an amount z according to (145), which accounts for the shifts in the electroweak couplings and the W mass as well as a triplet contact term. Generically, for any model with a heavy gauge triplet we have

$$z \equiv -\alpha\Delta T + \delta \tag{154}$$

with $\delta = (c^2 v/F)^2$, where c is the cosine of the mixing angle between the two $SU(2)$ and F the high scale.

In the presence of non-oblique corrections we may call the conventional definition of S, T, and U *effective* parameters, which in the linear approximation are given by [20]

$$\begin{aligned}\frac{M_W^2}{M_{W,0}^2} &= 1 + \frac{\alpha}{4s_w^2}(S_{\text{eff}} + U_{\text{eff}}),\\ \frac{M_Z^2}{M_{Z,0}^2} &= 1 + \frac{\alpha}{4s_w^2 c_w^2}S_{\text{eff}} - \alpha T_{\text{eff}},\\ \frac{\Gamma_Z}{M_Z^2\beta_Z} &= 1 + \alpha T_{\text{eff}},\end{aligned} \tag{155}$$

where $\beta_Z = \Gamma_{Z,0}/M_{Z,0}^3$. The quantities with the zero subscript are calculated in the pure SM, i.e, using the *measured* values of e, s_w, and G_F. While the extraction of e from electromagnetic data and s_w from Z-pole asymmetries is free of non-oblique corrections, G_F contains an extra contribution z (145) if related to the electroweak scale v.

In our effective theory, we obtain

$$\begin{aligned}\frac{M_W^2}{M_{W,0}^2} &= 1 + \frac{\alpha}{4s_w^2}\Delta S + \delta,\\ \frac{M_Z^2}{M_{Z,0}^2} &= 1 + \frac{\alpha}{4s_w^2 c_w^2}\Delta S - \alpha\Delta T + \delta,\\ \frac{\Gamma_Z}{M_Z^2\beta_Z} &= 1 + \alpha\Delta T - \delta.\end{aligned} \tag{156}$$

A comparison with (155) reveals the connection between the effective parameters and the ones we have calculated above,

$$S_{\text{eff}} = \Delta S, \tag{157}$$

$$T_{\text{eff}} = \Delta T - \frac{1}{\alpha}\delta, \tag{158}$$

$$U_{\text{eff}} = \frac{4s_w^2}{\alpha}\delta. \tag{159}$$

The result is somewhat unexpected: The choice of the low-energy parameter G_F as input mimics nonvanishing T_{eff} and U_{eff} even in the absence of custodial $SU(2)$ violation. Actually, low-energy neutral-current data or a precise measurement of the W width would allow for the identification of the non-oblique correction δ, but the present experimental accuracy is insufficient for this, given the small values of δ that are allowed by the direct constraints on heavy vector bosons.

In the Littlest Higgs model, for increasing mixing angle c the shift in T_{eff} (which is $\propto c^4$) compensates the positive ΔT contribution of the $SU(2)_c$-violating sector, eventually resulting in a negative T_{eff} value.

In Fig. 1, we depict the allowed region in the S_{eff}-T_{eff} plane for two different values of F. The contours are restricted by the direct limits on contact interactions. After having translated our ΔS, ΔT parameters to the effective ones, we can take the fit of the ST_{eff}-parameters for a given Higgs mass *as is* [20] and compare it with the prediction of the model under consideration. Looking at the figure, we can conclude that a light Higgs boson ($m_H = 120$ GeV) is consistent with the Littlest Higgs model only if $F \gtrsim 4$ TeV [4–6]. However, allowing for larger Higgs masses reduces this limit to less than 4 TeV. We should also keep in mind that radiative corrections [13] and unknown effects from new physics beyond the UV cutoff Λ will add extra small shifts to the S and T parameters which can slightly change this conclusion.

Our derivation shows how this picture looks like in more general models. In the triplet sector, any extension will only result in additional contributions which have a form identical to that in the Littlest Higgs model. These terms will contribute to ΔS only. In the singlet sector, there is more freedom: Removing the $U(1)$ boson, changing the hypercharge assignments, or extending this sector in some other way, allows for different values of ΔT [6, 7, 16], and more free parameters may enter the game. In particular, the spectrum can be arranged to be consistent with the custodial $SU(2)$ symmetry, such that the corrections to the T parameter cancel altogether. In such models, the bounds on F are significantly weaker than in the Littlest Higgs model [3].

V. NEW EFFECTS

New data from Tevatron, LHC, and a future Linear Collider will allow us to constrain the parameter space by measurements of new independent observables. Apart from improved limits on contact interactions, there will be precision data on vector-boson self-couplings, on Higgs and top-quark interactions. In this section, we derive the corresponding anomalous contributions, starting from the effective Lagrangian (A1).

The results given below apply directly to any Little Higgs model which contains no extra singlet current and only one light Higgs doublet. The complications which arise in the presence of an extra current have been discussed above. In addition to the anomalous Z couplings, the operator $\mathcal{O}_{VN}^{(1)}$ (115) induces $HZff$ interactions which add to the terms in (170). As far as the Higgs sector is concerned, many Little Higgs models predict more than one light Higgs doublet. However, in the present paper we do not attempt a discussion of extended Higgs sectors, and leave the general case to a future publication. If there are multiple physical Higgs states, the structure of anomalous couplings in the gauge and fermion sectors is unchanged, but the genuine Higgs-gauge, Higgs-fermion and Higgs self-couplings reflect the additional complications.

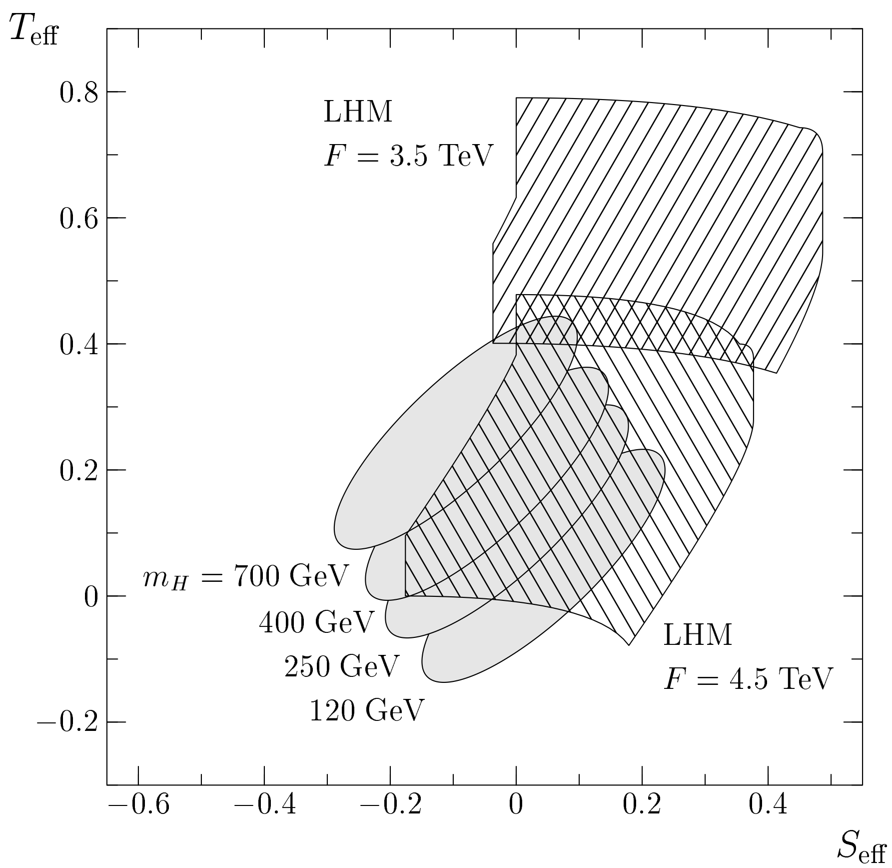

FIG. 1: *ST* predictions for the Littlest Higgs model with standard $U(1)$ charge assignments [2]. The shaded ellipses are the 68 % exclusion contours which follow from the electroweak precision data [16, 20], assuming four different Higgs masses. The hatched areas are the allowed parameter ranges of the Littlest Higgs model for two different values of the scale F. The limits from contact interactions have been taken into account.

A. Anomalous Triple-Gauge Couplings

Strictly speaking, tree-level contributions to triple gauge couplings from the exchange of heavy particles are impossible at the level of dimension-six operators [11]. In the absence of fermions the self-couplings of vector bosons define the gauge couplings g and g'. However, in practice the gauge couplings are measured in processes which involve external fermions. These interactions do receive tree-level corrections. We have used the equations of motion to canonically normalize the fermion gauge interactions. As a result, anomalous triple-gauge couplings appear.

We use the standard parameterization:

$$\begin{aligned}\mathcal{L}_{WWV} = &-\mathrm{i}e\frac{c_w}{s_w}\Big[g_1^Z\left(W^{+,\mu\nu}W_\mu^- - W_\mu^+W^{-,\mu\nu}\right)Z_\nu + \kappa_Z W_\mu^+W_\nu^- Z^{\mu\nu}\Big]\\ &-\mathrm{i}e\Big[g_1^\gamma\left(W^{+,\mu\nu}W_\mu^- - W_\mu^+W^{-,\mu\nu}\right)A_\nu + \kappa_\gamma W_\mu^+W_\nu^- A^{\mu\nu}\Big]\end{aligned} \tag{160}$$

Inserting our results into the formulae of [10], we obtain

$$g_1^Z = \kappa_Z = 1 - 2M_Z^2 f_{VW} \quad \text{and} \quad g_1^\gamma = \kappa_\gamma = 1. \tag{161}$$

Since the anomalous contributions are generated solely by a renormalization of gauge couplings, we have $g_1^Z = \kappa_Z$ and $g_1^\gamma = \kappa_\gamma$. Electromagnetic gauge invariance requires $g_1^\gamma = 1$, hence the photon couplings are unchanged. The $SU(2)_c$ relation

$$\Delta\kappa_\gamma = -\frac{c_w^2}{s_w^2}(\Delta\kappa_Z - \Delta g_1^Z) \tag{162}$$

is automatically satisfied in this case, so there is no contribution from the $SU(2)_c$-violating coefficient f_{VB}.

In our derivation of the effective Lagrangian, we have expressed everything in terms of the parameters e, s_w, and a dimensionful quantity, for which we either take M_W, M_Z, or v. As far as the coefficients of dimension-six operators are concerned, the particular scheme used for experimentally defining the input parameters is irrelevant, since differences are formally of higher order. However, when discussing interactions which also occur at tree level in the Standard Model, we have to be specific about the definition of the input parameters, since a shift in the SM contribution will be of the same order as the direct contribution of the anomalous interactions.

For instance, in the G_F-M_Z-α scheme, there is an additional contribution to g_1^Z and κ_Z since we have to define the three-gauge-boson vertices in terms of $\hat{s}_0$ and $\hat{c}_0$:

$$\mathcal{L}_{WWZ} = -\mathrm{i}e\frac{\hat{c}_0}{\hat{s}_0}\Big[g_1^Z\left(W^{+,\mu\nu}W_\mu^- - W_\mu^+W^{-,\mu\nu}\right)Z_\nu + \kappa_Z W_\mu^+W_\nu^- Z^{\mu\nu}\Big], \tag{163}$$

where now

$$g_1^Z = \kappa_Z = 1 - 2M_Z^2 f_{VW} - \frac{y+z}{2(\hat{c}_0^2 - \hat{s}_0^2)}. \tag{164}$$

B. Anomalous Higgs couplings

Anomalous Higgs couplings are induced both by vector and scalar exchange. Expanding the effective Lagrangian (A1), we obtain the following contributions:

1. Couplings to longitudinal gauge bosons:

$$\mathcal{L}_{HVV} = \chi_W\frac{2M_W^2}{v}HW^{+,\mu}W_\mu^- + \chi_Z\frac{M_Z^2}{v}HZ^\mu Z_\mu \tag{165}$$

These occur at tree-level in the SM and get anomalous contributions from various sources. The direct contributions are

$$\chi_W = 1 + 4M_W^2 f_{VW} + \frac{v^2}{2}f_{hh}, \tag{166}$$

$$\chi_Z = 1 + 4\left(M_W^2 f_{VW} + 2s_w^2 M_Z^2 f_{VB}\right) + \frac{v^2}{2}\left(f_{h,1} + f_{hh}\right). \tag{167}$$

These relations apply if we define the parameters M_W and M_Z in (165) as the measured values (136). Due to the appearance of v in (165), in the G_F-M_Z-α scheme there is also an indirect contribution to be added:

$$\Delta\chi_W = \Delta\chi_Z = -\frac{1}{2}z, \tag{168}$$

where z is given in (145).

2. Couplings to transversal gauge bosons:

$$\mathcal{L}'_{HVV} = h'_{WW}HW^{+,\mu\nu}W_{\mu\nu}^- + h'_{ZZ}HZ^{\mu\nu}Z_{\mu\nu} + h'_{ZA}HZ^{\mu\nu}A_{\mu\nu} + h'_{AA}HA^{\mu\nu}A_{\mu\nu} \tag{169}$$

While these terms are generated by the operators $\mathcal{O}_{BW}$, $\mathcal{O}'_{WW}$, and $\mathcal{O}'_{BB}$ individually, they all vanish in the linear combination present in (A1), and thus are induced at loop-level only, as one would expect.

3. Contact terms, i.e., direct couplings of Higgs bosons to vector bosons and the weak or hypercharge currents. These follow from the (A1), if the physical fields are inserted, derivatives acting on the Higgs field are eliminated by partial integration, and the equations of motion of the vector fields are applied. Alternatively, we can read them off directly from the V-J interactions in (45,46):

$$\begin{aligned}\mathcal{L}_{HVJ} = {} & M_W f_{VJ}^{(3)}\left(HW^{+,\mu}J_\mu^- + HW^{-,\mu}J_\mu^+\right) \\ & + M_Z f_{VJ}^{(3)} HZ^\mu J_\mu^{(3),0} \\ & - 2M_Z HZ^\mu\left(f_{VJ}^{(1)}J_{Y,\mu}^{(1)} + f_{VN}^{(1)}J_{N,\mu}^{(1)}\right).\end{aligned} \tag{170}$$

4. Anomalous couplings of the Higgs boson to the scalar current, i.e., to massive fermions. Due to the effect of the operator $\mathcal{O}'_{J_S}$, all such couplings are modified by the common factor

$$\chi_f = 1 - v^2 f_{J_S} = 1 + \frac{v^2}{2}\left(f_{VV}^{(3)} + 2f_{VV}^{(1)}\right). \tag{171}$$

If the fermions are mixed with new heavy particles, as it is the case for the top quark in Little Higgs models, there are extra contributions to (171). These will be considered below in Sec. V D.

C. Higgs pairs

In the effective Lagrangian (A1), various terms induce anomalous couplings which are relevant for Higgs pair production.

1. The quartic $HHWW$ and $HHZZ$ couplings are modified:

$$\mathcal{L}_{HHVV} = \eta_W\frac{M_W^2}{v^2}H^2W^{+,\mu}W_\mu^- + \eta_Z\frac{M_Z^2}{2v^2}H^2Z^\mu Z_\mu, \tag{172}$$

where

$$\eta_W = 1 + 20M_W^2 f_{VW} - \frac{v^2}{2}\left(f_{h,1} - 4f_{hh}\right), \tag{173}$$

$$\eta_Z = 1 + 20\left(M_W^2 f_{VW} + 2s_w^2 M_Z^2 f_{VB}\right) + 2v^2\left(f_{h,1} + f_{hh}\right). \tag{174}$$

are the direct contributions. Here, we have applied the equations of motion of the Higgs boson to eliminate derivative couplings. The indirect corrections in the G_F-M_Z-α scheme are

$$\Delta\eta_W = \Delta\eta_Z = -z. \tag{175}$$

2. The cubic Higgs self-coupling is directly affected by the presence of the operator $\mathcal{O}_{h,3}$. Furthermore, the operators $\mathcal{O}_{h,1}$ and $\mathcal{O}_{hh}$ contribute to this coupling if we eliminate derivative couplings by the equations of motion. Parameterizing the vertex by

$$\mathcal{L}_{HHH} = -\chi_H \frac{m_H^2}{2v} H^3, \qquad (176)$$

we have a direct contribution

$$\chi_H = 1 - \frac{v^2}{2}\left(f_{h,1} + f_{hh}\right) - \frac{2v^4}{3m_H^2} f_{h,3}. \qquad (177)$$

To determine the indirect contribution, we augment the set of independent parameters (G_F, M_Z, α) by the physical Higgs mass m_H, to get

$$\Delta\chi_H = -\frac{1}{2}z \qquad (178)$$

in this scheme.

D. Top-Quark Couplings

The presence of heavy vector-like quarks in the spectrum affects the interactions of the top quark with gauge bosons and Higgs bosons in a non-universal way.

1. The electroweak interactions of the top and bottom quarks are modified by the operators $\mathcal{O}_{Vq}$ and $\mathcal{O}_{Vt}$ (77, 78), which originate from heavy quark exchange, and by the redefinition of the vector fields due to heavy vector exchange. In the physical basis, the Z and W couplings are

$$\begin{aligned}\mathcal{L}_{tV} = &-\frac{e}{4c_w s_w}\left[\bar{t}\not{Z}\left(v_t - a_t\gamma_5\right)t - \bar{b}\not{Z}\left(v_b - a_b\gamma_5\right)b\right] \\ &- \frac{e}{2\sqrt{2}\,s_w} c_{tb}\left[\bar{t}\not{W}^+(1-\gamma_5)b + \bar{b}\not{W}^-(1-\gamma_5)t\right]\end{aligned} \qquad (179)$$

where the coefficients are given by

$$v_b = \left(1 - \frac{4}{3}s_w^2\right)\left[1 + M_Z^2\left(f_{VW} + 2f_{VB}\right)\right] \qquad (180)$$

$$a_b = 1 + M_Z^2\left(f_{VW} + 2f_{VB}\right) \qquad (181)$$

$$\begin{aligned}v_t = &\left(1 - \frac{8}{3}s_w^2\right)\left[1 + M_Z^2\left(f_{VW} + 2f_{VB}\right)\right] \\ &+ v^2(f_{Vq} + f_{Vt})\end{aligned} \qquad (182)$$

$$a_t = 1 + M_Z^2\left(f_{VW} + 2f_{VB}\right) + v^2(f_{Vq} - f_{Vt}) \qquad (183)$$

$$c_{tb} = 1 + M_W^2\left(f_{VW} + 2f_{VB}\right) + \frac{v^2}{2} f_{Vq} \qquad (184)$$

While the corrections proportional to $(f_{VW}+2f_{VB})$ are universal to all fermions and taken into account by the S-T fit of the SM, the corrections proportional to f_{Vq} and f_{Vt} are specific to the top-quark vertices.

The indirect corrections in the G_F-M_Z-α scheme are in this case

$$\Delta v_b = -\left(1 + \frac{4\hat{s}_0^2/3}{\hat{c}_0^2 - \hat{s}_0^2}\right)\frac{y+z}{2}, \qquad (185)$$

$$\Delta v_t = -\left(1 + \frac{8\hat{s}_0^2/3}{\hat{c}_0^2 - \hat{s}_0^2}\right)\frac{y+z}{2}, \qquad (186)$$

$$\Delta a_b = \Delta a_t = -\frac{y+z}{2}, \qquad (187)$$

$$\Delta c_{tb} = -\frac{\hat{c}_0^2}{\hat{c}_0^2 - \hat{s}_0^2}\,\frac{y+z}{2}. \qquad (188)$$

2. The top-quark Yukawa coupling is also modified by the heavy T-quark. There are further effects due to nonlinear Goldstone interactions and heavy-scalar exchange which altogether make up the coefficient of the operator $\mathcal{O}_{hq}$. Finally, there are the corrections from $\mathcal{O}_{VV}^{(3)}$ and $\mathcal{O}_{VV}^{(1)}$ which have been given already in (171). The resulting vertex is

$$\mathcal{L}_{tH} = -\frac{m_t}{v}\chi_t \bar{t}Ht, \qquad (189)$$

where

$$\chi_t = 1 + \frac{v^2}{2}\left(f_{VV}^{(3)} + 2f_{VV}^{(1)}\right) + \frac{v^3}{\sqrt{2}\,m_t} f_{hq}. \qquad (190)$$

The indirect contribution in the G_F-M_Z-α scheme is

$$\Delta\chi_t = -\frac{1}{2}z. \qquad (191)$$

3. There are also quartic $ttZH$ and $tbWH$ vertices:

$$\begin{aligned}\mathcal{L}_{tVH} = &-M_Z f_{Vq}\bar{t}H\not{Z}\left(1-\gamma_5\right)t \\ &- M_Z f_{Vt}\bar{t}H\not{Z}\left(1+\gamma_5\right)t \\ &- \frac{1}{\sqrt{2}} M_W f_{Vq}\Big[\bar{t}H\not{W}^+(1-\gamma_5)b \\ &\qquad + \bar{b}H\not{W}^-(1-\gamma_5)t\Big]\end{aligned} \qquad (192)$$

In the Littlest Higgs model, $f_{Vt} = 0$, and these couplings are purely left-handed, which is due to the fact that it is the right-handed top quark which mixes with the heavy T fermion.

VI. THE RECONSTRUCTION OF A LITTLE HIGGS MODEL

Despite the fact that Little Higgs Models are constrained by electroweak precision data, there remains a considerable parameter space where such models are viable. Assuming that such a mechanism is realized in nature, one should ask the question to what extent it is possible to derive the model and its parameters from experiments at future colliders.

To verify the generic mechanism that is common to all Little Higgs models, we would like to check two characteristic properties of the model, namely the cancellation of quadratic divergences as a result of the symmetry structure, and the Goldstone-boson nature of the Higgs boson. A direct check of the first property would require the measurement of the quartic couplings of Higgs bosons to heavy vectors, scalars, and fermions, which is out of reach of the next generation of colliders. However, the symmetry structure manifests itself also in relations of couplings which are accessible once the new particles have been discovered at the LHC [6, 16, 21]. The same couplings also enter the low-energy effective Lagrangian. For instance, the cancellation in the vector-boson triplet sector is reflected in the relation of the coefficients $f_{JJ}^{(3)}$ and f_{VW} (92, 93), once the scale F is known. Similar statements hold for the scalar and fermion sectors. Thus, a sufficiently accurate determination of the low-energy coefficients complements direct measurements at LHC. In cases where direct measurements are difficult (e.g., in the scalar sector), low-energy observables may be the only handle on the Little Higgs mechanism.

In order to establish the Goldstone nature of the Higgs boson, we should demonstrate the nonlinearity in the Higgs-boson representation above the scale F, i.e., the presence of nonrenormalizable terms in the Higgs interactions. The low-energy trace of this is encoded in terms which are independent of the mixing angles and masses of the Little Higgs spectrum. For instance, in the Littlest Higgs model there are a constant contribution $-1/6F^2$ in the coefficient $f_{VV}^{(3)}$ (94) and a similar term in the coefficient f_{hq} (81) (i.e., the constant 2/3 in β (73)).

Since the anomalous contributions we have calculated in the preceding sections all carry a common suppression factor v^2/F^2 relative to the SM result, for a meaningful measurement the low-energy observables have to be determined at least to this accuracy. If F happens to be rather high (e.g., $F \gtrsim 4$ TeV for the unmodified Littlest Higgs model), the required precision is in the per mil range. A high-luminosity e^+e^- Linear Collider can reach this level for a limited subset of observables which include contact terms and triple-gauge couplings. In the Higgs and top sectors, accuracies of the order of 1-2 % are possible for the observables of interest [22]. At LHC, the level of precision is generically weaker, but direct measurements are possible for new heavy particles in the spectrum. Thus, if the scale F is of the order 2 TeV or less, which is allowed in various Little Higgs models [3, 7], a complete coverage of the low-energy parameters becomes feasible. In any case, all observables will be included in a combined fit if signals of a Little Higgs model are found, once a sufficient data sample has been collected at LHC and a Linear Collider.

A. Vector Bosons

The new X and Y gauge bosons can be produced and detected at the LHC if they are not too heavy [6], and their couplings can be directly measured [21]. Indirect constraints from low-energy observables can be combined with those results for an improved fit and will help to disentangle the contributions of various sectors:

1. The measurement of contact terms, e.g., in the processes
$$e^+e^- \to e^+e^- \quad \text{and} \quad e^+e^- \to \mu^+\mu^- \tag{193}$$
will significantly improve the limits for a particular combination of the operator coefficients $f_{JJ}^{(1)}$ and $f_{JJ}^{(3)}$, equivalent to the detection of a Z' boson up to a mass of 5-10 TeV [22, 23]. For a separate measurement of the triplet contribution, one needs a charged-current channel. For instance, the cross section measurement of the process
$$e^+e^- \to \bar\nu\nu\gamma \tag{194}$$
allows for detecting the effect of W' bosons up to $M \sim 5$ TeV [22, 24]. These limits can be combined with the possible observation (or non-observation) of those states at LHC to extract the scale F and the mixing angles in the vector-boson sector.

2. Another probe of heavy-vector exchange is given by quartic $HZff$ and $HWff$ interactions, which depend on $f_{VJ}^{(3)}$ and $f_{VJ}^{(1)}$. The neutral component can be extracted by measuring the angular distribution and/or the energy dependence of the Higgs-strahlung process [25], while the charged component affects WW fusion. (A detailed experimental analysis of contact terms in Higgs production has not yet been performed.)

3. The triple-gauge couplings will be measured to better than per mil accuracy at a Linear Collider [22, 26]. Assuming that S and T and the contact terms are known, this allows for the extraction of the coefficient $f_{VJ}^{(3)}$ to a precision level comparable to the contact-term measurements. Thus, an independent check of the coupling relations in the vector-boson sector is feasible.

4. Once the Higgs mass is known, the existing precision data can be turned into measurements of ΔS and ΔT. If the Giga-Z option of a Linear Collider is realized, the accuracy of this measurement will improve by one order of magnitude [22]. In our context, the value of ΔS provides us with the parameter combination $f_{VW} + 2f_{VB}$. Turning the argument around, together with the measurement of triple gauge couplings one gets an independent constraint on $f_{JJ}^{(3)}$.

Combining those measurements in a single fit, all parameters in the gauge sector can be derived. In particular, if LHC and Linear Collider data are taken together, there will be enough redundancy to go beyond the assumption of a specific model, such that the complete set of heavy vector bosons (singlets and triplets) and their couplings can be reconstructed.

B. Scalars

In the effective Lagrangian (A1), Higgs-boson operators are affected both by the scalar and by the vector boson sector. Since the vector-boson contributions can be extracted by the methods described above, we get an indirect handle on the scalar sector, which is difficult to access directly. The statistical and systematic uncertainties for Higgs production and decay measurements at LHC and a Linear Collider limit the achievable accuracy to 1-2 % or worse, depending on the channel and on the Higgs mass [22, 27]. The following arguments show that a complete coverage of the scalar sector is possible in principle. In practice, this exercise can be successful if the scale F is of the order 2 TeV or lower, while for higher scales the accessible information becomes limited.

1. ΔT depends on $f_{VV}^{(1)}$ and a correction due to heavy-scalar exchange. Once the Higgs mass and the properties of new $U(1)$ vector bosons are known, we can isolate this piece. This quantity (i.e., the ρ parameter) has been measured with per mil accuracy. At GigaZ this can be improved by another order of magnitude.

2. The couplings of the Higgs boson to gauge bosons will be measured in Higgs-strahlung and vector boson fusion. Combining this with the information on ΔT, we can constrain the coefficient $f_{VV}^{(3)}$.

3. The ratio of the branching ratios $H \to ff$ and $H \to WW, ZZ$ also depends on the coefficients $f_{VV}^{(3)}$ and $f_{VV}^{(1)}$. Thus, Higgs decay measurements will add independent information on those coefficients. (Here, we need the assumption that the fermions are not mixed with any heavy partners, which is likely true for the b quark, and even more for τ and c.)

4. Finally, double Higgs production depends on the coefficient $f_{h,3}$, the Higgs potential correction. Both at a Linear Collider and at LHC this measurement is severely statistics-limited [22, 28, 29], and in Little Higgs models the small corrections to the trilinear Higgs coupling are unobservable even for very low F.

If sufficient precision can be reached from a combination of all available data, we can isolate the contribution of the heavy scalar ϕ (and thus confirm its existence) and detect the constant contribution in $f_{VV}^{(3)}$ (94) which stems from the last term of (40). As discussed above, this would be direct evidence for the Goldstone-boson nature of the Higgs boson.

Our discussion has been centered on the Littlest Higgs model with its obvious generalizations, which contains just a single Higgs doublet in its low-energy spectrum. Other Little Higgs models predict a richer structure: Apart from extra doublets, there could also be light scalar singlets and triplets, which have to be pair-produced and thus are difficult to access. While this complication will not invalidate our treatment of the vector-boson sector, the reconstruction of the scalar sector in such models is beyond the scope of the present paper.

C. Top-Quark Observables

The top quark will be studied both at LHC and at a Linear Collider. In addition, LHC opens the opportunity to produce new states in the quark sector directly (e.g., the heavy quark T of the Littlest Higgs model) and study their decays [6, 16]. Here, we consider the information on this sector which low-energy observables can provide.

1. While $\bar{t}t$ production at threshold is dominated by QCD effects, continuum production of top pairs allow for an accurate determination of the form factors v_t and a_t and thus provides a measurement of the operator coefficients f_{Vq} and f_{Vt}. The achievable accuracy is of the order 1-2 % [22, 30].

2. The same coefficients are probed by measurements of the tbW vertex in single-top production and in top decays.

3. A measurement of the top Yukawa coupling (or the ratio g_{ttH}/g_{bbH}) complements this by information on the scalar couplings to the top sector, i.e., the coefficient f_{hq}. Similar to $f_{VV}^{(3)}$, this anomalous coupling contains a constant contribution which is not due to heavy-particle exchange, but a consequence of the nonlinear Goldstone nature of the Higgs boson. Here, a Linear Collider could reach a precision of up to 2.5 % (depending on the Higgs mass) [22].

The sensitivity to the top-quark sector of Little Higgs models from low-energy observables is similar to the scalar sector. However, the prospects for direct measurements at the LHC are more favorable in this case due to the fact that new heavy quarks can be produced by strong interactions.

VII. CONCLUSIONS

Using the effective-theory formalism, we have given a complete account of the anomalous couplings that are present in Little Higgs models below the new-particle threshold. In models without extra gauged $U(1)$ groups, all present-day constraints on the parameter space are encoded in the parameters S and T and the coefficients of four-fermion contact interactions. If new $U(1)$ vector bosons with non-canonical hypercharge assignments are present, they complicate this picture and introduce shifts in the individual Z-fermion couplings, so the electroweak fit has to be adapted accordingly.

The existing constraints on S and T, if combined with the limits on contact interactions, push the expected scale F of the minimal Little Higgs model (the Littlest Higgs model in its original version) up to 4 TeV and higher, where for a scale less than about 5 TeV a high Higgs mass is necessary to fit the electroweak data. This lower limit is mainly caused by the large amount of custodial $SU(2)$ violation in this model, and it can be evaded by a different treatment of the hypercharge sector. The limits on the S parameter are similarly constraining and restrict the mixing angles in the vector-boson sector.

Beyond the electroweak precision observables which have been measured so far, new anomalous couplings exhibit traces of all sectors of the model. While some masses and couplings can be determined directly in the production and decay of heavy particles at the LHC, it becomes possible to derive the full structure of the model by combining this with precision measurements at a future Linear Collider. If the achievable precision is sufficient (this strongly depends on the actual value of the new-physics scale F), we will be able to check the coupling relations that are responsible for the cancellation of divergences and the nonlinear nature of the Higgs interactions, thus verifying or excluding the Little Higgs mechanism as the correct model of electroweak symmetry breaking.

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APPENDIX A: THE EFFECTIVE LAGRANGIAN

In this appendix we collect the dimension-six operators which are present in the low-energy theory after integrating out the heavy degrees of freedom. The result can be written in the form

$$\begin{aligned}\mathcal{L}_6 = {} & f_{JJ}^{(3)}\mathcal{O}_{JJ}^{(3)} + f_{JJ}^{(1)}\mathcal{O}_{JJ}^{(1)} \\ & + f_{VN}^{(1)}\mathcal{O}_{VN} - \frac{1}{2}(f_{VV}^{(3)} + 2f_{VV}^{(1)})\mathcal{O}'_{J_S} \\ & - 2(f_{VW} + 2f_{VB})\mathcal{O}_{BW} \\ & - 4f_{VW}\mathcal{O}_W - 2f_{VW}\mathcal{O}'_{WW} \\ & - 8f_{VB}\mathcal{O}_B - 4f_{VB}\mathcal{O}'_{BB} \\ & + f_{h,1}\mathcal{O}'_{h,1} + f_{hh}\mathcal{O}'_{hh} + f_{h,3}\mathcal{O}'_{h,3} \\ & + f_{Vq}\mathcal{O}_{Vq} + f_{Vt}\mathcal{O}_{Vt} + f_{hq}\mathcal{O}'_{hq}.\end{aligned} \tag{A1}$$

For the operators, we adopt the basis of Ref. [10] with some minor modifications. The operators are defined as follows:

$$\mathcal{O}_{JJ}^{(3)} = \operatorname{tr} J^{(3),\mu} J_\mu^{(3)}, \tag{A2a}$$
$$\mathcal{O}_{JJ}^{(1)} = J^{(1),\mu} J_\mu^{(1)}, \tag{A2b}$$
$$\mathcal{O}_{VN}^{(1)} = V^{(1),\mu} J_{N,\mu}^{(1)}, \tag{A2c}$$
$$\mathcal{O}'_{J_S} = (h^\dagger h - v^2/2)(h^\dagger J_S + J_S^\dagger h), \tag{A2d}$$
$$\mathcal{O}_{BW} = -\frac{1}{2} B_{\mu\nu} h^\dagger W^{\mu\nu} h, \tag{A2e}$$
$$\mathcal{O}_W = i(D_\mu h)^\dagger W^{\mu\nu}(D_\nu h), \tag{A2f}$$
$$\mathcal{O}'_{WW} = -\frac{1}{2}(h^\dagger h - v^2/2) \operatorname{tr} W_{\mu\nu} W^{\mu\nu}, \tag{A2g}$$
$$\mathcal{O}_B = \frac{i}{2}(D_\mu h)^\dagger (D_\nu h) B^{\mu\nu}, \tag{A2h}$$
$$\mathcal{O}'_{BB} = -\frac{1}{4}(h^\dagger h - v^2/2) B_{\mu\nu} B^{\mu\nu}, \tag{A2i}$$
$$\mathcal{O}'_{h,1} = \left((D_\mu h)^\dagger h\right)\left(h^\dagger (D^\mu h)\right) - (v^2/2)(D_\mu h)^\dagger (D^\mu h), \tag{A2j}$$
$$\mathcal{O}'_{hh} = (h^\dagger h - v^2/2)\left((D_\mu h)^\dagger (D^\mu h)\right), \tag{A2k}$$
$$\mathcal{O}'_{h,3} = \frac{1}{3}(h^\dagger h - v^2/2)^3, \tag{A2l}$$
$$\mathcal{O}_{Vq} = \bar{\tilde{q}}_L \not{V}^T \tilde{q}_L, \tag{A2m}$$
$$\mathcal{O}_{Vt} = \bar{t}_R \not{V}^{(1)} t_R, \tag{A2n}$$
$$\mathcal{O}'_{hq} = (h^\dagger h - v^2/2)\left(\bar{t}_R h^T \tilde{q}_L + \text{h.c.}\right), \tag{A2o}$$

where V_μ is the vector field

$$V_\mu = i\left[h(D_\mu h)^\dagger - (D_\mu h)h^\dagger\right] \tag{A3}$$

with triplet and singlet parts

$$V_\mu^{(1)} = \operatorname{tr} V_\mu, \quad V_\mu^{(3)} = V_\mu - \frac{1}{2}\operatorname{tr} V_\mu. \tag{A4}$$

The triplet fermion current $J^{(3)}$ is the usual isospin current, $J_Y^{(1)}$ is the hypercharge current, and J_S is the fermion current coupled to the SM Higgs doublet. The exact form of the currents $J^{(1)}$ and $J_N^{(1)}$ is model-dependent. In Little Higgs models where only one $U(1)$ gauge boson is coupled to fermions, J_N vanishes, and $J^{(1)}$ is proportional to the hypercharge current.

In the Littlest Higgs model, the values of the coefficients in (A1) are

$$f_{JJ}^{(3)} = -\frac{4c^4}{F^2}, \tag{A5a}$$

$$f_{JJ}^{(1)} = -\frac{10}{F^2}, \tag{A5b}$$

$$f_{VN}^{(1)} = \frac{5(c'^2 - s'^2)}{F^2}, \tag{A5c}$$

$$f_{VV}^{(3)} = \frac{(1+2c^2)(c^2-s^2)}{4F^2} - \frac{1}{6F^2}, \tag{A5d}$$

$$f_{VV}^{(1)} = \frac{5(1+2c'^2-4a)(c'^2-s'^2)}{8F^2}, \tag{A5e}$$

$$f_{VW} = -\frac{1}{2g^2} f_{VJ}^{(3)} = -\frac{c^2(c^2-s^2)}{g^2F^2}, \tag{A5f}$$

$$f_{VB} = -\frac{1}{2g'^2} f_{VJ}^{(1)} = -\frac{5(c'^2-a)(c'^2-s'^2)}{2g'^2F^2}, \tag{A5g}$$

$$f_{h,1} = 4 f_{VV}^{(1)} + \frac{4\lambda_{2\phi}^2}{M_\phi^4}, \tag{A5h}$$

$$f_{hh} = 3 f_{VV}^{(3)} + 2 f_{VV}^{(1)} + \frac{4\lambda_{2\phi}^2}{M_\phi^4}, \tag{A5i}$$

$$f_{h,3} = -3\frac{m_H^2}{v^2}\left(f_{VV}^{(3)} + 2 f_{VV}^{(1)} - \frac{1}{3F^2}\right), \tag{A5j}$$

$$f_{Vq} = -\frac{s_t^4}{F^2}, \tag{A5k}$$

$$f_{Vt} = 0, \tag{A5l}$$

$$f_{hq} = \frac{\sqrt{2}\lambda_t}{F^2}\left(c_t^2 s_t^2 - \frac{2}{3} + \frac{\sqrt{2}\lambda_{2\phi}F}{M_\phi^2}\right), \tag{A5m}$$

while the expressions for the Coleman-Weinberg potential parameters are given in (50–51e). The singlet current $J^{(1)}$ is given by

$$J_\mu^{(1)} = (c'^2 - a) J_{Y,\mu}^{(1)} + J_{N,\mu}^{(1)}, \tag{A6}$$

where J_N collects the terms that cannot be absorbed into the hypercharge current by shifting the parameter a. In the original version of the model [2], J_N and a both vanish.

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