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Vertex diagrams for the QED form factors at the 2-loop level

R. Bonciani* †,

Facultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg, D-79104 Freiburg, Germany

P. Mastrolia‡,

Dipartimento di Fisica dell'Università di Bologna, I-40126 Bologna, Italy and Institut für Theoretische Teilchenphysik, Universität Karlsruhe, D-76128 Karlsruhe, Germany

E. Remiddi§

Dipartimento di Fisica dell'Università di Bologna and INFN Sezione di Bologna, I-40126 Bologna, Italy

Abstract

We carry out a systematic investigation of all the 2-loop integrals occurring in the electron vertex in QED in the continuous D-dimensional regularization scheme, for on-shell electrons, momentum transfer $t = -Q^2$ and finite squared electron mass $m_e^2 = a$. We identify all the Master Integrals (MI's) of the problem and write the differential equations in Q^2 which they satisfy. The equations are expanded in powers of $\epsilon = (4 - D)/2$ and solved by the Euler's method of the variation of the constants. As a result, we obtain the coefficients of the Laurent expansion in ϵ of the MI's up to zeroth order expressed in close analytic form in terms of Harmonic Polylogarithms.

Key words:Feynman diagrams, Multi-loop calculations, Vertex diagrams
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*Email: Roberto.Bonciani@physik.uni-freiburg.de
†This work was supported by the European Union under contract HPRN-CT-2000-00149
‡Email: Pierpaolo.Mastrolia@bo.infn.it
§Ettore.Remiddi@bo.infn.it

1 Introduction

The QED electron form factors at two loops were considered in [1], for massive on-shell electrons and arbitrary momentum transfer within the Pauli-Villars regularization scheme and giving a fictitious small mass λ to the photon for the parametrization of infrared divergences. The main results of [1] are the analytic calculation of the imaginary parts of the form factors for arbitrary momentum transfer in terms of Nielsen's polylogarithms [2, 3], and of the charge slope of the electron at two loops (besides the check of the magnetic anomaly). The analytic evaluation of the real parts, expected to involve a class of functions wider than Nielsen's polylogarithms, was not attempted in [1].

To our knowledge, the full analytic calculation of the real parts of the 2-loop QED form factors, for arbitrary momentum transfer and finite electron mass, has not yet been carried out, despite a great number of papers dealing with a variety of kinematical configurations (neglecting typically the electron mass at large momentum transfer).

In this paper we work out a systematic investigation of all the 2-loop integrals occurring in the electron vertex in QED in the continuous D-dimensional regularization scheme [4] (using the same D for ultraviolet and infrared regularization) for on shell electrons of finite squared mass $m_e^2 = a$ and arbitrary momentum transfer $t = -Q^2$. We identify all the Master Integrals (MI's) occurring in all the graphs and evaluate them analytically, in terms of Harmonic Polylogarithms [5, 6], for arbitrary value of Q^2. We present the results for spacelike momentum transfer, i.e. $t < 0$ or $Q^2 > 0$; the case of timelike t can be obtained by standard analytic continuation.

The extraction of the form factors and their expression in terms of the MI's will be carried out in a subsequent paper.

The diagrams involved are those shown in Fig. (1).

Following a by now standard approach, we first express all the scalar integrals associated to each graph in terms of the Master Integrals (MI's) by using the integration by parts [7] and Lorentz invariance [8] identities, then write the differential equations on the momentum transfer which are satisfied by the MI's, and expand the equations in powers of $\epsilon = (4 - D)/2$ around $\epsilon = 0$ ($D = 4$) up to the required order. We obtain in that way a system of chained differential equations for the coefficients of the ϵ-expansion of the MI's and finally solve the system for the coefficients by Euler's variation of constants method. As a result, we express the coefficients in close analytic form in terms of Harmonic Polylogarithms [5, 6].

Let us recall that the Euler's method requires the solution of the associated homogeneous equation. Even if general algorithms for the solution of differential equations are not available, it is to be stressed here that all the homogeneous equations which we had to solve came out to be essentially trivial (typically, first order homogeneous equations with rational coefficients).

The present paper is structured as follows. In section 2 we review briefly the techniques for reducing the calculation of generic multi-loop Feynman graph integrals to the calculation of the MI's. Integration by Parts, Lorentz Invariance and general symmetry relations are recalled and the application of this approach to our

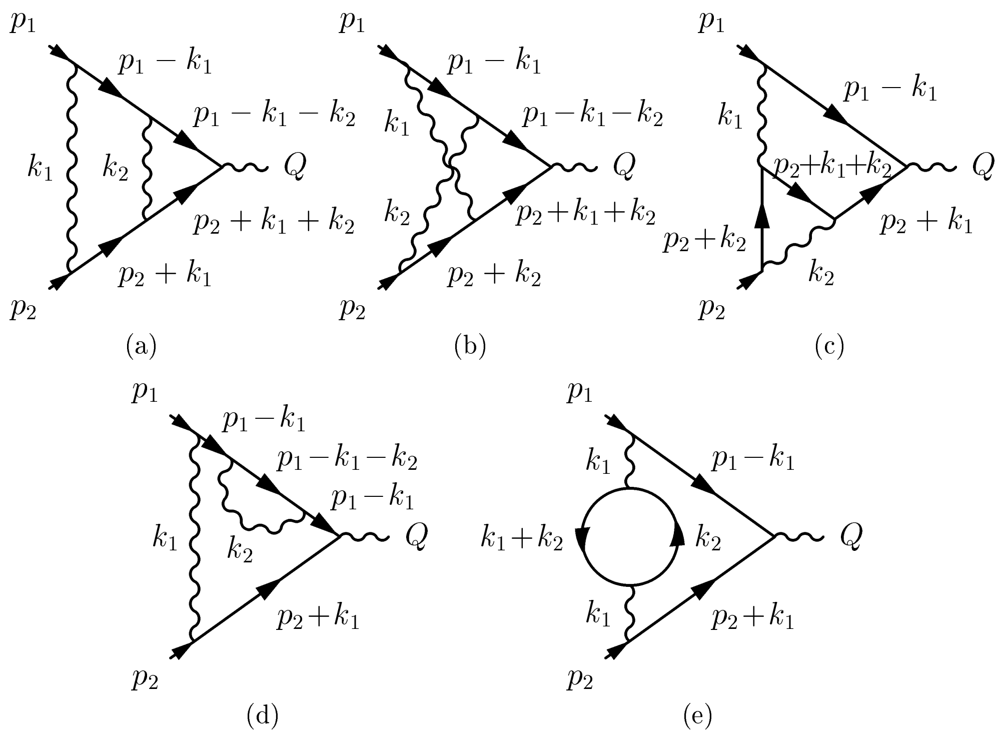


Figure 1: 2-loop vertex diagrams for the QED form factor. The fermionic external lines are on the mass-shell $p_1^2 = p_2^2 = -a$, while the wavy line on the r.h.s. has momentum $Q = p_1 + p_2$, with $Q^2 = -s$.

case is discussed. In section 3 we review the method of differential equations for the calculation of the MI's. In section 4 we describe exhaustively the case of three typical integrals, giving in some details the system of differential equations and the steps for obtaining the solution. In section 5 we present the results for all the MI's encountered in the calculation of the 2-loop vertex diagrams and in section 6 we give the results for the scalar 6-denominator vertex diagrams which are not MI's. Sections 7 and 8 contain respectively the expansions of the vertex 6-denominator diagrams in the region of great and low momentum transfer. Finally, after the summary, section 9, and appendix A, where we give the routing used for the explicit calculations, in appendix B we give the results of the 1-loop diagrams involved in our calculations and in appendix C we list the results for all the reducible diagrams appearing in the calculation.

2 The reduction to master integrals

The aim of this paper is the evaluation of all the possible scalar integrals which can occurr in the calculation of the Feynman diagrams of Fig. (1). They imply two loop momenta, k_1 and k_2, and three external momenta, p_1, p_2, Q; among them only two are independent, because of the momentum conservation law: $p_1 + p_2 = Q$. With two external momenta and two loop momenta, we can construct three Mandelstam invariant variables, p_1^2, p_2^2 and $Q^2 = (p_1 + p_2)^2$ (we use the Euclidean metric, so that the mass shell conditions are $p_1^2 = p_2^2 = -a$, where a is the squared electron mass) and seven different scalar products involving the loop momenta, namely $(p_1 \cdot k_1)$, $(p_1 \cdot k_2)$, $(p_2 \cdot k_1)$, $(p_2 \cdot k_2)$, $(k_1 \cdot k_2)$, (k_1^2) and (k_2^2).

The graphs of Fig. (1) involve up to 6 different propagators: more exactly, the graphs (a,b,c) have 6 different propagators, while graph (d) contains 2 equal electron propagators, graph (e) 2 equal photon propagators, so that two graphs, (d,e), involve only 5 different propagators. In the following we will not consider anymore graphs but topologies: topologies will be drawn exactly as Feynman graphs, except that all propagators are different. To make an example, when a graph contains twice some propagator, as the two equal photon propagators of graph (e) above, the corresponding topology contains that propagator only once; indeed, the topology of graph (e) of Fig. (1) is given by the topology (e) of Fig. (4). Besides those topologies, we will also encounter all the subtopologies obtained by removing from the graphs one or more propagators in all possible ways.

Let t be the number of the propagators in any of the topologies or subtopologies; we can express t of the 7 scalar products containing the loop momenta in terms of the propagators, (the remaining $(7-t)$ scalar products will be called irreducible) and correspondingly the most general scalar integral associated to that topology or subtopology has the form

$$I(p_1, p_2) = \int \{d^D k_1\}\{d^D k_2\} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} , \qquad (1)$$

where $\{d^D k\}$ is the loop integration measure (its explicit expression, irrelevant here, will be given in section 5), the integer $m_i, i = 1, t$ are the powers of the t propagators, with $m_1 \geq 1$, and the integer $n_j, j = 1, q,\ q = (7-t)$, with $n_j \geq 0$, are the powers of the irreducible scalar products. Let us further recall that the continuous dimensional regularization makes the definition meaningful for any values of the 7 integer n_i, m_j.

We will denote with $I_{t,r,s}$ the family of the integrals with a same set of t propagators, a total of $r = \sum_i (m_i - 1)$ powers of the t propagators and $s = \sum_j n_j$ powers of the $(7-t)$ irreducible scalar products. The number of the integrals contained in the family is

$$N\left[I_{t,r,s}\right] = \begin{pmatrix} r+t-1 \\ t-1 \end{pmatrix} \begin{pmatrix} s-t+6 \\ 6-t \end{pmatrix} . \quad (2)$$

As we will see more in detail in a moment, one can establish several identities involving integrals of the type of Eq. (1) with different sets of the 7 indices m_i, n_j. The identities can be written in the form of a sum of a finite number of terms set equal to zero, where each term is a polynomial (of finite order and with integer coefficients in the variable D, a and the Mandelstam invariants) times an integral of the family, as will be seen explicitely in the example of next section.

The identities can be used to express as many as possible integrals of a given family in terms of as few as possible suitably choosen integrals of that family – called the Master Integrals of that family.

The identities will be generated by using Integration by Parts, Lorentz Invariance (or rotational invariance in D dimensions) and symmetry considerations.

2.0.1 Integration by Parts Identities

Integration by Parts Identities (IBP-Id's) are among the most remarkable properties of dimensionally regularized Feynman integrals [7]. In our case, for each of the integrals defined in Eq. (1) one can write

$$\int \{d^D k_1\}\{d^D k_2\} \frac{\partial}{\partial k_1^\mu} \left\{ v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0 \,, \quad (3)$$

$$\int \{d^D k_1\}\{d^D k_2\} \frac{\partial}{\partial k_2^\mu} \left\{ v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right\} = 0 \,, \quad (4)$$

where thanks to the dimensional regularization everything is well defined and the identity holds trivially. In the above identities the vector v^μ can be any of the 4 independent vectors of the problem: k_1, k_2, p_1, or p_2, so that for each integrand there are 8 IBP-Id's. When evaluating explicitly the derivatives, one obtains a combination of integrands with a total power of the irreducible scalar products equal to $(s-1)$, s and $(s+1)$ and total powers fo the propagators in the denominator equal to $(t+r)$ and $(t+r+1)$, therefore involving, besides the integrals of the family $I_{t,r,s}$, also the families $I_{t,r,s-1}$, $I_{t,r+1,s}$ and $I_{t,r+1,s+1}$.

Simplifications between reducible scalar products and propagators in the denominator may also occur, giving lower powers of the propagators. It may happen that some propagator disappears at all in this process; the resulting term will then give an integral of a simpler family (or subtopology) with $(t-1)$ propagators.

As an explicit example let us consider the case of the 4-denominator topology of Fig. (2). We have three irreducible scalar products, in this topology; we choose

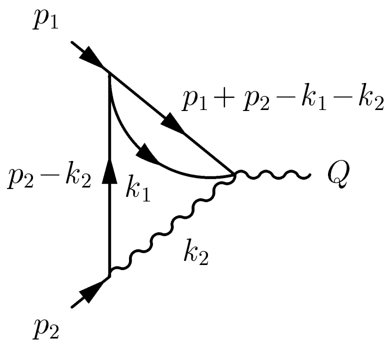
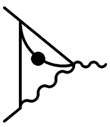
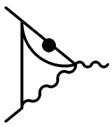
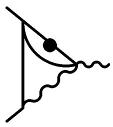


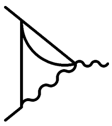
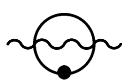
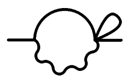
Figure 2: A 4-denominator topology.

$(p_1 \cdot k_1)$, $(p_2 \cdot k_1)$ and $(k_1 \cdot k_2)$. Eq. (3), for generic values of the indices m_i, n_i and for a generic independent vector v_μ, reads:

$$\int \{d^D k_1\}\{d^D k_2\} \frac{\partial}{\partial k_1^\mu} \left\{ \frac{v^\mu \, (p_1 \cdot k_1)^{n_1} (p_2 \cdot k_1)^{n_2} (k_1 \cdot k_2)^{n_3}}{[k_1^2+a]^{m_1}[k_2^2]^{m_2}[(p_2-k_2)^2+a]^{m_3}[(p_1+p_2-k_1-k_2)^2+a]^{m_4}} \right\} = 0\,. \tag{5}$$

Let us take for simplicity $m_1 = \cdots = m_4 = 1$, $n_1 = \cdots n_3 = 0$ and $v^\mu = p_1^\mu$. Performing the derivative with respect to k_1 and simplifying the reducible scalar products with the corresponding denominator, we write Eq. (5) as follows:

$0 \;=\; -2$  $(p_1 \cdot k_1) - 2$  $(k_1 \cdot k_2) + 2$  $(p_2 \cdot k_1)$

$+$  $-$  $+ \dfrac{(1-\epsilon)}{a}$  , (6)

where a dot on a propagator line means that the propagator is squared and irreducible scalar products left are explicitely written.

2.0.2 Lorentz invariance identities

Another class of identities can be derived from the fact that the integrals $I(p_i)$, Eq. (1), are Lorentz scalars (or rather D-dimensional rotational invariant) [8]. If we consider an infinitesimal Lorentz transformation on the external momenta, $p_i \to p_i + \delta p_i$, where $\delta p_i^\mu = \epsilon^\mu_\nu p_i^\nu$, and ϵ^μ_ν is a completely antisymmetric tensor, we have

$$I(p_i + \delta p_i) \;=\; I(p_i)\,, \tag{7}$$

Because of the antisymmetry of ϵ^μ_ν and because

$$\begin{aligned} I(p_i + \delta p_i) &= I(p_i) + \sum_n \delta p_n^\mu \frac{\partial I(p_i)}{\partial p_n^\mu} \\ &= I(p_i) + \epsilon^\mu_\nu \left[\sum_n p_n^\nu \frac{\partial I(p_i)}{\partial p_n^\mu} \right]\,, \end{aligned} \tag{8}$$

we can write the following relation:

$$\sum_n \left[p_n^\nu \frac{\partial}{\partial p_n^\mu} - p_n^\mu \frac{\partial}{\partial p_n^\nu} \right] I(p_i) \;=\; 0\,. \tag{9}$$

Eq. (9) can be contracted with all possible antisymmetric combination of the external momenta $p_i^\mu p_j^\nu$, to obtain other identities for the considered integrals.

In our case we have two external independent momenta and we can thus construct, besides Eqs. (3,4), the further identity

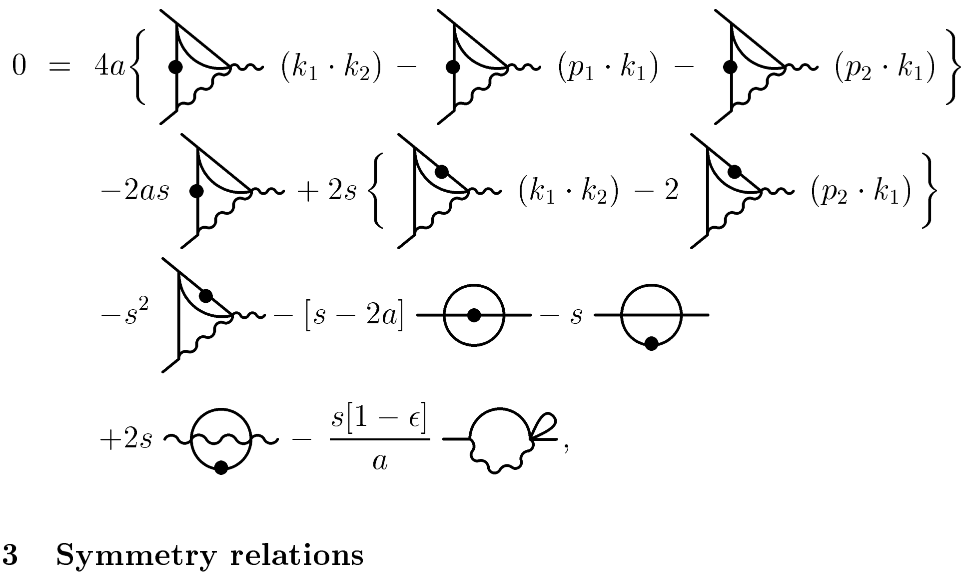
$$\left[(p_1 \cdot p_2) \left(p_1^\mu \frac{\partial}{\partial p_1^\mu} - p_2^\mu \frac{\partial}{\partial p_2^\mu} \right) + p_2^2 \; p_1^\mu \frac{\partial}{\partial p_2^\mu} - p_1^2 \; p_2^\mu \frac{\partial}{\partial p_1^\mu} \right] I(p_i) \;=\; 0\,. \tag{10}$$

Let us note that, in order to obtain non–trivial identities from Eq. (10), the derivative with respect to the momentum p_i has to be intended as a differentiation under the loop integral in the definition, Eq. (1), i.e. directly on the integrand of $I_{t,r,s}$. Again, this is allowed by the dimensional regularization.

As an explicit example let us consider the same topology as in the previous section. If we take $n_1 = n_2 = n_3 = 0$ and $m_1 = m_2 = m_3 = m_4 = 1$, we have

$$I(p_1, p_2) = \int \{d^D k_1\}\{d^D k_2\} \frac{1}{[k_1^2 + a] k_2^2 [(p_2 - k_2)^2 + a][(p_1 + p_2 - k_1 - k_2)^2 + a]}, \tag{11}$$

and Eq. (10) reads as follows:



(12)

2.0.3 Symmetry relations

In general further identities among Feynman graph integrals can arise when the Feynman graph has some symmetry. In such a case there can be a trasformation of the loop momenta which does not change the value of the integral, but transforms the integrand in a combination of different integrands. By imposing the identity of the initial integral to the combination of integrals resulting from the change of loop momenta one obtains further identities relating integrals corresponding to a same graph.

As an example, let us consider again the topology of Fig. (2), with generic indices on the numerator and on the denominator:

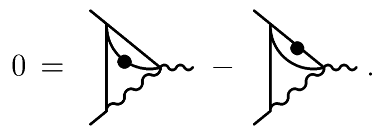
$$I(p_1,p_2) = \int \{d^Dk_1\}\{d^Dk_2\} \frac{(p_1\cdot k_1)^{n_1}(p_2\cdot k_1)^{n_2}(k_1\cdot k_2)^{n_3}}{[k_1^2+a]^{m_1}\,[k_2^2]^{m_2}[(p_2-k_2)^2\ +a]^{m_3}\,[(p_1+p_2-k_1-k_2)^2+a]^{m_4}}. \tag{13}$$

The two propagators with momentum k_2 and $(p_1+p_2-k_1-k_2)$ have the same mass. The following redefinition of the integration momentum

$$k_1 = p_1 + p_2 - k_1^{'} - k_2\,, \tag{14}$$

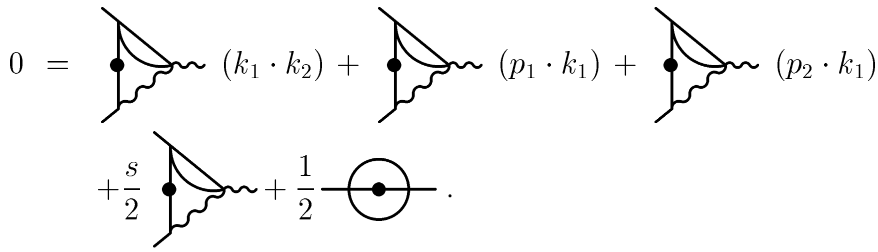
that consists in the interchange of the two propagators in the closed electron loop, does not affect of course the value of the integral; nevertheless, the explicit form of the integrand can change, generating non-trivial identities.

Taking for instance $n_1 = n_2 = n_3 = 0$ and $m_1 = m_2 = m_4 = 1$, $m_3 = 2$ in Eq.(13), the substitution (14) gives, for example, the following very simple relation:



(15)

Taking $n_1 = n_2 = 0$, $n_3 = 1$ and $m_1 = m_2 = m_4 = 1$, $m_3 = 2$, we get the more complicated idetity



(16)

Summarizing, for each of the $N[I_{t,r,s}]$ integrals of the family $I_{t,r,s}$, Eq. (2), we have the 9 identities, Eqs. (3,4) and Eq. (10), involving integrals of the families up to $I_{t,r+1,s+1}$. For $(r = 0, s = 0)$ the number of all the integrals involved in the identities (for $t = 6$ they are 14) exceeds the number of the equations obtained (which in this case is 9), but when writing systematically all the equations for increasing values of r and s, $r = 0,1,..\,,\ s = 0,1,...$, the number of the equations grows faster than the number of the integrals [9], so that at some point one deals with more equations than involved integrals – generating an apparently overconstrained set of linear equations for the unknown integrals. At this stage one can use the symmetry relations, somewhat reducing the number of the unknown integrals, after which one is left with the problem of solving the linear system of the identities. The problem is in principle trivial, but algebraically very lengthy, so that some organization is required for obtaining the solution.

To that aim, one can order the integrals in some lexicographic order (which means giving a "weight" to each integral; the weight can be almost any increasing function of the indices m_i, n_j, such that integrals with higher indices have bigger weights) and then solve the system by the Gauss substitution rule by considering one by one, in some order, the equations of the system and using each equation for expressing the integral with highest weight present in that equation in terms of the other integrals of lower weight, and then substituting in the rest of the system. The algorithm is straightforward, but its execution requires of course a great amount of algebra; indeed, it was implemented as a chain of programs, written in the computer language `C`, which automatically runs programs written for the algebraic computer languages `FORM` [10] and `Maple` [11], reads the outputs and generates new input programs till all the equations are solved (and the solutions are written as a modulus of `FORM` code).

One finds that several equations are identically satisfied (the system is only apparently overconstrained), and all the appearing unknown integrals are expressed in terms of very few independent integrals, the Master Integrals (MI's) for that family of integrals. In so doing, the resulting MI's correspond to the integrals of lowest weight; but as the choice of the weight is to a large extent arbitrary, there is also some freedom in the choice of the integrals to pick up as actual MI's (not in their number, of course!). Concerning in particular the calculation described in this paper, there are several cases in which two MI's are found for a given topology or subtopology, while sometimes only one MI is present. It may also happen that no MI for the considered topology is left – i.e. all the integrals corresponding to the given t-propagator (sub)topology can be expressed in terms of MI's of its subtopolgies with $(t-1)$ propagators.

As a last remark, strictly speaking we are not able to prove that the MI's we find are really the minimal set of MI's, i.e. that they are all independent from each other (in the sense of the combination with polynomial factors described above); but in any case the number of the MI's which we find is small, so that reducing the several hundred of integrals occurring in the calculation of the vertex graphs form factors to a few (in fact 16 MI's, see section 2.1) is in any case a great progress. The (unlikely!) discovery that one of our MI's can be expressed as combination of the others would just simplify even further the calculation – without spooling, however, the correctness of the already obtained results.

Concerning the number of the subtopologies, a topology with t propagators has $(t-1)$ subtopology with $(t-1)$ propagators, $(t-1)(t-2)$ subtopologies with $(t-2)$ propagators etc. It turns out, however, that most subtopologies are in fact equal due to symmetry relations, and the subtopologies coming from different graphs overlap to a great extent. For those reasons, the actual number of all the different subtopologies is relatively small.

We show in Figs. (3–6) all the different topologies (and subtopologies; we will refer to them as topologies as well). In all the figures, a straight external line stands for an electron on the mass-shell, while the wavy external line carries the momentum Q.

There are 3 independent topologies involving 6 denominators, those of Fig. (3),

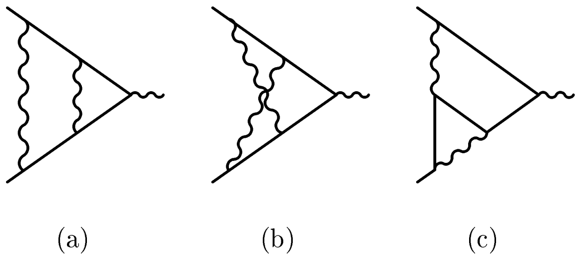


Figure 3: The set of 3 independent 6-denominator topologies present in the graphs of Fig.(1).

corresponding to the original graphs (a,b,c) of Fig. (1). One then finds the 8 independent topologies involving 5 denominators shown in Fig. (4); note that the topologies (b,e) of Fig. (4) correspond to the 6-propagator vertex graphs (d,e) of Fig. (1), the topology (f) of Fig. (4) corresponds to a vacuum polarization graph, (g) is a constant (i.e. does not depend on the momentum transfer Q^2, but on a squared electron momentum on the mass-shell, $p^2 = -a$) and (h) factorizes into two 1-loop topologies.

Fig. (5) contains all the 12 independent 4-denominator topologies; again, only the topologies (a,b,c,d) correspond to genuine 2-loop vertex topologies, while (e) corresponds to a vacuum polarization, (f,g,h) are constants and (i,j,k,l) factorize into two 1-loop topologies.

Finally, Fig. (6) contains all the 6 independent 3-denominator topologies; (a) corresponds to a vacuum polarization, the others are constants or factorizable.

At the 2-denominator level, the only topology giving a non-vanishing contribution corresponds to the product of two 1-loop tadpoles with squared mass a shown in Fig. (6) (g).

2.1 The MI's.

For each independent topology we write the identities among the integrals of the associated family and solve them in terms of MI's according to the previous discussion, see section 2. The resulting MI's are shown in Fig. (7). The MI's are represented with a graph very much equal to the graph representing their topology, but occasional with some "decoration". When no decoration is present, the corresponding MI is nothing but the corresponding full scalar graph (first power of all the propagators, numerator equal to 1); that is the case, for instance, of (a), (c), (e) etc. of Fig. (7). When a propagator line is decorated with a dot, it appears squared in the MI, as in the case of (d),(h) of Fig. (7). Finally, the decoration can be a scalar product involving at least one loop momentum; the scalar product appears in the numerator of the integrand of the corresponding MI, as in (b), (e) etc. of Fig. (7).

As anticipated, there are topologies with two MI's, as the topology (b) of Fig.

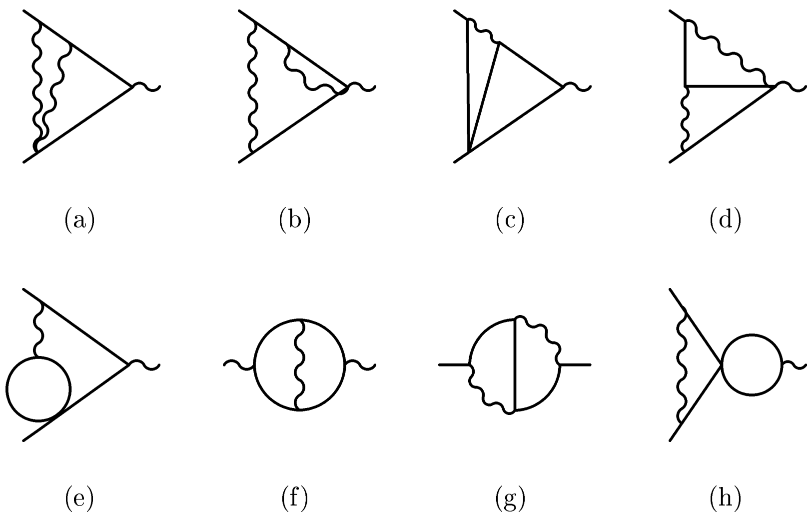


Figure 4: The set of 8 independent 5-denominator topologies contained in the graphs of Fig.(1). External fermion lines are put on the mass-shell $p_1^2 = p_2^2 = -a$, while external wavy lines carry an off-shell momentum $Q = p_1 + p_2$. The topology (g) is evaluated on the mass-shell.

(3) which has the two MI's (a), (b) of Fig. (7), topologies with a single MI, as (a) of Fig. (5) which has the MI (i) of Fig. (7), and topologies without MI's, i.e. topologies whose associated integrals can all be expressed in terms of the MI's of their subtopologies; that is the case, for instance, of the topology (a) of Fig. (3).

As a last remark, let us recall that there is some arbitrariness on the actual scalar integrals to be choosen as MI's; in the case of Fig. (7) some of the MI's correspond to graphs decorated with dots, other to graphs decorated by scalar products. The choice, by no means mandatory but rather somewhat accidental, was suggested by the convenience of later use.

3 Calculation of the MIs. The system of differential equations

Once all the MIs of a given topology are obtained, the problem of their calculation arises. We will address the problem by the differential equations method, which turns out to be a really very powerful tool. The use of differential equations in one of the internal masses was first proposed out in [12], then extended to more general differential equations in any of Mandelstam variables in [13] and successively used in [14] for the MI's of the sunrise diagram with arbitrary internal masses. An application to the 4-point functions with massless internal propagators was worked out in [8] for the 2-loop case and it brought to the complete evaluation of the master integrals for the planar [8, 15, 16] and non planar topologies [17]. In this paper, we will write (and solve) the differential equations in the momentum transfer in the

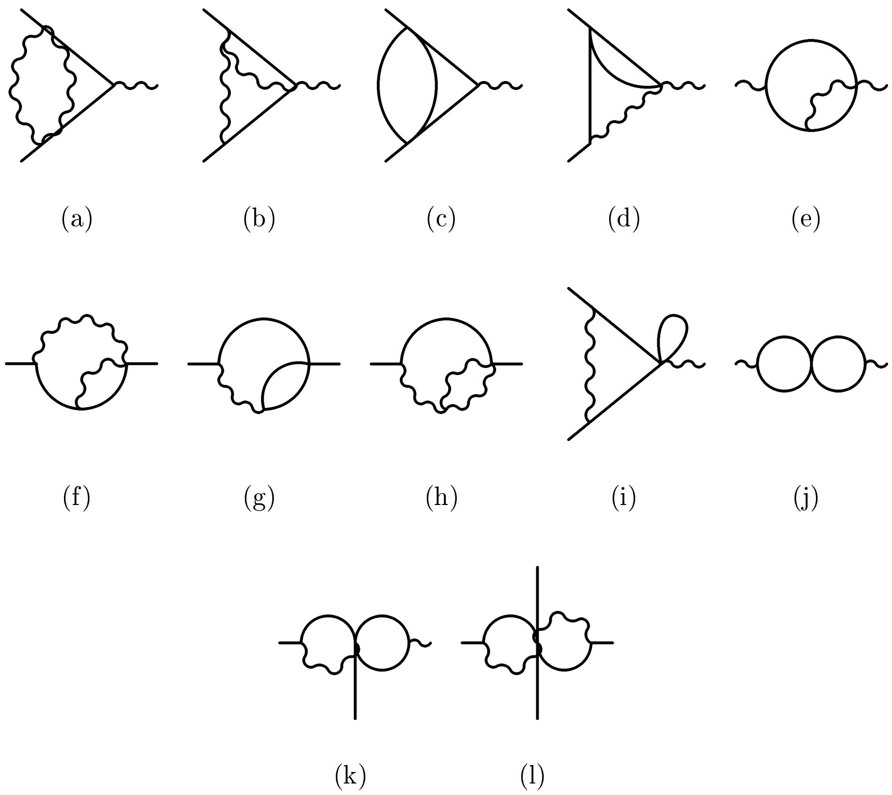


Figure 5: The set of 12 independent 4-denominator (sub)topologies coming from the 5-denominator topologies of Fig.(4).

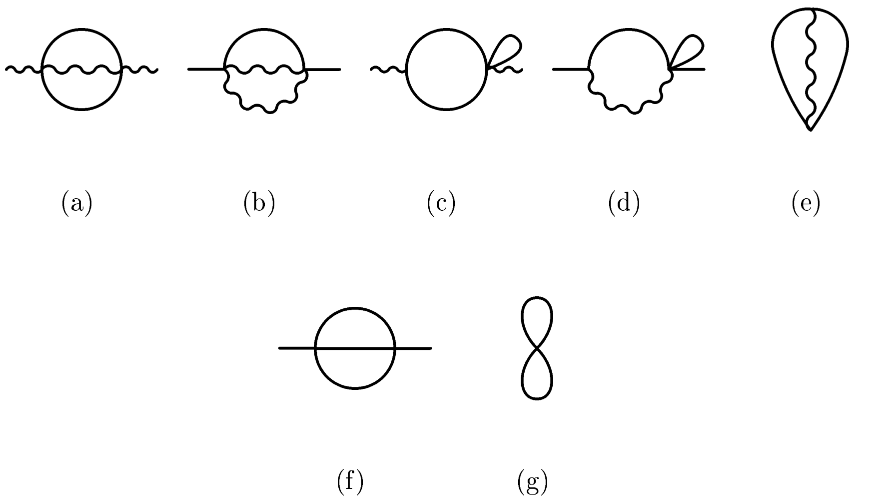


Figure 6: The set of 6 independent 3-denominator (sub)topologies, (a)–(f), coming from the 4-denominator topologies of Fig.(5) and the only non-vanishing topology at 2-denominator level, (g), product of two tadpoles of squared mass a.

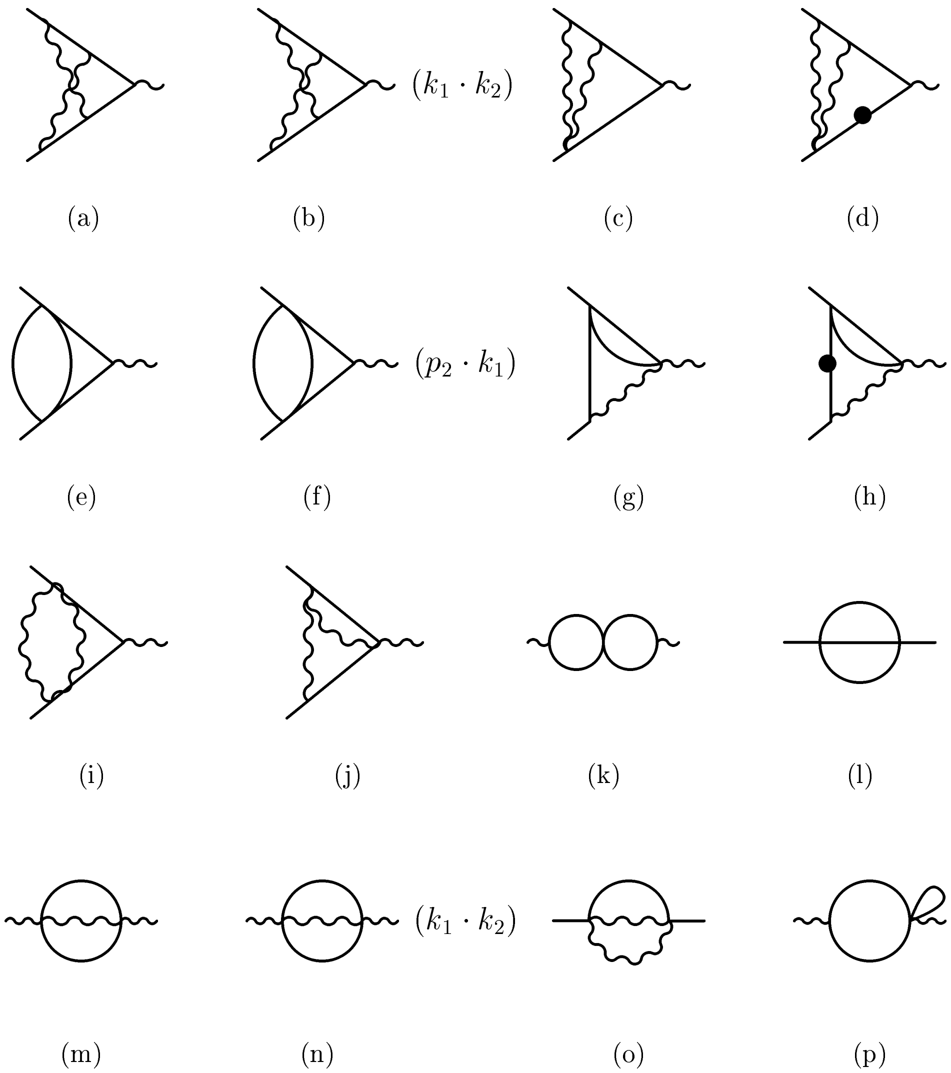


Figure 7: The set of 16 Master Integrals (MIs). As explained in section 2.1, the diagrams shown are a graphical representation of the corresponding D-regularized integral. A dot on a propagator line means that the corresponding propagator is squared and an explicitly written scalar product means that the corresponding D-regularized integral has that scalar product in the numerator of the integrand.

case of 3-point functions with massive fermionic propagators in QED, keeping the external electron legs on the mass shell.

Let us summarize briefly the idea of the method. To begin with, consider any scalar integral $F(s_i)$, (we will be interested here in the MI's, but what follows applies to any scalar integral as well) defined as

$$F(s_i) \;=\; \int \{d^D k_1\}\{d^D k_2\}\, \frac{S_1^{n_1}\cdots S_q^{n_q}}{\mathcal{D}_1^{m_1}\cdots \mathcal{D}_t^{m_t}}\,; \tag{17}$$

$F(s_i)$ depends in general on the three external kinematical invariants $s_1 = -p_1^2,\ s_2 = -p_2^2$ and $s_3 = -Q^2 = -(p_1+p_2)^2$, where $p_1^2,\ p_2^2$ will be later constrained on the mass shell $p_1^2 = p_2^2 = -a$.

Let us construct the following quantities:

$$O_{jk}(s_i) = p_j^\mu \frac{\partial}{\partial p_k^\mu} F(s_i)\,. \tag{18}$$

As $F(s_i)$ depends on the Mandelstam invariants s_i, by the chain differentiation rule we have

$$O_{jk}(s_i) = p_j^\mu \sum_\xi \frac{\partial s_\xi}{\partial p_k^\mu}\frac{\partial}{\partial s_\xi} F(s_i) = \sum_\xi a_{\xi,jk}(s_l)\frac{\partial}{\partial s_\xi} F(s_i)\,, \tag{19}$$

where the functions $a_{\xi,jk}(s_l)$ are linear combinations of the Mandelstam invariants s_i.

As j and k take the two values $1, 2$ we obtain in that way a system of 4 linear equations (not all linear independent), which we can solve for the three derivatives $\frac{\partial}{\partial s_\xi}F(s_i)$; we have in particular

$$\frac{\partial}{\partial Q^2}F(s_i) = \left[A\left(p_1^\mu\frac{\partial}{\partial p_1^\mu} + p_2^\mu\frac{\partial}{\partial p_2^\mu}\right) + B\left(p_1^\mu\frac{\partial}{\partial p_2^\mu} + p_2^\mu\frac{\partial}{\partial p_1^\mu}\right)\right] F(s_i)\,, \tag{20}$$

where

$$A \;=\; \frac{1}{4}\left[\frac{1}{Q^2} + \frac{1}{Q^2+4a}\right]\,, \tag{21}$$

$$B \;=\; \frac{1}{4}\left[\frac{1}{Q^2} - \frac{1}{Q^2+4a}\right]\,. \tag{22}$$

Assume now that $F(s_i)$ is a master integral for some given topology. We can now substitute the right-hand side of Eq. (17) in the right-hand side of Eq. (20) and perform the direct differentiation of the integrand. It is clear that we obtain a combination of several integrals, all belonging to the same topology as $F(s_i)$; therefore, we can use the solutions of the IBP and other identities for that topology and express everything in the r.h.s. of Eq. (20) in terms of the MI's for the considered topology and its subtopologies. If there are several different MI's for that topology, the procedure can be repeated for all the other MI's as well. In so doing one obtains a system of linear differential equations in Q^2 for $F(s_i)$ and the other MI's (if any), expressing their Q^2-derivatives in terms of the MI's of the considered topology and of its subtopologies; due to the presence of the MI's of the subtopologies the equations are in general non-homogeneous.

At this point we can impose the mass-shell conditions, and the general structure of the system reads

$$\begin{aligned} \frac{\partial}{\partial Q^2} M_i(D,a,Q^2) &= \sum_j A_j(D,a,Q^2) M_j(D,a,Q^2) \\ &\quad + \sum_k B_k(D,a,Q^2) N_k(D,a,Q^2) \ , \end{aligned} \tag{23}$$

where the $M_i(D,a,Q^2)$ are the MI's of the topology with the electron legs on the mass-shell, $N_k(D,a,Q^2)$ the MI's of the subtopologies, we have made explicit the dipendence on D for later use (note that the above equations are exact in D) and the coefficients $A_j(D,a,Q^2), B_k(D,a,Q^2)$ are rational factors depending on D, Q^2 and the electron squared mass a. As will be apparent by the examples, the singularities of $A_j(D,a,Q^2), B_k(D,a,Q^2)$ in the variable Q^2, such as $1/(Q^2+4a)$ and $1/Q^2$, and correspond to the thresholds and pseudothresholds of the corresponding Feynman graphs.

It is clear that the procedure can be repeated in principle for the other Mandelstam variables as well. We are not interested in this further equations, as in the case we are considering the other Mandelstam variables are the invariant masses of the electrons, which we keep frozen on the mass-shell.

As already observed, the Eqs. (23) for the MI's $M_i(D,a,Q^2)$ of a given topology are not homogeneous, as they may involve the MI's $N_k(D,a,Q^2)$ of the subtopologies. It is therefore natural to proceed bottom up, starting from the equations for the MI's of the simplest topologies (i.e. with less denominators), solving those equations and using the results within the equations for the MI's of the more complicated topologies with additional propagators, whose non-homogeneous part can then be considered as known.

3.0.1 The boundary conditions

Some comments on boundary conditions. As already observed, the coefficients of the differential equations Eq. (23) are in general singular at $Q^2 = 0$ and $Q^2 = -4a$; correspondingly, the solutions of the equations can have a singular behaviour in those points. But we know that the Vertex integrals are regular in Q^2 at $Q^2 = 0$ when the electron lines are on the mass shell, a qualitative result which can easily verified, when needed, by direct inspection of the very definition of the amplitudes as loop integrals. It turns out that the qualitative information provided by the regularity behaviour (implying the absence, in the $Q^2 \to 0$ limit, of terms in $1/Q^2$ or $\ln Q^2$) is completely sufficient for the quantitative determination of the otherwise arbitrary integration constants which naturally arise when solving a system of differential equations.

3.1 Laurent series expansion in ϵ

The system of differential equations Eq. (23) is exact in D, but we are interested, in any case, on the Laurent expansion of the solutions in powers of $\epsilon = (4-D)/2$. It turns out that our 2-loop integrals have up to a double pole in ϵ (which can be of ultraviolet or infrared origin), so that quite in general we will expand the two loop MI's as

$$M_i(D,a,Q^2) = \sum_{j=-2}^{n} \epsilon^j M_i^{(j)}(a,Q^2) + \mathcal{O}(\epsilon^{(n+1)}) \; , \tag{24}$$

where n is the required order in ϵ. We will present for all the considered integrals the coefficients of the ϵ expansion up to the zeroth order included. In some cases, however, we had to expand intermediate results up to the term of fourth order in ϵ. That depends on the fact that some of the MI's, which appears in the non-homogeneous part of a system of differential equations for more complicated MI's, can be multiplied by coefficients which are also singular in ϵ.

When expanding systematically in ϵ all the MI's (including those appearing in the non-homogeneous part) and all the D-dependent coefficients of Eq. (23), one obtains a system of chained equations formed by the equations valid for each power of ϵ. The first equation corresponds to the coefficient of double pole in ϵ of the equation, and involves only the coefficients $M_i^{(-2)}(a,Q^2)$ as unknown; the next equation, corresponding to the single pole in ϵ, involves the $M_i^{(-1)}(a,Q^2)$ as unknown, but can in general involve $M_i^{(-2)}(a,Q^2)$ if some of the coefficients contains a power of ϵ; such a term in $M_i^{(-2)}(a,Q^2)$ can be considered as known once the equation for the double pole has been solved. For the subsequent equations we have the same structure: they involves the coefficient $M_i^{(j)}(a,Q^2)$ at the order j in ϵ as unknown and the previous coefficients as known non-homogeneous terms.

Let us note that the homogeneous part of all the equations arising from the ϵ expansion of Eq. (23) is always the same and obviously identical to the homogeneous part of Eq. (23) at $D=4$, i.e. $\epsilon = 0$. It is natural to look for the solutions of the chained equations by means of the Euler's method of the variation of the constants, using repeatedly the solutions of the homogeneous equation, as we will show in some examples in the next section.

General algorithms for the solution of the homogeneous equations are not available; it turns out however that in all the cases considered in this paper the homogeneous equations at $D=4$ have almost trivial solutions, so that Euler's formula can immediately be written. With the change of variable

$$x = \frac{\sqrt{Q^2+4a} - \sqrt{Q^2}}{\sqrt{Q^2+4a} + \sqrt{Q^2}} \; , \tag{25}$$

all integrations can further be carried out in closed analytic form, the result being a combination of the Harmonic Polylogarithms introduced in [5] (see also [6] for their numerical evaluation), a generalization of the already widely used Nielsen's Polylogarithms [2, 3].

As a last remark some comments on the arbitrariness of the choice of the MI's. For the topologies we considered, we had at most 2 MI's, which means that we had to solve in principle a linear system of two equations or an equivalent second-order differential equation. However, the freedom in the explicit choice of the MI's can play an essential role in simplifying the calculation. It turns out, in fact, that if we choose the two MI's with a different leading singular behaviour in ϵ, the system of the two coupled first-order linear differential equations does in fact decouple.[5] As a result, instead of solving a second-order differential equation we can solve simply two first-order equations. We will show how to exploit this possibility in the solution of the systems considered in sections 4.2 and 4.3.

4 Explicit calculations

In this section the equations for three topologies and their solutions are discussed in some details. We chose a 4-, a 5- and a 6-dominator topology, shown respectively in Fig. (5) (a), Fig. (6) (a) and Fig. (3) (b), to illustrate the algorithm for the solution of the corresponding system of differential equations.

The 4-denominator topology is the simpler among the three cases, since it has only one MI. Correspondingly, the system in Eq. (23) reduces to a single first-order linear differential equation – whose solution is therefore trivial.

The other two topologies are more difficult. They have both two MI's and therefore, in both cases, we must solve in principle a system of two first-order coupled linear differential equations. As we have already remarked in section 3.1, the explicit form of the system depends on the choice of the pair of MI's. Indeed, we choose in both cases two MI's with different leading behaviour in ϵ, such that the system decouples order by order in ϵ.

Before to go on, two remarks have to be done.

The first one is the following. We are interested in really 2-loop diagrams, but, as we have seen, in the reduction to the MI's we encountered topologies which factorize in the product of two 1-loop topologies. In particular, two MI's have this structure; they are shown in Fig. (7) (k) and (p) and they consist respectively in a product of two bubbles in the Q-channel and of one bubble in the Q-channel and a Tadpole. The same algorithm explained in sections 2–3 was applied, therefore, to the 1-loop problem and the results are shown in appendix B, where we discuss in particular the solution of the differential equation for the bubble with two massive propagators with equal squared mass a.

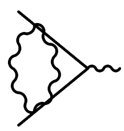
The second remark concerns the 3-denominator MI's. They constitute the simplest non-trivial 2-loop topologies of the entire pyramid of MI's, and thus they are present in the non-homogeneous part of the systems of differential equations for all the other topologies. For the two MI's of the Sunrise with two equal-mass and

[5] The decoupling can be, in some cases, exact in ϵ, which means that a combination of integrals diagonalize the system without expanding it in powers of ϵ. More in general, the homogeneous equation at $D = 4$ is brought to acquire a triangular form, with the first differential equation which contains only one of the MI's and the second equation which involves both MI's.

one mass-less propagator, Fig. (7) (m) and (n), a system of two non-homogeneous first-order differential equations can be established, the non-homogeneous part consisting essentially on the product of two massive 1-loop Tadpoles, times a ratio of polynoms in Q^2, a and ϵ. But, if we consider the Sunrise with two mass-less propagators, analogous to that one in Fig. (7) (o), but with the external leg off-shell, this is no longer possible. The resulting system is, in fact, homogeneous, as we can understand from the fact that contracting a propagator line we have at least a product with a mass-less Tadpole, which vanishes in dimensional regularization. In this case the conditions of regularity of the integrals in $Q^2 = 0$ are not sufficient to determine the boundary conditions, as explained in section 3.0.1. In this situation we are forced to evaluate the integrals by direct integration, as we did for the MI in Fig. (7) (o); but that is not a problem, given the simplicity of the integrals.

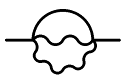
4.1 The full calculation for the topology in Fig. (5) (a)

The topology in Fig. (5) (a) has only one MI. We choose the simpler one, i.e. the scalar integral itself, Fig. (7) (e):

$$F(\epsilon, a, Q^2) =$$  $$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_{14} \mathcal{D}_{15}} \,, \qquad (26)$$

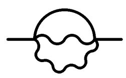
where the exlicit expressions of the denominators $\mathcal{D}_i$ are given in appendix A.

The first-order linear differential equation in the variable Q^2, which we obtain by the methods described in the previous sections, reads

$$\frac{dF(\epsilon, a, Q^2)}{dQ^2} = -\frac{1}{2}\left[\frac{1}{Q^2} - \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F(\epsilon, a, Q^2) - \frac{(2-3\epsilon)}{4a}\left[\frac{1}{Q^2} - \frac{1}{(Q^2+4a)}\right]$$  $$, \qquad (27)$$

where the 3-denominator diagram on the non-homogeneous part of the equation is the MI of Fig. (7) (o), function only of the squared mass a; its expansion in Laurent series of ϵ is given in Eqs. (85–87).

As we can notice, from Eq. (27) we see that its solutions can be singular at $Q^2 = -4a$ and $Q^2 = 0$. The integral we are considering, Eq. (26), is indeed singular at the physical threshold $Q^2 = -4a$, but regular at the pseudothreshold $Q^2 = 0$. This allows us to get the initial condition directely from the differential equation. In fact, multiplying Eq. (27) for Q^2 and taking the limit $Q^2 \to 0$, the left-hand side simply vanishes; the right-hand side gives us:

$$F(\epsilon, a, Q^2 = 0) = -\frac{(2-3\epsilon)}{2a}$$  $$\qquad (28)$$

Even if in this particular case it is possible to find a solution of Eq. (27) exact in $D = 4 - 2\epsilon$, [18], in this section we look for a solution expanded in Laurent series of ϵ:

$$F(\epsilon, a, Q^2) = \sum_{i=-2}^{0} \epsilon^i F_i(a, Q^2) + \mathcal{O}(\epsilon) \ , \tag{29}$$

The homogeneous equation at $D = 4$, i.e. $\epsilon = 0$, is

$$\frac{df(a,y)}{dy} = -\frac{1}{2}\left[\frac{1}{y} - \frac{1}{(y+4a)}\right] f(a,y) \, , \tag{30}$$

whose solution is

$$f(a,y) = k\sqrt{1 + \frac{4a}{y}} \, , \tag{31}$$

where k is a normalization constant.

Order by order in ϵ, we obtain the solution of the non-homogeneous equation by means of the method of the variation of the constant k (Euler's method). Substituting, into Eq. (27), Eq. (29) and the expansion in ϵ of the 3-denominator integral on the non-homogeneous part, Eq. (83), the result reads as follows:

$$\begin{aligned} F_i(a, Q^2) \;\; &= \;\; \sqrt{1+\frac{4a}{Q^2}} \, \left\{ \int^{Q^2} \frac{dy}{\sqrt{1+\frac{4a}{y}}} \left[-\frac{1}{y+4a} F_{i-1}(a,y) \right. \right. \\ & \left. \left. - \frac{1}{2a}\left(\frac{1}{y} - \frac{1}{y+4a}\right) C_i + \frac{3}{4a}\left(\frac{1}{y} - \frac{1}{y+4a}\right) C_{i-1} \right] + k_i \right\} \, , \end{aligned} \tag{32}$$

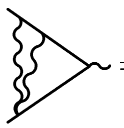
where the explicit values of the constants C_i are given in Eqs.(85–87).

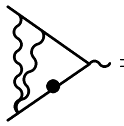
The determination of the constants k_i is made by imposing that the solution, Eq. (32) satisfies the initial condition, Eq. (28), or, which is the same thing, imposing the regularity of the solution at $Q^2 = 0$.

It is then useful to express the result in terms of the variable x, defined in Eq. (25). The explicit form of the solution up to the zeroth order in ϵ is given in Eqs. (105–107).

4.2 The full calculation for the topology in Fig. (4) (a)

The topology in Fig. (4) (a) has two MI's. In order to decouple the system of differential equations order by order in ϵ, we choose the couple of MI's given by Fig. (7) (c) and (d), i.e. the fully scalar integral and the scalar integral with a squared propagator:

$F_1(\epsilon, a, Q^2) \;=$  $= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_9 \mathcal{D}_{14} \mathcal{D}_{15}} \, , \quad (33)$

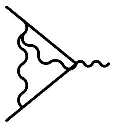
$F_2(\epsilon, a, Q^2) \;=$  $= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_9 \mathcal{D}_{14} \mathcal{D}_{15}^2} \, . \quad (34)$

The corresponding system of first-order linear differential equations in the variable Q^2 reads

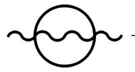
$$
\begin{aligned}
\frac{dF_1(\epsilon,a,Q^2)}{dQ^2} &= -\frac{1}{2}\left[\frac{1}{Q^2}+\frac{(1-2\epsilon)}{(Q^2+4a)}\right] F_1(\epsilon,a,Q^2) \\
&- F_2(\epsilon,a,Q^2)+\Omega^{(1)}(\epsilon,a,Q^2)\,, \qquad (35)\\
\frac{dF_2(\epsilon,a,Q^2)}{dQ^2} &= \frac{\epsilon^2}{2a}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right] F_1(\epsilon,a,Q^2) \\
&- \left[\frac{1}{Q^2}+\frac{(1+2\epsilon)}{(Q^2+4a)}\right] F_2(\epsilon,a,Q^2)+\Omega^{(2)}(\epsilon,a,Q^2)\,, \qquad (36)
\end{aligned}
$$

where the functions $\Omega^{(i)}(\epsilon,a,Q^2)$ are the following combinations of MI's:

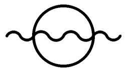
$$
\Omega^{(1)}(\epsilon,a,Q^2) = \frac{1}{2aQ^2}\frac{(1-3\epsilon)(1-2\epsilon)}{(1-4\epsilon)}
$$



$$
+\frac{1}{2\epsilon}\Bigg\{\frac{(1-2\epsilon)^2(3-4\epsilon)}{16a^2(1-4\epsilon)Q^2}
-\frac{(3-20\epsilon^2+16\epsilon^3)}{16a^2(1-4\epsilon)(Q^2+4a)}-\frac{(11-108\epsilon+256\epsilon^2-168\epsilon^3)}{4a(1-4\epsilon)(Q^2+4a)^2}
+\frac{3(5-16\epsilon+12\epsilon^2)}{(Q^2+4a)^3}\Bigg\}
$$

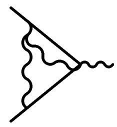


$$
-\frac{1}{\epsilon}\Bigg\{\frac{3(1-2\epsilon)^2(1-\epsilon)}{16a^3(1-4\epsilon)Q^2}
-\frac{3(1-2\epsilon)^2(1-\epsilon)}{16a^3(1-4\epsilon)(Q^2+4a)}-\frac{3(1-\epsilon-4\epsilon^2+4\epsilon^3)}{4a^2(1-4\epsilon)(Q^2+4a)^2}
-\frac{9(1-3\epsilon+2\epsilon^2)}{a(Q^2+4a)^3}\Bigg\}
$$

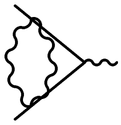


$$
(k_1\cdot k_2)-\Bigg\{\frac{(3-5\epsilon+2\epsilon^2)}{16a^3(1-4\epsilon)Q^2}
-\frac{(3-5\epsilon+2\epsilon^2)}{16a^3(1-4\epsilon)(Q^2+4a)}-\frac{(2-7\epsilon+5\epsilon^2)}{4a^2(1-4\epsilon)(Q^2+4a)^2}
-\frac{(3-9\epsilon+6\epsilon^2)}{a\epsilon(Q^2+4a)^3}\Bigg\}T^2(\epsilon,a)\,, \qquad (37)
$$

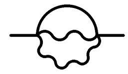
$$
\Omega^{(2)}(\epsilon,a,Q^2) = \frac{\epsilon(1-2\epsilon)(1-3\epsilon)}{2a^2(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]
$$



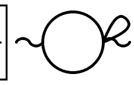
$$
+\frac{(1-2\epsilon)(1-3\epsilon)}{4a}\left[\frac{1}{4aQ^2}-\frac{1}{4a(Q^2+4a)}
-\frac{1}{(Q^2+4a)^2}\right]
$$



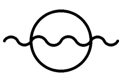
$$
+\frac{(1-2\epsilon)(1-3\epsilon)(2-3\epsilon)}{8a^2(1-4\epsilon)}\left[\frac{1}{4aQ^2}
-\frac{1}{4a(Q^2+4a)}-\frac{1}{(Q^2+4a)^2}\right]
$$



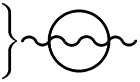
$$
+\frac{(1-\epsilon)(1-2\epsilon)}{4a^2}\left[\frac{1}{4aQ^2}-\frac{1}{4a(Q^2+4a)}-\frac{1}{(Q^2+4a)^2}\right]
$$



$$
\begin{aligned}
&-\Bigg\{\frac{(1-2\epsilon)(1-11\epsilon+12\epsilon^2)}{32a^3(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]\\
&-\frac{(1-2\epsilon)}{4\epsilon}\Bigg[\frac{(3-4\epsilon)(1-4\epsilon)}{16a^2Q^2}-\frac{(3-36\epsilon+296\epsilon^2-288\epsilon^3)}{16a^2(1-4\epsilon)(Q^2+4a)^2}\\
&-\frac{3(1-16\epsilon+20\epsilon^2)}{2a(1-4\epsilon)(Q^2+4a)^3}+\frac{3(5-6\epsilon)(1-6\epsilon)}{(Q^2+4a)^4}\Bigg]\Bigg\}
\end{aligned}
$$



$$
\begin{aligned}
&-\Bigg\{\frac{3(1-\epsilon)(1-2\epsilon)(1+4\epsilon)}{64a^4(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]\\
&+\frac{3(1-\epsilon)(1-2\epsilon)}{2a\epsilon}\Bigg[\frac{(1-4\epsilon)}{16a^2Q^2}-\frac{(1-6\epsilon+24\epsilon^2)}{16a^2(1-4\epsilon)(Q^2+4a)^2}\\
&-\frac{(1-7\epsilon+36\epsilon^2)}{2a(1-4\epsilon)(Q^2+4a)^3}+\frac{3(1-6\epsilon)}{(Q^2+4a)^4}\Bigg]\Bigg\}
\end{aligned}
$$



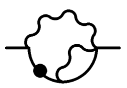
$$(k_1 \cdot k_2)$$

$$
\begin{aligned}
&-\Bigg\{\frac{(1-\epsilon)(3-31\epsilon+122\epsilon^2-104\epsilon^3)}{128a^4(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]\\
&-\frac{(1-\epsilon)}{2a\epsilon}\Bigg[\frac{\epsilon(1-4\epsilon)}{16a^2Q^2}+\frac{(3-32\epsilon+130\epsilon^2-120\epsilon^3)}{16a^2(1-4\epsilon)(Q^2+4a)^2}\\
&+\frac{(3-29\epsilon+134\epsilon^2-168\epsilon^3)}{2a(1-4\epsilon)(Q^2+4a)^3}+\frac{3(1-2\epsilon)(1-6\epsilon)}{(Q^2+4a)^4}\Bigg]\Bigg\}T^2(\epsilon,a), \qquad (38)
\end{aligned}
$$

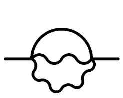
where $T(\epsilon, a)$ stands for the tadpole explicitly definited in Eq. (70).

For what initial conditions are concerned, we know that $F_1(\epsilon, a, Q^2)$ is analytic in $Q^2 = 0$. For Euclidean momenta, the limit $Q^2 \to 0$ (which implies $Q \to 0$) can be recovered by the limit $p_2 \to -p_1$. Taking this limit directly within the integrand of Eq. (33) we obtain

$$F_1(\epsilon, a, Q^2 = 0) \; = $$



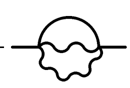
$$= -\frac{(2-3\epsilon)(1-3\epsilon)}{2a^2(1-4\epsilon)}$$



$$-\frac{(1-\epsilon)^2(1-2\epsilon)}{2a^3(1-2\epsilon)}T^2(\epsilon,a). \qquad (39)$$

Once we have the initial condition for $F_1(\epsilon, a, Q^2)$, we can calculate the initial condition for the second MI, $F_2(\epsilon, a, Q^2)$, directly from Eq. (35). In fact, because of the analyticity of $F_1(\epsilon, a, Q^2)$ in $Q^2 = 0$, we can multiply Eq. (35) by Q^2 and take the limit $Q^2 \to 0$. The left-hand side vanishes and we find the following relation:

$$F_2(\epsilon, a, Q^2 = 0) \; = \; -\frac{3\epsilon(1-3\epsilon)(2-3\epsilon)}{8a^3(1-4\epsilon)}$$



$$+\frac{(16\epsilon^4-16\epsilon^3-76\epsilon^2+226\epsilon-171)}{32a^4}T^2(\epsilon,a)\,. \qquad (40)$$

We look for a solution of the system of Eqs. (35–36) in terms of the coefficients of the Laurent series in ϵ:

$$
\begin{aligned}
F_1(\epsilon, a, Q^2) &= \sum_{i=-2}^{0} \epsilon^i F_i^{(1)}(a, Q^2) + \mathcal{O}(\epsilon) \ , && (41)\\
F_2(\epsilon, a, Q^2) &= \sum_{i=-2}^{0} \epsilon^i F_i^{(2)}(a, Q^2) + \mathcal{O}(\epsilon) \ , && (42)
\end{aligned}
$$

As immediately seen by direct inspection of Eqs. (35,36), the systematic expansion in powers of ϵ gives a triangular system in which the second equation, at order i in the ϵ expansion, consists of a homogeneous part with $F_i^{(2)}(a, Q^2)$ only, and a non-homogeneous part which contains expansion terms of order lower than i. The first equation for $F_i^{(1)}(a, Q^2)$, on the contrary, contains both $F_i^{(1)}(a, Q^2)$ and $F_i^{(2)}(a, Q^2)$ in the homogeneous part; but as a first step we can solve the second equation for $F_i^{(2)}(a, Q^2)$, substitute the result in the first equation and split again the first equation in a new homogeneous part, which now involves only $F_i^{(1)}(a, Q^2)$, and a non-homogeneous part which is known. As a result, the original system splits into two homogeneous decoupled equations, which are

$$
\begin{aligned}
\frac{d f_1(a,y)}{dy} &= -\frac{1}{2}\left[\frac{1}{y} + \frac{1}{(y+4a)}\right] f_1(a,y) \, , && (43)\\
\frac{d f_2(a,y)}{dy} &= -\left[\frac{1}{y} + \frac{1}{(y+4a)}\right] f_2(a,y) \, , && (44)
\end{aligned}
$$

with solutions

$$
\begin{aligned}
f_1(a,y) &= \frac{k_1}{\sqrt{y(y+4a)}} \, , && (45)\\
f_2(a,y) &= \frac{k_2}{y(y+4a)} \, . && (46)
\end{aligned}
$$

By means of the Euler method we can find, order by order in ϵ, the solution of the non-homogeneous system, solving the two first order differential equations in the order of the two quadrature formulas:

$$
\begin{aligned}
F_i^{(2)}(a, Q^2) &= \frac{1}{Q^2(Q^2+4a)} \Bigg\{ \int^{Q^2} dy \, y(y+4a) \Big[\frac{1}{2a}\left(\frac{1}{y} - \frac{1}{(y+4a)}\right) F_{i-2}^{(1)}(a,y) \\
&\quad - \frac{2}{(y+4a)} F_{i-1}^{(2)} + \Omega_i^{(2)}(a,y) \Big] + k_i^{(2)} \Bigg\} \, , && (47)\\
F_i^{(1)}(a, Q^2) &= \frac{1}{\sqrt{Q^2(Q^2+4a)}} \Bigg\{ \int^{Q^2} dy \sqrt{y(y+4a)} \Big[\frac{1}{(y+4a)} F_{i-1}^{(1)} - F_i^{(2)}(a,y) \\
&\quad + \Omega_i^{(1)}(a,y) \Big] + k_i^{(1)} \Bigg\} , && (48)
\end{aligned}
$$

where, for simplicity, we put:

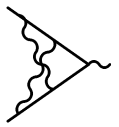
$$\Omega^{(1)}(\epsilon, a, Q^2) = \sum_{i=-2}^{0} \epsilon^i \Omega_i^{(1)}(a, Q^2) + \mathcal{O}(\epsilon) \, , \tag{49}$$

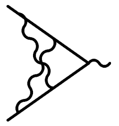
$$\Omega^{(2)}(\epsilon, a, Q^2) = \sum_{i=-2}^{0} \epsilon^i \Omega_i^{(2)}(a, Q^2) + \mathcal{O}(\epsilon) \, , \tag{50}$$

The determination of the constants $k_i^{(1)}$ and $k_i^{(2)}$ is made imposing the initial conditions Eqs. (39, 40). The solution, expressed in terms of the variable x, is given in Eqs. (137-139).

4.3 The full calculation for the topology in Fig. (3) (b)

The topology in Fig. (3) (b) has two MI's. We choose the MI's corresponding to Fig. (7) (a) and (b), i.e. the fully scalar integral and the scalar integral with the scalar product $(k_1 \cdot k_2)$ in the numerator of the integrand. Also in this case the choice of the MI's diagonalizes the system in the limit $\epsilon \to 0$.

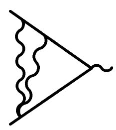
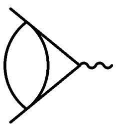
$$F_1(\epsilon, a, Q^2) =$$  $$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{11}\mathcal{D}_{14}\mathcal{D}_{15}} \, , \tag{51}$$

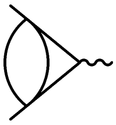
$$F_2(\epsilon, a, Q^2) =$$  $$(k_1 \cdot k_2) = \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{k_1 \cdot k_2}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{11}\mathcal{D}_{14}\mathcal{D}_{15}} . \tag{52}$$

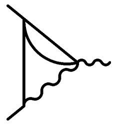
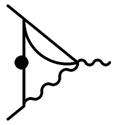
The system of first-order linear differential equations is the following:

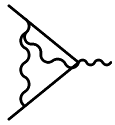
$$\begin{aligned}
\frac{dF_1(\epsilon, a, Q^2)}{dQ^2} &= -(1+2\epsilon)\left[\frac{1}{Q^2} + \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F_1(\epsilon, a, Q^2) \\
&\quad -\frac{2\epsilon}{a}\left[\frac{1}{Q^2} - \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F_2(\epsilon, a, Q^2) + \Omega^{(1)}(\epsilon, a, Q^2) \, , \qquad (53) \\
\frac{dF_2(\epsilon, a, Q^2)}{dQ^2} &= \epsilon \; F_1(\epsilon, a, Q^2) \\
&\quad -\frac{(1-2\epsilon)}{2}\left[\frac{1}{Q^2} + \frac{1}{(Q^2+4a)}\right] F_2(\epsilon, a, Q^2) + \Omega^{(2)}(\epsilon, a, Q^2), \qquad (54)
\end{aligned}$$

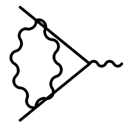
where the functions $\Omega^{(i)}(\epsilon, a, Q^2)$ are defined as follows:

$$\Omega^{(1)}(\epsilon, a, Q^2) = \frac{\epsilon}{a}\left[\frac{1}{Q^2} - \frac{1}{(Q^2+4a)}\right]$$  $$- \frac{3(1-2\epsilon)}{8a^2}\left[\frac{4a}{Q^4} - \frac{1}{Q^2} + \frac{1}{(Q^2+4a)}\right]$$  $$+ \frac{3(2-3\epsilon)}{4a^3}\left[\frac{4a}{Q^4} - \frac{1}{Q^2}\right.$$

$$+\frac{1}{(Q^2+4a)}\Bigg]$$  $$(p_2\cdot k_1)\;+\frac{5(1-3\epsilon)}{4a^2}\Bigg[\frac{1}{Q^2}$$

$$-\frac{1}{(Q^2+4a)}\Bigg]$$  $$+\frac{5}{2a}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]$$ 

$$-\frac{3(1-2\epsilon)(1-3\epsilon)}{4a^2(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]$$ 

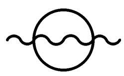
$$+\frac{(1-2\epsilon)(1-3\epsilon)}{16a^2\epsilon}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}-\frac{4a}{(Q^2+4a)^2}\right]$$ 

$$-\Bigg\{\frac{(17-116\epsilon+248\epsilon^2-128\epsilon^3)}{64a^3\epsilon(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]$$

$$+\frac{(3-4\epsilon)(1-4\epsilon)(1-22\epsilon)}{64a^2\epsilon^2}\frac{1}{Q^4}$$

$$+\frac{(3-162\epsilon+968\epsilon^2-2240\epsilon^3+1536\epsilon^4)}{64a^2\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}$$

$$+\frac{3(1-2\epsilon)(1+34\epsilon-140\epsilon^2+120\epsilon^3}{8a\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^3}$$

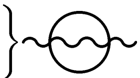
$$-\frac{3(1-2\epsilon)(5-6\epsilon)(1-6\epsilon)}{4\epsilon^2}\frac{1}{(Q^2+4a)^4}\Bigg\}$$ 

$$+\Bigg\{\frac{3(1-\epsilon)(9-2\epsilon-112\epsilon^2)}{64a^4\epsilon(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]$$

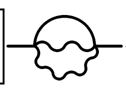
$$-\frac{3(1-\epsilon)(1-4\epsilon)(1-22\epsilon)}{32a^3\epsilon^2}\frac{1}{Q^4}$$

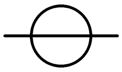
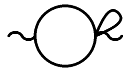
$$+\frac{3(1-\epsilon)(1-48\epsilon+196\epsilon^2-128\epsilon^3)}{32a^3\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}$$

$$+\frac{3(1-\epsilon)(1-2\epsilon)(1-37\epsilon+96\epsilon^2)}{4a^2\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^3}$$

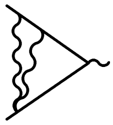
$$+\frac{9(1-\epsilon)(1-2\epsilon)(1-6\epsilon)}{2a\epsilon^2}\frac{1}{(Q^2+4a)^4}\Bigg\}$$  $$(k_1\cdot k_2)$$

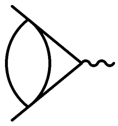
$$+\frac{(1-2\epsilon)(1-3\epsilon)(2-3\epsilon)}{32a^3\epsilon(1-4\epsilon)}\Bigg[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}$$

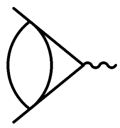
$$-\frac{4a}{(Q^2+4a)^2}\Bigg]$$  $$-\frac{3(2-3\epsilon)}{16a^3}\left[\frac{4a}{Q^4}-\frac{1}{Q^2}+\frac{1}{(Q^2+4a)}\right]\times$$

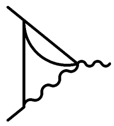
$\times$  $- \dfrac{3(2-3\epsilon)}{32a^3}\left[\dfrac{4a}{Q^4}-\dfrac{1}{Q^2}+\dfrac{1}{(Q^2+4a)}\right]$ 

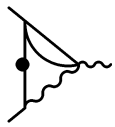
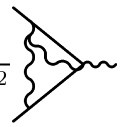
$$
\begin{aligned}
&-\left\{\frac{(1-\epsilon)(3-89\epsilon+436\epsilon^2-576\epsilon^3+16\epsilon^4)}{128a^4\epsilon^2(1-2\epsilon)(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)}\right]\right.\\
&-\frac{(1-\epsilon)(1-38\epsilon+100\epsilon^2)}{32a^3\epsilon(1-2\epsilon)}\frac{1}{Q^4}\\
&-\frac{(1-\epsilon)(3-84\epsilon+310\epsilon^2-208\epsilon^3)}{32a^3\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}\\
&-\frac{(1-\epsilon)(3-89\epsilon+374\epsilon^2-408\epsilon^3)}{8a^2\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^3}\\
&\left.-\frac{3(1-\epsilon)(1-2\epsilon)(1-6\epsilon)}{2a\epsilon^2}\frac{1}{(Q^2+4a)^4}\right\} T^2(\epsilon,a)\,, \qquad (55)
\end{aligned}
$$

$\Omega^{(2)}(\epsilon,a,Q^2) \;=\; \epsilon$  $+\left\{\dfrac{(1-2\epsilon)(1-4\epsilon)}{2\epsilon}\left[\dfrac{4a}{Q^4}+\dfrac{1}{(Q^2+4a)}\right]\right.$

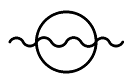
$\left.-\dfrac{(1-2\epsilon)}{8a\epsilon}\dfrac{1}{Q^2}\right\}$  $+\left\{\dfrac{(2-3\epsilon)(1-4\epsilon)}{a\epsilon}\dfrac{1}{Q^4}\right.$

$\left.-\dfrac{(1-2\epsilon)(2-3\epsilon)}{4a^2\epsilon}\left[\dfrac{1}{Q^2}-\dfrac{1}{(Q^2+4a)}\right]\right\}$  $(p_2\cdot k_1)$

$-\dfrac{(1-3\epsilon)}{a}\dfrac{1}{(Q^2+4a)}$  $-\left[\dfrac{(1-2\epsilon)}{2\epsilon}\dfrac{1}{Q^2}\right.$

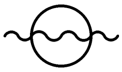
$\left.+\dfrac{(1-6\epsilon)}{2\epsilon}\dfrac{1}{(Q^2+4a)}\right]$  $+\dfrac{(1-2\epsilon)(1-3\epsilon)}{2a(1-4\epsilon)}\dfrac{1}{Q^2}$ 

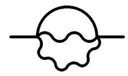
$$
\begin{aligned}
&+\left\{\frac{(1-2\epsilon)(3-4\epsilon)(1-4\epsilon)}{16a\epsilon^2}\frac{1}{Q^4}\right.\\
&+\frac{(3-18\epsilon+4\epsilon^2+136\epsilon^3-160\epsilon^4)}{64a^2\epsilon^2(1-4\epsilon)}\frac{1}{Q^2}\\
&+\frac{3(1-2\epsilon)(5-6\epsilon)(1-6\epsilon)}{4\epsilon^2}\frac{1}{(Q^2+4a)^3}\\
&-\frac{(3-74\epsilon+410\epsilon^2-816\epsilon^3+504\epsilon^4)}{8a\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}
\end{aligned}
$$

$\left.-\dfrac{(3-18\epsilon+36\epsilon^2+40\epsilon^3-96\epsilon^4)}{64a^2\epsilon^2(1-4\epsilon)}\dfrac{1}{(Q^2+4a)}\right\}$ 

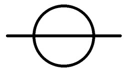
$$
-\frac{3(1-\epsilon)}{2a\epsilon^2}\left\{\frac{(1-2\epsilon)(1-4\epsilon)}{4a}\frac{1}{Q^4}+\frac{(1-32\epsilon^2+72\epsilon^3)}{16a^2(1-4\epsilon)}\left(\frac{1}{Q^2}\right.\right.
$$

$$
-\frac{1}{(Q^2+4a)}\Bigg) - 3(1-2\epsilon)(1-6\epsilon)\frac{1}{(Q^2+4a)^3}
$$

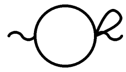
$$
-\frac{(1-5\epsilon+4\epsilon^2+12\epsilon^3)}{2a(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}\Bigg\}
$$

$(k_1 \cdot k_2)$

$$
-\frac{(1-2\epsilon)(2-3\epsilon)(1-3\epsilon)}{16a^2\epsilon(1-4\epsilon)}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)^2}\right]
$$


$$
-\Bigg\{\frac{(2-3\epsilon)(1-4\epsilon)}{8a\epsilon}\frac{1}{Q^4}-\frac{(1-2\epsilon)(2-3\epsilon)}{8a\epsilon}\Bigg[\frac{1}{Q^2}
$$

$$
-\frac{1}{(Q^2+4a)^2}\Bigg]\Bigg\}
$$


$$
+\Bigg\{\frac{(1-\epsilon)(1-4\epsilon)}{2a\epsilon}\frac{1}{Q^4}
$$

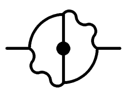
$$
+\frac{(1-\epsilon)(1-2\epsilon)}{8a^2\epsilon}\left[\frac{1}{Q^2}-\frac{1}{(Q^2+4a)^2}\right]\Bigg\}
$$


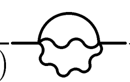
$$
-\frac{(1-\epsilon)}{2a}\Bigg\{\frac{(1-4\epsilon)}{4a(1-2\epsilon)}\frac{1}{Q^4}+\frac{(3-25\epsilon-71\epsilon^2+58\epsilon^3+32\epsilon^4)}{16a^2\epsilon^2(1-2\epsilon)(1-4\epsilon)}\Bigg(\frac{1}{Q^2}
$$

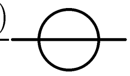
$$
-\frac{1}{(Q^2+4a)}\Bigg)-\frac{3(1-2\epsilon)(1-6\epsilon)}{\epsilon^2}\frac{1}{(Q^2+4a)^3}
$$

$$
-\frac{3(1-7\epsilon+14\epsilon^2)}{4a\epsilon^2(1-4\epsilon)}\frac{1}{(Q^2+4a)^2}\Bigg\}T^2(\epsilon,a)\,. \tag{56}
$$

For what concerns initial conditions, we know that $F_1(\epsilon, a, Q^2)$ is analytic in $Q^2 = 0$. For Euclidean momenta, the limit $Q^2 \to 0$ (which implies $Q \to 0$) can be recovered by the limit $p_2 \to -p_1$. Taking this limit directly within the integrand of Eq. (51) we have:

$$
F_1(\epsilon, a, Q^2 = 0) \;=\;
$$

,

$$
= -\frac{3\epsilon(2-3\epsilon)(1-3\epsilon)}{4a^3(1-4\epsilon)(1+2\epsilon)}
$$


$$
+\frac{3(2-3\epsilon)(1-3\epsilon)}{64a^3\epsilon}
$$


$$
+\frac{(1-\epsilon)^2(9+3\epsilon-160\epsilon^2-196\epsilon^3)}{64a^4\epsilon(1-2\epsilon)(1+2\epsilon)}\,T^2(\epsilon,a). \tag{57}
$$

We can find the initial condition for F_2 from Eq. (53), multiplying by Q^2 and taking the limit $Q^2 \to 0$, or performing the limit directely inside the integral of Eq. (52). What we get is the following expression:

$$
F_2(\epsilon, a, Q^2 = 0) \;=\; \frac{(2-3\epsilon)(1-3\epsilon)}{8a^2\epsilon}
$$

$$
+\frac{(2-3\epsilon)(1-3\epsilon)}{32a^2\epsilon^2}
$$

$$+\frac{(1-\epsilon)^2(3-15\epsilon+16\epsilon^2)}{32a^3\epsilon^2(1-2\epsilon)}\,T^2(\epsilon,a). \tag{58}$$

We look for a solution of the system of Eqs. (53–54) expanded in Laurent series of ϵ:

$$F_1(\epsilon,a,Q^2) = \sum_{i=-2}^{0}\epsilon^i F_i^{(1)}(a,Q^2)+\mathcal{O}(\epsilon)\,, \tag{59}$$

$$F_2(\epsilon,a,Q^2) = \sum_{i=-2}^{0}\epsilon^i F_i^{(2)}(a,Q^2)+\mathcal{O}(\epsilon)\,, \tag{60}$$

According to the previous remarks, the solution can be built, order by order, by the method of the variation of the constants of the associated homogeneous system, which reads

$$\frac{df_1(a,y)}{dy} = -\left[\frac{1}{y}+\frac{1}{(y+4a)}\right] f_1(a,y)\,, \tag{61}$$

$$\frac{df_2(a,y)}{dy} = -\frac{1}{2}\left[\frac{1}{y}+\frac{1}{(y+4a)}\right] f_2(a,y)\,, \tag{62}$$

As we can see, the homogeneous system is completely diagonalized in the limit $\epsilon\to 0$. The solution of the system is the following:

$$f_1(a,y) = \frac{k_1}{y(y+4a)}\,, \tag{63}$$

$$f_2(a,y) = \frac{k_2}{\sqrt{y(y+4a)}}\,. \tag{64}$$

By means of the Euler's method we can find, order by order in ϵ, the solution of the non-homogeneous system:

$$\begin{aligned}F_i^{(1)}(a,Q^2) = \frac{1}{Q^2(Q^2+4a)}\Bigg\{\int^{Q^2} dy\, y(y+4a)\Bigg[&-2\left(\frac{1}{y}+\frac{1}{(y+4a)}\right)F_{i-1}^{(1)}(a,y)\\ &+\frac{4}{(y+4a)}F_{i-2}^{(1)}(a,y)-\frac{2}{a}\left(\frac{1}{y}-\frac{1}{(y+4a)}\right)F_{i-1}^{(2)}(a,y)\\ &-\frac{4}{a(y+4a)}F_{i-2}^{(2)}(a,y)+\Omega_i^{(1)}(a,y)\Bigg]+k_i^{(1)}\Bigg\},\end{aligned} \tag{65}$$

$$\begin{aligned}F_i^{(2)}(a,Q^2) = \frac{1}{\sqrt{Q^2(Q^2+4a)}}\Bigg\{\int^{Q^2} dy\,\sqrt{y(y+4a)}\Bigg[&F_{i-1}^{(1)}(a,y)\\ &+\left(\frac{1}{y}-\frac{1}{(y+4a)}\right)F_{i-1}^{(2)}(a,y)+\Omega_i^{(2)}(a,y)\Bigg]+k_i^{(2)}\Bigg\}\,,\end{aligned} \tag{66}$$

where, for simplicity, we put:

$$\Omega^{(1)}(\epsilon,a,Q^2) = \sum_{i=-2}^{0}\epsilon^i\Omega_i^{(1)}(a,Q^2)+\mathcal{O}(\epsilon)\,, \tag{67}$$

$$\Omega^{(2)}(\epsilon,a,Q^2) = \sum_{i=-2}^{0}\epsilon^i\Omega_i^{(2)}(a,Q^2)+\mathcal{O}(\epsilon)\,, \tag{68}$$

The determination of the constants $k_i^{(1)}$ and $k_i^{(2)}$ is made imposing the initial conditions Eqs. (57, 58). The solution, expressed in terms of the variable x, is given in Eqs. (144-146).

5 Results for the MI's

We list in this section all the MI's necessary for the calculation of the 2-loop vertex diagrams of Fig. (1).

We give them as a Laurent series in $\epsilon = (4-D)/2$ and we express the coefficients of the series in terms of harmonic polylogarithms of the variable x, already introduced in Eq. (25)

$$x = \frac{\sqrt{Q^2+4a} - \sqrt{Q^2}}{\sqrt{Q^2+4a} + \sqrt{Q^2}} \, .$$

For brevity we present the results only up to the zeroth order in ϵ, but of course the method allows us to calculate any order in ϵ. All the coefficients of the ϵ expansion depend of course on a and Q^2 (or the above variable x); for reducing the size of the formulas, we will not write anymore the dependence of the coefficients on those variables. The analytic results are expressed in terms of Harmonic Polylogarithms of argument x. Definition, notation and properties of the Harmonic Polylogarithms can be found in [5, 6].

The scale μ_0 is the regularization scale and $a = m_e^2$ is the only mass involved (in our case the mass of the electron).

The explicit values of the $\mathcal{D}_i$ is given in appendix A.

For what concerns the normalization of our integrals, we define the 1-loop Tadpole with mass a as

$$T(\epsilon, a) = \mu_0^{2(4-D)} \int \{d^D k\} \frac{1}{k^2+a} \, . \tag{69}$$

We further define $\{d^D k\}$ in order to have[6]:

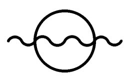
$$T(\epsilon, a) = \left(\frac{a}{\mu_0^2}\right)^{-\epsilon} \frac{a}{\epsilon(\epsilon-1)} \, , \tag{70}$$

so that

$$\{d^D k\} = \frac{d^D k}{\pi^{\frac{D}{2}} \Gamma\left(3 - \frac{D}{2}\right)} \, . \tag{71}$$

with $D = 4 - 2\epsilon, \ \epsilon = (4-D)/2$.

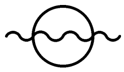
5.1 Topologies with $t = 3$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{16}} \tag{72}$$

[6]With this normalization the tadpole $T(\epsilon, a)$ of the present paper is 4 times the corresponding quantity of [13], [14] and [19].

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i A_i + \mathcal{O}\left(\epsilon\right), \tag{73}$$



$$(k_1 \cdot k_2) = \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{k_1 \cdot k_2}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{16}} \tag{74}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i B_i + \mathcal{O}\left(\epsilon\right). \tag{75}$$

As already said above, from now on we write for short A_i, B_i instead of $A_i(a, Q^2)$, and $B_i(a, Q^2)$. Referring to [18] for more details, we find

$$\frac{A_{-2}}{a} = -1\,, \tag{76}$$

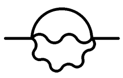
$$\frac{A_{-1}}{a} = -\frac{5}{2} - \frac{1}{4}\left[x + \frac{1}{x}\right], \tag{77}$$

$$\frac{A_0}{a} = -\frac{11}{4} - \frac{13}{8}\left[\frac{1}{x} + x\right] - \left[1 + \frac{1}{2}\Big(\frac{1}{x} - x\Big) - \frac{2}{(1-x)}\right] H(0,x) + 2\left[1 - \frac{1}{(1-x)} + \frac{1}{(1-x)^2}\right] H(0,0,x)\,; \tag{78}$$

$$\frac{B_{-2}}{a^2} = -\frac{1}{4}\left[x + \frac{1}{x}\right], \tag{79}$$

$$\frac{B_{-1}}{a^2} = -\frac{1}{24}\left[\frac{1}{x^2} + x^2\right] - \frac{1}{24}\left[\frac{1}{x} + x\right], \tag{80}$$

$$\frac{B_0}{a^2} = -\frac{11}{12} - \frac{1}{48}\left[13\Big(\frac{1}{x^2} + x^2\Big) - 11\Big(\frac{1}{x} + x\Big)\right] - \frac{1}{2}\left[1 - \frac{2}{(1-x)}\right] H(0,x) - \frac{1}{12}\left[\Big(\frac{1}{x^2} - x^2\Big) - 7\Big(\frac{1}{x} - x\Big)\right] H(0,x) + \frac{1}{2}\left[\frac{1}{x} + x - \frac{2}{(1-x)} + \frac{2}{(1-x)^2}\right] H(0,0,x)\,. \tag{81}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_{15}} \tag{82}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i C_i + \mathcal{O}\left(\epsilon\right), \tag{83}$$

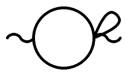
$$\tag{84}$$

As mentioned in section 5, we calculated this MI directely by means of Feynman parameters. We found:

$$\frac{C_{-2}}{a} = -\frac{1}{2}, \tag{85}$$

$$\frac{C_{-1}}{a} = \frac{5}{4}\,, \tag{86}$$

$$\frac{C_0}{a} = -\frac{11}{8} - 2\zeta(2)\,. \tag{87}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_{12}} \tag{88}$$

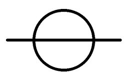
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i + \mathcal{O}(\epsilon)\,, \tag{89}$$

where:

$$\frac{E_{-2}}{a} = -1\,, \tag{90}$$

$$\frac{E_{-1}}{a} = -3 + \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, \tag{91}$$

$$\frac{E_0}{a} = -7 - \left[1 - \frac{2}{(1-x)}\right] \Big\{\zeta(2) - 3H(0,x) - H(0,0,x) + 2H(-1,0,x)\Big\}\,. \tag{92}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_{15}} \tag{93}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i F_i + \mathcal{O}(\epsilon)\,, \tag{94}$$

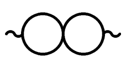
where [19]:

$$\frac{F_{-2}}{a} = -\frac{3}{2}\,, \tag{95}$$

$$\frac{F_{-1}}{a} = -\frac{17}{4}\,, \tag{96}$$

$$\frac{F_0}{a} = -\frac{59}{8}\,. \tag{97}$$

5.2 Topologies with $t = 4$

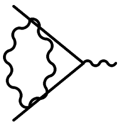


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_{12} \mathcal{D}_{13}} \tag{98}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i G_i + \mathcal{O}(\epsilon)\,, \tag{99}$$

where:

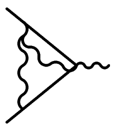
$$\begin{aligned}
G_{-2} &= 1, && (100)\\
G_{-1} &= 4-2\left[1-\frac{2}{(1-x)}\right]H(0,x)\,, && (101)\\
G_{0} &= 12+2\left[1-\frac{2}{(1-x)}\right]\Big\{\zeta(2)-4H(0,x)+2H(-1,0,x) \\
&\quad -\frac{2}{(1-x)}H(0,0,x)\Big\}. && (102)
\end{aligned}$$



$$\begin{aligned}
&= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{14}\mathcal{D}_{15}} && (103)\\
&= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-2}^{0}\epsilon^i I_i+\mathcal{O}\left(\epsilon\right), && (104)
\end{aligned}$$

where:

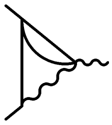
$$\begin{aligned}
I_{-2} &= \frac{1}{2}\,, && (105)\\
I_{-1} &= \frac{5}{2}-\left[1-\frac{2}{(1-x)}\right]H(0,x), && (106)\\
I_{0} &= \frac{19}{2}+2\zeta(2)+\left[1-\frac{2}{(1-x)}\right]\{2\zeta(2)-5H(0,x)-2H(0,0,x) \\
&\quad +4H(-1,0,x)\}. && (107)
\end{aligned}$$



$$\begin{aligned}
&= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{10}\mathcal{D}_{14}} && (108)\\
&= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-2}^{0}\epsilon^i J_i+\mathcal{O}\left(\epsilon\right), && (109)
\end{aligned}$$

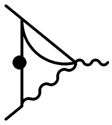
where:

$$\begin{aligned}
J_{-2}(x) &= \frac{1}{2}\,, && (110)\\
J_{-1}(x) &= \frac{5}{2}\,, && (111)\\
J_{0}(x) &= \frac{19}{2}+2\Big[\frac{1}{(1-x)}-\frac{1}{(1+x)}\Big]\Big\{\zeta(2)H(0,x)+H(0,0,0,x)\Big\} \\
&\quad -H(0,0,x). && (112)
\end{aligned}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{11} \mathcal{D}_{16}} \tag{113}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i K_i + \mathcal{O}(\epsilon)\,, \tag{114}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{11}^2 \mathcal{D}_{16}} \tag{115}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i L_i + \mathcal{O}(\epsilon)\,, \tag{116}$$

where:

$$K_{-2} = \frac{1}{2}\,, \tag{117}$$

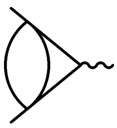
$$K_{-1} = \frac{5}{2}\,, \tag{118}$$

$$\begin{aligned} K_0 = {} & \frac{19}{2} - 2\zeta(2) + 2\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right][\zeta(2)H(0,x) + H(0,0,0,x)] \\ & - H(0,0,x)\,. \end{aligned} \tag{119}$$

$$aL_{-2} = -\frac{1}{2}\,, \tag{120}$$

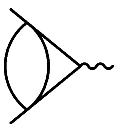
$$aL_{-1} = -1 + \frac{1}{2}\left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, \tag{121}$$

$$\begin{aligned} aL_0 = {} & -2 + \frac{1}{2}\left[1 - \frac{2}{(1-x)}\right]\Big\{\zeta(2) + 2H(0,x) + 7H(0,0,x) + 2H(1,0,x) \\ & - 3H(-1,0,x)\Big\} + 10H(0,0,x)\,. \end{aligned} \tag{122}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_{14} \mathcal{D}_{15}} \tag{123}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i M_i + \mathcal{O}(\epsilon)\,, \tag{124}$$



$$(p_2 \cdot k_1) = \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{p_2 \cdot k_1}{\mathcal{D}_6 \mathcal{D}_7 \mathcal{D}_{14} \mathcal{D}_{15}} \tag{125}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i N_i + \mathcal{O}(\epsilon)\,, \tag{126}$$

where:

$$M_{-2} = \frac{1}{2}\,, \tag{127}$$

$$M_{-1} = \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, \tag{128}$$

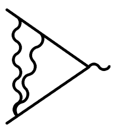
$$\begin{aligned} M_{0} = {} & \frac{19}{2} - 2\zeta(2) + 2\left[1 - \frac{2}{(1-x)}\right] [\zeta(2) - 5H(0,x) + 2H(-1,0,x)] \\ & + \frac{2}{(1-x)} H(0,0,x) + \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) H(0,x) \\ & + H(0,0,0,x)]\,. \end{aligned} \tag{129}$$

$$\frac{N_{-2}}{a} = \frac{1}{8} + \frac{1}{16}\left[x + \frac{1}{x}\right], \tag{130}$$

$$\frac{N_{-1}}{a} = \frac{9}{32}\left[2 + x + \frac{1}{x}\right] - \frac{1}{8}\left[4 + x - \frac{1}{x}\right] H(0,x) + \frac{1}{(1-x)} H(0,x)\,, \tag{131}$$

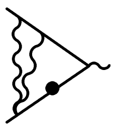
$$\begin{aligned} \frac{N_0}{a} = {} & \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\left[(1+16\zeta(2))x + \frac{1}{x}\right] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\left[32 + 9x \right. \\ & \left. - \frac{9}{x}\right] H(0,x) + \frac{(16+\zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\left[2 - \frac{1}{x} \right. \\ & \left. - \frac{4}{(1-x)}\right] H(0,0,x) + \frac{1}{4}\left[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\right] H(-1,0,x) \\ & + \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,0,0,x)\,. \end{aligned} \tag{132}$$

5.3 Topologies with $t = 5$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_9 \mathcal{D}_{14} \mathcal{D}_{15}} \tag{133}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} P_0 + \mathcal{O}(\epsilon)\,, \tag{134}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_9 \mathcal{D}_{14} \mathcal{D}_{15}^2} \tag{135}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-1}^{0} \epsilon^i Q_i + \mathcal{O}(\epsilon)\,, \tag{136}$$

where:

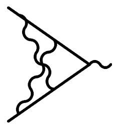
$$\begin{aligned} aP_0 = {} & \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \left\{ \frac{17\zeta^2(2)}{10} - 4\zeta(3) H(0,x) + \zeta(2) H(0,0,x) \right. \\ & + 4\zeta(2) H(1,0,x) + 4H(1,0,0,0,x) - 2H(0,1,0,0,x) \end{aligned}$$

$$-2H(0,0,-1,0,x)+4H(0,-1,0,0,x)\Big\}\,, \qquad (137)$$

$$a^2 Q_{-1} = \frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]H(0,0,x)\,, \qquad (138)$$

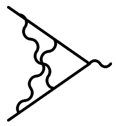
$$a^2 Q_0 = \frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]\Big\{\zeta(3)+5H(0,0,0,x)$$
$$+2H(0,1,0,x)-4H(0,-1,0,x)-4H(-1,0,0,x)\Big\}\,. \qquad (139)$$

5.4 Topologies with $t=6$



$$= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{11}\mathcal{D}_{14}\mathcal{D}_{15}} \qquad (140)$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-1}^{0}\epsilon^i R_i + \mathcal{O}(\epsilon)\,, \qquad (141)$$



$$(k_1\cdot k_2) = \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{k_1\cdot k_2}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{11}\mathcal{D}_{14}\mathcal{D}_{15}} \qquad (142)$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} S_0 + \mathcal{O}(\epsilon)\,, \qquad (143)$$

where:

$$a^2 R_{-1} = -\frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right][\zeta(3)+\zeta(2)H(0,x)$$
$$+2H(0,0,0,x)+2H(0,1,0,x)-2H(0,-1,0,x)]\,, \qquad (144)$$

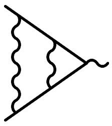
$$a^2 R_0 = -\frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]\left[\frac{37\zeta^2(2)}{10}+H(0,x)\right.$$
$$-4H(-1,x)+\zeta(3)H(1,x)-2\zeta(2)H(0,0,x)$$
$$-4\zeta(2)H(-1,0,x)-2\zeta(2)H(0,-1,x)-2\zeta(2)H(0,1,x)$$
$$+4\zeta(2)H(1,0,x)+12H(0,0,0,0,x)+8H(-1,0,-1,0,x)$$
$$-8H(-1,0,0,0,x)-8H(-1,0,1,0,x)+20H(0,-1,-1,0,x)$$
$$-16H(0,-1,0,0,x)-12H(0,-1,1,0,x)-24H(0,0,-1,0,x)$$
$$-16H(0,0,1,0,x)-12H(0,1,-1,0,x)+8H(0,1,0,0,x)$$
$$+4H(0,1,1,0,x)-8H(1,0,-1,0,x)+8H(1,0,0,0,x)$$
$$\left.+8H(1,0,1,0,x)\right]\,, \qquad (145)$$

$$aS_0 = \left[\frac{1}{(1+x)}-\frac{1}{(1-x)}\right]\left\{\frac{\zeta^2(2)}{10}-\zeta(3)H(0,x)+\zeta(2)(2H(1,0,x)\right.$$
$$+3H(0,-1,x))+\frac{1}{2}H(0,0,0,0,x)+H(0,-1,0,0,x)$$

$$+H(0,0,-1,0,x)+H(0,1,0,0,x)+2H(1,0,0,0,x)\Big\}. \qquad (146)$$

6 The 6-denominator reducible diagrams

The other diagrams of Fig. (1) are all reducible diagrams. We give in this section their result.



$$= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{10}\mathcal{D}_{14}\mathcal{D}_{15}} \qquad (147)$$

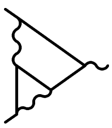
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-2}^{0}\epsilon^i F_i^{(1)}+\mathcal{O}(\epsilon)\,, \qquad (148)$$

where:

$$a^2F_{-2}^{(1)} = -\frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]H(0,0,x), \qquad (149)$$

$$a^2F_{-1}^{(1)} = \frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]\Big\{\zeta(3)+2\zeta(2)H(0,x)$$
$$-H(0,0,0,x)+4H(-1,0,0,x)+2H(0,1,0,x)\Big\}, \qquad (150)$$

$$a^2F_{0}^{(1)} = -\frac{1}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1-x)^2}+\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]\Big\{\frac{12}{5}\zeta^2(2)$$
$$-\zeta(3)[H(0,x)-4H(-1,x)]+\zeta(2)[7H(0,0,x)+2H(0,1,x)$$
$$+8H(-1,0,x)+4H(0,-1,x)]+5H(0,0,0,0,x)$$
$$+16H(-1,-1,0,0,x)-4H(-1,0,0,0,x)+8H(-1,0,1,0,x)$$
$$-16H(0,-1,-1,0,x)+6H(0,-1,0,0,x)+12H(0,-1,1,0,x)$$
$$+14H(0,0,-1,0,x)-12H(0,0,1,0,x)+12H(0,1,-1,0,x)$$
$$-8H(0,1,0,0,x)-4H(0,1,1,0,x)\Big\}. \qquad (151)$$



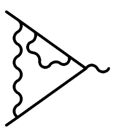
$$= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{10}\mathcal{D}_{11}\mathcal{D}_{15}} \qquad (152)$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-2}^{0}\epsilon^i F_i^{(2)}+\mathcal{O}(\epsilon)\,, \qquad (153)$$

where:

$$a^2F_{-2}^{(2)} = -\frac{1}{8}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]H(0,x)\,, \qquad (154)$$

$$
\begin{aligned}
a^2F^{(2)}_{-1} &= \frac{1}{8}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[\zeta(2) + 2H(0,x) + 2H(0,0,x)\right. \\
&\left. -2H(-1,0,x) + 2H(1,0,x)\right], \qquad (155) \\
a^2F^{(2)}_{0} &= -\frac{3\zeta(2)\ln 2}{(1+x)}\left[1 - \frac{1}{(1+x)}\right] - \frac{\zeta(2)}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \\
&-\frac{\zeta(3)}{8}\left[\frac{7}{(1-x)} - \frac{1}{(1+x)} - \frac{6}{(1+x)^2}\right] - \frac{1}{2}\left[\frac{(1+2\zeta(2))}{(1-x)}\right. \qquad (156) \\
&\left.-\frac{(1-\zeta(2))}{(1+x)} - \frac{3\zeta(2)}{(1+x)^2}\right]H(0,x) - \frac{\zeta(2)}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]H(1,x) \\
&+\frac{\zeta(2)}{4}\left[\frac{1}{(1-x)} + \frac{11}{(1+x)} - \frac{12}{(1+x)^2}\right]H(-1,x) \\
&-\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[H(0,0,x) - H(-1,0,x) + H(1,0,x)\right] \\
&+\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{7}{(1+x)} + \frac{12}{(1+x)^2}\right]H(0,0,0,x) \\
&-\frac{1}{2}\left[\frac{5}{(1-x)} - \frac{7}{(1+x)} + \frac{2}{(1+x)^2}\right]H(-1,0,0,x) \\
&-\frac{1}{2}\left[\frac{5}{(1-x)} - \frac{9}{(1+x)} + \frac{4}{(1+x)^2}\right]H(0,-1,0,x) \\
&+\left[\frac{1}{(1-x)} - \frac{2}{(1+x)} + \frac{1}{(1+x)^2}\right]H(0,1,0,x) \\
&+\left[\frac{1}{(1-x)} - \frac{2}{(1+x)^2}\right]H(1,0,0,x) + \frac{1}{2}\left[\frac{1}{(1-x)}\right. \\
&\left.-\frac{1}{(1+x)}\right]\left[7H(-1,-1,0,x) - 3H(-1,1,0,x) - 3H(1,-1,0,x)\right. \\
&\left.+H(1,1,0,x)\right]. \qquad (157)
\end{aligned}
$$



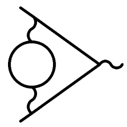
$$
= \mu_0^{2(4-D)} \int \{d^Dk_1\}\{d^Dk_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9^2\mathcal{D}_{10}\mathcal{D}_{14}} \qquad (158)
$$

$$
= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i F^{(3)}_i + \mathcal{O}(\epsilon), \qquad (159)
$$

where:

$$
\begin{aligned}
a^2F^{(3)}_{-2} &= \frac{1}{8}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]H(0,x), \qquad (160) \\
a^2F^{(3)}_{-1} &= -\frac{\zeta(2)}{8}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] - \frac{1}{(1+x)} + \frac{1}{(1+x)^2} + \frac{1}{2}\left[\frac{1}{(1-x)}\right. \\
&\left.-\frac{3}{(1+x)^2} + \frac{3}{(1+x)^3}\right]H(0,x) - \frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right]\left[H(0,0,x)\right.
\end{aligned}
$$

$$
\begin{aligned}
& -H(-1,0,x)+H(1,0,x)\big], \qquad (161)\\
a^2F_0^{(3)} \;=\;& -\frac{\zeta(2)}{2}\left[\frac{1}{(1-x)}-\frac{3}{(1+x)^2}+\frac{2}{(1+x)^3}\right]+\frac{7\zeta(3)}{8}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\\
&+\Bigg\{\left[\frac{1}{(1-x)}-\frac{3}{(1+x)^2}+\frac{2}{(1+x)^3}\right]+\frac{7\zeta(3)}{4}\Bigg[\frac{1}{(1-x)}\\
&-\frac{1}{(1+x)}\Bigg]\Bigg\}H(0,x)-\frac{\zeta(2)}{4}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\big[H(-1,x)\\
&-H(1,x)\big]+\Bigg\{\frac{1}{2}\left[\frac{1}{(1+x)}-\frac{3}{(1+x)^2}+\frac{2}{(1+x)^3}\right]-\frac{1}{4}\Bigg[\frac{1}{(1-x)}\\
&-\frac{1}{(1+x)}\Bigg]\Bigg\}H(0,0,x)-\left[\frac{1}{(1+x)}-\frac{3}{(1+x)^2}+\frac{2}{(1+x)^3}\right]H(-1,0,x)\\
&-\frac{1}{2}\left[\frac{1}{(1-x)}-\frac{1}{(1+x)}\right]\big[H(1,0,x)-H(0,0,0,x)+7H(-1,-1,0,x)\\
&-5H(-1,0,0,x)-3H(-1,1,0,x)-5H(0,-1,0,x)+2H(0,1,0,x)\\
&-3H(1,-1,0,x)+2H(1,0,0,x)+H(1,1,0,x)\big]. \qquad (162)
\end{aligned}
$$



$$
\begin{aligned}
&= \mu_0^{2(4-D)}\int\{d^Dk_1\}\{d^Dk_2\}\frac{1}{\mathcal{D}_1^2\mathcal{D}_7\mathcal{D}_8\mathcal{D}_9\mathcal{D}_{10}} \qquad (163)\\
&= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-1}^{0}\epsilon^i F_i^{(4)}+\mathcal{O}(\epsilon), \qquad (164)
\end{aligned}
$$

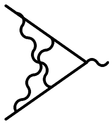
where:

$$
\begin{aligned}
a^2F_{-1}^{(4)} \;=\;& \frac{1}{(1+x)}-\frac{1}{(1+x)^2}-\frac{1}{6}\Bigg[\frac{2}{(1-x)}+\frac{1}{(1+x)}-\frac{9}{(1+x)^2}\\
&+\frac{9}{(1+x)^3}\Bigg]H(0,x), \qquad (165)\\
a^2F_0^{(4)} \;=\;& -\frac{8}{3}\left[\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]+\frac{\zeta(2)}{3}\Bigg[\frac{1}{(1-x)}+\frac{2}{(1+x)}\\
&-\frac{12}{(1+x)^2}+\frac{15}{(1+x)^3}-\frac{6}{(1+x)^4}\Bigg]\\
&+\frac{1}{36}\left[\frac{11}{(1-x)}-\frac{5}{(1+x)}-\frac{18}{(1+x)^2}+\frac{12}{(1+x)^3}-\right]H(0,x)\\
&+\frac{1}{3}\left[\frac{2}{(1-x)}+\frac{1}{(1+x)}-\frac{9}{(1+x)^2}+\frac{6}{(1+x)^3}-\right]H(-1,0,x)\\
&-\frac{1}{3}\left[\frac{1}{(1-x)}-\frac{2}{(1+x)^2}-\frac{1}{(1+x)^3}+\frac{2}{(1+x)^4}\right]H(0,0,x). \qquad (166)
\end{aligned}
$$

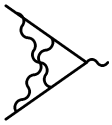
7 Expansion for $Q^2 \gg a$

We list, in this section, the asymptotic expansion of the 6-denominator vertex diagrams given in the previous sections, in order to show their behaviour for momentum transfer larger than the mass.

Putting $y = Q^2/a$, $L = \ln y$ and keeping terms up to the order $(1/y)^5$, we have:



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-1}^{0} \epsilon^i \Big[\sum_{j=2}^{5} \frac{A^{(i)}_{(j)}}{y^j}\Big] \,, \tag{167}$$



$$(k_1 \cdot k_2) \simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{j=1}^{5} \frac{B^{(0)}_{(j)}}{y^j} \,, \tag{168}$$

where:

$$a^2 A^{(-1)}_{(2)} = \zeta(3) - \zeta(2)L - \frac{1}{3}L^3 \,, \tag{169}$$

$$a^2 A^{(-1)}_{(3)} = -2\zeta(2) - 4\zeta(3) + 4\zeta(2)L - 2L^2 + \frac{4}{3}L^3 \,, \tag{170}$$

$$a^2 A^{(-1)}_{(4)} = -1 + 11\zeta(2) + 16\zeta(3) + 4\zeta(2)L - 2L^2 + \frac{4}{3}L^3 \,, \tag{171}$$

$$a^2 A^{(-1)}_{(5)} = \frac{10}{3} - \frac{152}{3}\zeta(2) - 64\zeta(3) + 36L + 64\zeta(2)L - \frac{152}{3}L^2 + \frac{64}{3}L^3, \tag{172}$$

$$a^2 A^{(0)}_{(2)} = \frac{37}{10}\zeta^2(2) - \zeta(3)L - \zeta(2)L^2 + \frac{1}{2}L^4 \,, \tag{173}$$

$$\begin{aligned} a^2 A^{(0)}_{(3)} = {} & -4\zeta(2) - 2\zeta(3) - \frac{74}{5}\zeta^2(2) - 4\zeta(2)L + 4\zeta(3)L - 8L \\ & +4\zeta(2)L^2 - 4L^2 - \frac{88}{3}\zeta(2)L^3 + 4L^3 - 2L^4 \,, \end{aligned} \tag{174}$$

$$\begin{aligned} a^2 A^{(0)}_{(4)} = {} & +18\zeta(2) + \frac{296}{5}\zeta^2(2) + 15\zeta(3) - 20 + 18\zeta(2)L - 16\zeta(3)L \\ & +31L - 16\zeta(2)L^2 + 37L^2 - \frac{70}{3}L^3 + 8L^4 \,, \end{aligned} \tag{175}$$

$$\begin{aligned} a^2 A^{(0)}_{(5)} = {} & -\frac{724}{9}\zeta(2) - \frac{1184}{5}\zeta^2(2) - \frac{248}{3}\zeta(3) + 138 - \frac{208}{3}\zeta(2)L \\ & +64\zeta(3)L - \frac{2204}{27}L + 64\zeta(2)L^2 - \frac{1948}{9}L^2 + 112L^3 \\ & -32L^4 \,, \end{aligned} \tag{176}$$

$$a B^{(0)}_{(1)} = -\frac{1}{5}\zeta^2(2) - 2\zeta(3)L - \frac{1}{24}L^4 \,, \tag{177}$$

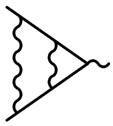
$$\begin{aligned} a B^{(0)}_{(2)} = {} & -2 - 2\zeta(2) + \frac{2}{5}\zeta^2(2) - 4\zeta(3) - 2 + 4\zeta(2)L + 4\zeta(3)L - 2L \\ & +\frac{1}{3}L^3 - \frac{1}{4}L^4 \,, \end{aligned} \tag{178}$$

$$a B^{(0)}_{(3)} = \frac{31}{8} + \frac{37}{2}\zeta(2) - \frac{6}{5}\zeta^2(2) + 14\zeta(3) + \frac{33}{4}L - 14\zeta(2)L$$

$$-12\zeta(3)L + \frac{7}{2}L^2 - \frac{7}{6}L^3 - \frac{1}{4}L^4\,, \tag{179}$$

$$aB^{(0)}_{(4)} = -\frac{2195}{324} - \frac{767}{9}\zeta(2) + 4\zeta^2(2) - \frac{148}{3}\zeta(3) + \frac{148}{3}\zeta(2)L + 40\zeta(3)L$$
$$-\frac{1237}{54}L - 18L^2 + \frac{37}{9}L^3 + \frac{5}{6}L^4\,, \tag{180}$$

$$aB^{(0)}_{(5)} = \frac{146447}{10368} + \frac{25325}{72}\zeta(2) - 14\zeta^2(2) + \frac{533}{3}\zeta(3) + \frac{52955}{864}L$$
$$-\frac{533}{3}\zeta(2)L - 140\zeta(3)L + \frac{615}{8}L^2 - \frac{533}{36}L^3 - \frac{35}{12}L^4\,. \tag{181}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i \Big[\sum_{j=2}^{5} \frac{C^{(i)}_{(j)}}{y^j}\Big]\,, \tag{182}$$

where:

$$a^2C^{(-2)}_{(2)} = \frac{L^2}{2}\,, \tag{183}$$

$$a^2C^{(-2)}_{(3)} = 2L - 2L^2\,, \tag{184}$$

$$a^2C^{(-2)}_{(4)} = 2 - 11L + 8L^2\,, \tag{185}$$

$$a^2C^{(-2)}_{(5)} = -14 + \frac{152}{3}L - 32L^2\,, \tag{186}$$

$$a^2C^{(-1)}_{(2)} = -\zeta(3) + 2\zeta(2)L - \frac{1}{6}L^3\,, \tag{187}$$

$$a^2C^{(-1)}_{(3)} = 4\zeta(2) + 4\zeta(3) - 2L - 8\zeta(2)L - 3L^2 + \frac{2}{3}L^3\,, \tag{188}$$

$$a^2C^{(-1)}_{(4)} = -3 - 22\zeta(2) - 16\zeta(3) + \frac{7}{2}L + 32\zeta(2)L + \frac{37}{2}L^2 - \frac{8}{3}L^3\,, \tag{189}$$

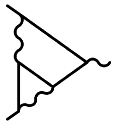
$$a^2C^{(-1)}_{(5)} = \frac{47}{3} + \frac{304}{3}\zeta(2) + 64\zeta(3) + \frac{70}{9}L - 128\zeta(2)L - 92L^2$$
$$+\frac{32}{3}L^3\,, \tag{190}$$

$$a^2C^{(0)}_{(2)} = \frac{12}{5}\zeta^2(2) + \zeta(3)L + \frac{7}{2}\zeta(2)L^2 + \frac{5}{24}L^4\,, \tag{191}$$

$$a^2C^{(0)}_{(3)} = -8 - 2\zeta(2) - \frac{48}{5}\zeta^2(2) + 6\zeta(3) - 2L + 6\zeta(2)L - 4\zeta(3)L$$
$$-14\zeta(2)L^2 + L^2 + \frac{7}{3}L^3 - \frac{5}{6}L^4, \tag{192}$$

$$a^2C^{(0)}_{(4)} = 48 + \frac{23}{2}\zeta(2) + \frac{192}{5}\zeta^2(2) - 37\zeta(3) + \frac{89}{4}L - 25\zeta(2)L + 16\zeta(3)L$$
$$+\frac{19}{4}L^2 + 56\zeta(2)L^2 - \frac{27}{2}L^3 + \frac{10}{3}L^4\,, \tag{193}$$

$$a^2C^{(0)}_{(5)} = -\frac{36349}{162} - \frac{362}{9}\zeta(2) - \frac{768}{5}\zeta^2(2) + 184\zeta(3) - \frac{3395}{27}L + 88\zeta(2)L$$
$$-64\zeta(3)L - \frac{503}{9}L^2 - 224\zeta(2)L^2 + \frac{580}{9}L^3 - \frac{40}{3}L^4\,. \tag{194}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i \Big[\sum_{j=1}^{5} \frac{E_{(j)}^{(i)}}{y^j}\Big]\,, \tag{195}$$

where:

$$a^2 E_{(1)}^{(-2)} = +\frac{L}{4}\,, \tag{196}$$

$$a^2 E_{(2)}^{(-2)} = \frac{1}{2} - \frac{L}{2}\,, \tag{197}$$

$$a^2 E_{(3)}^{(-2)} = -\frac{7}{4} + \frac{3}{2}L\,, \tag{198}$$

$$a^2 E_{(4)}^{(-2)} = \frac{37}{6} - 5L\,, \tag{199}$$

$$a^2 E_{(5)}^{(-2)} = -\frac{533}{24} + \frac{35}{2}L\,, \tag{200}$$

$$a^2 E_{(1)}^{(-1)} = \frac{1}{4}\zeta(2) - \frac{1}{2}L + \frac{1}{4}L^2, \tag{201}$$

$$a^2 E_{(2)}^{(-1)} = -1 - \frac{\zeta(2)}{2} + 2L - \frac{L^2}{2}\,, \tag{202}$$

$$a^2 E_{(3)}^{(-1)} = \frac{17}{4} + \frac{3}{2}\zeta(2) - 7L + \frac{3}{2}L^2, \tag{203}$$

$$a^2 E_{(4)}^{(-1)} = -\frac{101}{6} - 5\zeta(2) + \frac{76}{3}L - 5L^2\,, \tag{204}$$

$$a^2 E_{(5)}^{(-1)} = \frac{3157}{48} + \frac{35}{2}\zeta(2) - \frac{281}{3}L + \frac{35}{2}L^2, \tag{205}$$

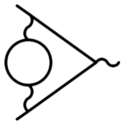
$$\begin{aligned} a^2 E_{(1)}^{(0)} &= -3\zeta(2)\ln(2) - \frac{1}{2}\zeta(2) - \frac{5}{2}\zeta(3) + L + \frac{7}{2}\zeta(2)L - \frac{1}{2}L^2 \\ &\quad + \frac{1}{6}L^3, \end{aligned} \tag{206}$$

$$\begin{aligned} a^2 E_{(2)}^{(0)} &= 5 + 12\zeta(2)\ln(2) + 11\zeta(2) + \frac{13}{2}\zeta(3) - 3L - 10\zeta(2)L + \frac{3}{2}L^2 \\ &\quad - \frac{5}{6}L^3, \end{aligned} \tag{207}$$

$$\begin{aligned} a^2 E_{(3)}^{(0)} &= -\frac{161}{8} - 48\zeta(2)\ln(2) - \frac{107}{2}\zeta(2) - \frac{45}{2}\zeta(3) + \frac{31}{4}L + 36\zeta(2)L \\ &\quad - \frac{31}{4}L^2 + \frac{7}{2}L^3, \end{aligned} \tag{208}$$

$$\begin{aligned} a^2 E_{(4)}^{(0)} &= \frac{2677}{36} + 192\zeta(2)\ln(2) + \frac{715}{3}\zeta(2) + 83\zeta(3) - \frac{221}{9}L + 136\zeta(2)L \\ &\quad + \frac{113}{3}L^2 - \frac{43}{3}L^3, \end{aligned} \tag{209}$$

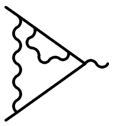
$$\begin{aligned} a^2 E_{(5)}^{(0)} &= -\frac{156965}{576} - 768\zeta(2)\ln(2) - \frac{12323}{12}\zeta(2) - \frac{629}{2}\zeta(3) + \frac{13319}{144}L \\ &\quad + 524\zeta(2)L - \frac{1387}{8}L^2 + \frac{349}{6}L^3. \end{aligned} \tag{210}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-1}^{0} \epsilon^i \Big[\sum_{j=1}^{5} \frac{F^{(i)}_{(j)}}{y^j}\Big]\,, \tag{211}$$

where:

$$
\begin{aligned}
a^2 F^{(-1)}_{(1)} &= 1 + \frac{1}{6}L, && (212)\\
a^2 F^{(-1)}_{(2)} &= -\frac{11}{3} + \frac{5}{3}L, && (213)\\
a^2 F^{(-1)}_{(3)} &= \frac{113}{6} - 11L, && (214)\\
a^2 F^{(-1)}_{(4)} &= -\frac{809}{9} + \frac{170}{3}L, && (215)\\
a^2 F^{(-1)}_{(5)} &= \frac{14779}{36} - \frac{805}{3}L, && (216)\\
a^2 F^{(0)}_{(1)} &= -\frac{8}{3} + \frac{2}{3}\zeta(2) - \frac{4}{9}L, && (217)\\
a^2 F^{(0)}_{(2)} &= \frac{85}{9} - \frac{7}{3}\zeta(2) - \frac{1}{9}L - \frac{3}{2}L^2, && (218)\\
a^2 F^{(0)}_{(3)} &= -\frac{1589}{36} + 13\zeta(2) - \frac{43}{6}L + \frac{19}{2}L^2, && (219)\\
a^2 F^{(0)}_{(4)} &= \frac{1123}{6} - \frac{214}{3}\zeta(2) + \frac{574}{9}L - \frac{149}{3}L^2, && (220)\\
a^2 F^{(0)}_{(4)} &= -\frac{109397}{144} + \frac{1115}{3}\zeta(2) - \frac{14345}{36}L + \frac{1445}{6}L^2, && (221)
\end{aligned}
$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i \Big[\sum_{j=1}^{5} \frac{G^{(i)}_{(j)}}{y^j}\Big]\,, \tag{222}$$

where:

$$
\begin{aligned}
a^2 G^{(-2)}_{(1)} &= -\frac{1}{4}L, && (223)\\
a^2 G^{(-2)}_{(2)} &= -\frac{1}{2} + \frac{1}{2}L, && (224)\\
a^2 G^{(-2)}_{(3)} &= \frac{7}{4} - \frac{3}{2}L, && (225)\\
a^2 G^{(-2)}_{(4)} &= -\frac{37}{6} + 5L, && (226)\\
a^2 G^{(-2)}_{(5)} &= \frac{533}{24} - \frac{35}{2}L, && (227)\\
a^2 G^{(-1)}_{(1)} &= -1 - \frac{1}{4}\zeta(2) - \frac{1}{2}L - \frac{1}{4}L^2, && (228)\\
a^2 G^{(-1)}_{(2)} &= 3 + \frac{1}{2}\zeta(2) - 2L + \frac{1}{2}L^2, && (229)
\end{aligned}
$$

$$a^2 G^{(-1)}_{(3)} = -\frac{69}{4} - \frac{3}{2}\zeta(2) + 13L - \frac{3}{2}L^2, \quad (230)$$

$$a^2 G^{(-1)}_{(4)} = \frac{517}{6} + 5\zeta(2) - \frac{196}{3}L + 5L^2, \quad (231)$$

$$a^2 G^{(-1)}_{(5)} = -\frac{19309}{48} - \frac{35}{2}\zeta(2) + \frac{911}{3}L - \frac{35}{2}L^2, \quad (232)$$

$$a^2 G^{(0)}_{(1)} = -\frac{1}{2}\zeta(2) + \frac{7}{4}\zeta(3) - L - \frac{7}{2}\zeta(2)L - \frac{1}{2}L^2 + \frac{1}{6}L^3, \quad (233)$$

$$a^2 G^{(0)}_{(2)} = -35 - 8\zeta(2) - \frac{7}{2}\zeta(3) - 4L + 7\zeta(2)L + \frac{5}{2}L^2 + \frac{1}{3}L^3, \quad (234)$$

$$a^2 G^{(0)}_{(3)} = \frac{127}{8} + 34\zeta(2) + \frac{21}{2}\zeta(3) + 39L - 21\zeta(2)L - \frac{53}{4}L^2 - L^3, \quad (235)$$

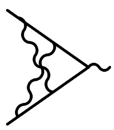
$$a^2 G^{(0)}_{(4)} = -\frac{1183}{36} - \frac{418}{3}\zeta(2) - 35\zeta(3) - \frac{701}{3}L + 70\zeta(2)L + \frac{367}{6}L^2 + \frac{10}{3}L^3, \quad (236)$$

$$a^2 G^{(0)}_{(5)} = -\frac{3397}{576} + \frac{3421}{6}\zeta(2) + \frac{245}{2}\zeta(3) + \frac{14389}{12}L - 245\zeta(2)L - \frac{6443}{24}L^2 + \frac{35}{3}L^3, \quad (237)$$

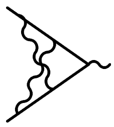
8 Expansion for $Q^2 \ll a$

We list, in this section, the expansion of the vertex diagrams around $Q^2 = 0$.

Putting $y = Q^2/a$, and keeping terms up to the order y^3, we have:



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-1}^{0} \epsilon^i \Big[\sum_{j=0}^{2} A^{(i)}_{(j)} y^j\Big], \quad (238)$$



$$(k_1 \cdot k_2) \simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{j=0}^{2} B^{(0)}_{(j)} y^j, \quad (239)$$

where:

$$a^2 A^{(-1)}_{(0)} = -\frac{1}{4}, \quad (240)$$

$$a^2 A^{(-1)}_{(1)} = \frac{5}{72}, \quad (241)$$

$$a^2 A^{(-1)}_{(2)} = -\frac{377}{21600}, \quad (242)$$

$$a^2 A^{(0)}_{(0)} = 1 - \frac{3}{4}\zeta(2), \quad (243)$$

$$a^2 A^{(0)}_{(1)} = -\frac{49}{216} + \frac{1}{4}\zeta(2), \quad (244)$$

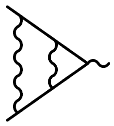
$$a^2 A^{(0)}_{(2)} = \frac{16717}{324000} - \frac{17}{240}\zeta(2), \quad (245)$$

$$aB^{(0)}_{(0)} = -2\zeta(2) - \frac{3}{4}\zeta(3) + 3\zeta(2)\ln 2\,, \tag{246}$$

$$aB^{(0)}_{(1)} = -\frac{7}{36} + \frac{29}{72}\zeta(2) + \frac{1}{8}\zeta(3) - \frac{1}{2}\zeta(2)\ln 2\,, \tag{247}$$

$$aB^{(0)}_{(2)} = \frac{37}{720} - \frac{1247}{14400}\zeta(2) - \frac{1}{40}\zeta(3) + \frac{1}{10}\zeta(2)\ln 2\,, \tag{248}$$

$$\tag{249}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i \Big[\sum_{j=0}^{2} C^{(i)}_{(j)} y^j\Big]\,, \tag{250}$$

where:

$$a^2C^{(-2)}_{(0)} = \frac{1}{8}\,, \tag{251}$$

$$a^2C^{(-2)}_{(1)} = -\frac{1}{24}\,, \tag{252}$$

$$a^2C^{(-2)}_{(2)} = \frac{17}{1440}\,, \tag{253}$$

$$a^2C^{(-1)}_{(0)} = \frac{1}{4}\,, \tag{254}$$

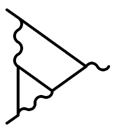
$$a^2C^{(-1)}_{(1)} = -\frac{1}{9}\,, \tag{255}$$

$$a^2C^{(-1)}_{(2)} = \frac{797}{21600}\,, \tag{256}$$

$$a^2C^{(0)}_{(0)} = -\frac{1}{2} + \frac{3}{2}\zeta(2)\,, \tag{257}$$

$$a^2C^{(0)}_{(1)} = \frac{17}{108} - \frac{1}{2}\zeta(2)\,, \tag{258}$$

$$a^2C^{(0)}_{(2)} = -\frac{2993}{81000} + \frac{17}{120}\zeta(2)\,,\,. \tag{259}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i \Big[\sum_{j=0}^{2} E^{(i)}_{(j)} y^j\Big]\,, \tag{260}$$

where:

$$a^2E^{(-2)}_{(0)} = \frac{1}{8}\,, \tag{261}$$

$$a^2E^{(-2)}_{(1)} = -\frac{1}{48}\,, \tag{262}$$

$$a^2E^{(-2)}_{(2)} = \frac{1}{240}\,, \tag{263}$$

$$a^2E^{(-1)}_{(0)} = 0\,, \tag{264}$$

$$a^2E^{(-1)}_{(1)} = \frac{1}{36}\,, \tag{265}$$

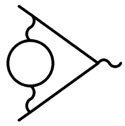
$$a^2E^{(-1)}_{(2)} = -\frac{29}{3600}\,, \tag{266}$$

$$a^2E^{(0)}_{(0)} = -\frac{1}{2}+\frac{3}{4}\zeta(2)\,, \tag{267}$$

$$a^2E^{(0)}_{(1)} = \frac{67}{432}-\frac{1}{32}\zeta(2)\,, \tag{268}$$

$$a^2E^{(0)}_{(2)} = -\frac{3407}{108000}-\frac{13}{1280}\zeta(2)\,, \tag{269}$$

$$\tag{270}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-1}^{0}\epsilon^i\Big[\sum_{j=0}^{2}F^{(i)}_{(j)}y^j\Big]\,, \tag{271}$$

where:

$$a^2F^{(-1)}_{(0)} = \frac{7}{12}\,, \tag{272}$$

$$a^2F^{(-1)}_{(1)} = -\frac{13}{72}\,, \tag{273}$$

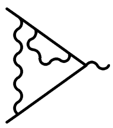
$$a^2F^{(-1)}_{(2)} = \frac{19}{360}\,, \tag{274}$$

$$a^2F^{0}_{(0)} = -\frac{35}{36}\,, \tag{275}$$

$$a^2F^{0}_{(1)} = \frac{79}{432}+\frac{1}{32}\zeta(2)\,, \tag{276}$$

$$a^2F^{0}_{(2)} = -\frac{61}{2160}-\frac{1}{64}\zeta(2)\,, \tag{277}$$

$$\tag{278}$$



$$\simeq \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon}\sum_{i=-2}^{0}\epsilon^i\Big[\sum_{j=0}^{2}G^{(i)}_{(j)}y^j\Big]\,, \tag{279}$$

where:

$$a^2G^{(-2)}_{(0)} = -\frac{1}{8}\,, \tag{280}$$

$$a^2G^{(-2)}_{(1)} = \frac{1}{48}\,, \tag{281}$$

$$a^2G^{(-2)}_{(2)} = -\frac{1}{240}\,, \tag{282}$$

$$a^2G^{(-1)}_{(0)} = -1\,, \tag{283}$$

$$a^2G^{(-1)}_{(1)} = \frac{2}{9}\,, \tag{284}$$

$$
\begin{aligned}
a^2 G^{(-1)}_{(2)} &= -\frac{211}{3600}\,, &(285)\\
a^2 G^{(0)}_{(0)} &= -1 - \frac{3}{2}\zeta(2)\,, &(286)\\
a^2 G^{(0)}_{(1)} &= \frac{19}{216} + \frac{1}{4}\zeta(2)\,, &(287)\\
a^2 G^{(0)}_{(2)} &= -\frac{3613}{108000} - \frac{1}{20}\zeta(2)\,. &(288)
\end{aligned}
$$

9 Summary

We have carried out a complete investigation of the scalar integrals associated to all the QED 2-loop vertex graphs, for on-shell electrons and arbitrary momentum transfer $t = -Q^2$ in the D-continuous regularization scheme. After identifying all the occurring Master Integrals (MI's), we have written the linear, non-homogeneous differential equations in Q^2 satisfied by the MI's, expanded them in $\epsilon = (4-D)/2$ and solved the equations by means of the method of the variation of the constants of Euler. The method requires the solution of the associated homogeneous equations. It turns out that all the associated homogenous equations are trivial, or became trivial after a suitable choice of the MI's for the graph topologies involving more than a single MI; typically one had to solve a first order homogeneous differential equation with simple rational coefficients.

The repeated integrations implied by Euler's method are immediately performed, in close analytic form, in terms of Harmonic Polylogarithms of increasing weight; the maximum weight occurring in the results presented in this paper was 4 (as in the case of the zeroth order term in ϵ of the double cross topology). By further iterations of the approach, one could almost mechanically obtain any additional term in the ϵ expansion of the MI's.

The explicit analytic evaluation of the QED vertex form factors in terms of the MI's will be presented elsewhere.

10 Acknowledgement

We are grateful to J. Vermaseren for his kind assistance in the use of the algebra manipulating program `FORM` [10], by which all our calculations were carried out.

We thank T. Gehrmann, R. Heinesch and Y. Schroder for pointing out several misprints in a preliminary version of the manuscript.

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A Propagators

We list here the denominators of the integral expressions appeared in the paper.

$$\mathcal{D}_1 = k_1^2\,, \tag{289}$$
$$\mathcal{D}_2 = k_2^2\,, \tag{290}$$
$$\mathcal{D}_3 = (k_1+k_2)^2\,, \tag{291}$$
$$\mathcal{D}_4 = (p_1-k_1)^2\,, \tag{292}$$
$$\mathcal{D}_5 = (p_2-k_2)^2\,, \tag{293}$$
$$\mathcal{D}_6 = [k_1^2+a]\,, \tag{294}$$
$$\mathcal{D}_7 = [k_2^2+a]\,, \tag{295}$$
$$\mathcal{D}_8 = [(k_1+k_2)^2+a]\,, \tag{296}$$
$$\mathcal{D}_9 = [(p_1-k_1)^2+a]\,, \tag{297}$$
$$\mathcal{D}_{10} = [(p_2+k_1)^2+a]\,, \tag{298}$$
$$\mathcal{D}_{11} = [(p_2-k_2)^2+a]\,, \tag{299}$$
$$\mathcal{D}_{12} = [(p_1+p_2-k_1)^2+a]\,, \tag{300}$$
$$\mathcal{D}_{13} = [(p_1+p_2-k_2)^2+a]\,, \tag{301}$$
$$\mathcal{D}_{14} = [(p_1-k_1-k_2)^2+a]\,, \tag{302}$$
$$\mathcal{D}_{15} = [(p_2+k_1+k_2)^2+a]\,, \tag{303}$$
$$\mathcal{D}_{16} = [(p_1+p_2-k_1-k_2)^2+a]\,. \tag{304}$$

B 1-loop ingredients

We recall in this appendix some useful results about 1-loop diagrams, fundamental ingredients for the 2-loop calculations, obtained with the method of differential equations. They appear in the 2-loop integrals which factorize in the products of two 1-loop integral; due to the possible presence of extra powers of $1/\epsilon$ in their coefficients, we give the results of the ϵ expansion up to the second order in ϵ.

The case of the massive bubble is exaustively examined. The differential equation is presented and solved, as usual in the $\epsilon \to 0$ expansion.

B.0.1 Tadpole

$$= \mu_0^{(4-D)} \int \{d^D k\} \frac{1}{(k^2+a)}$$
$$= \left(\frac{a}{\mu_0^2}\right)^{-\epsilon} \sum_{i=-1}^{2} \epsilon^i A_i + \mathcal{O}\left(\epsilon^3\right)\,, \tag{305}$$

where:

$$\frac{A_{-1}}{a} = -1\,, \tag{306}$$

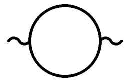
$$\frac{A_0}{a} = -1\,, \tag{307}$$

$$\frac{A_1}{a} = -1\,, \tag{308}$$

$$\frac{A_2}{a} = -1\,. \tag{309}$$

B.0.2 Fully massive bubble

The topology under consideration has one MI. We choose the scalar integral itself:

$F(\epsilon, a, Q^2) =$ 

$$= \mu_0^{(4-D)} \int \{d^D k\} \frac{1}{[k^2 + a]\,[(Q-k)^2 + a]}\,, \tag{310}$$

The corresponding first-order linear differential equation is the following:

$$\frac{dF(\epsilon, a, Q^2)}{dQ^2} = -\frac{1}{2}\left[\frac{1}{Q^2} - \frac{(1-2\epsilon)}{(Q^2+4a)}\right] F(\epsilon, a, Q^2) - \frac{(1-\epsilon)}{2a}\left[\frac{1}{Q^2} - \frac{1}{(Q^2+4a)}\right] T(\epsilon, a), \tag{311}$$

where $T(\epsilon, a)$ is the Tadpole.

As in the cases previously discussed, we use our knowledge on the analytical behaviour of the solution in order to find the initial condition. In fact, Eq. (311) shows two possible singularities for the function $F(\epsilon, a, Q^2)$, for $Q^2 = 0$ and for $Q^2 = -4a$. Only the second, nevertheless, is indeed a singularity for F, corresponding to the physical threshold. Multiplying Eq. (311) for Q^2 and taking the limit $Q^2 \to 0$, we obtain:

$$F(\epsilon, a, Q^2 = 0) = -\frac{(1-\epsilon)}{a} T(\epsilon, a)\,. \tag{312}$$

We look for a solution of Eq. (311), with initial condition (312), expanded in Laurent series around $\epsilon = 0$:

$$F(\epsilon, a, Q^2) = \sum_{i=-1}^{2} \epsilon^i F_i(a, Q^2) + \mathcal{O}(\epsilon^3)\,. \tag{313}$$

The associated homogeneous equation at $\epsilon = 0$ is

$$\frac{df(a,y)}{dy} = -\frac{1}{2}\left[\frac{1}{y} - \frac{1}{(y+4a)}\right] f(a,y)\,, \tag{314}$$

which has the following solution:

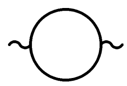
$$f(\epsilon, a, y) = k\sqrt{1 + \frac{4a}{y}}\,. \tag{315}$$

We can find the solution of the non-homogeneous equation, order by order in ϵ, by means of the Euler's method of the variation of the constant k. We have:

$$F_i(a,Q^2) = \sqrt{1+\frac{4a}{Q^2}}\Bigg\{\int^{Q^2}\frac{dy}{\sqrt{1+\frac{4a}{y}}}\Bigg[\frac{1}{(y+4a)}F_{i-1}(a,y)-\frac{1}{2a}\Bigg(\frac{1}{y} -\frac{1}{(y+4a)}\Bigg)[A_i-A_{i-1}]\Bigg]+k_i\Bigg\}, \quad (316)$$

where the coefficients A_i are those of Eqs. (306–309). The determination of the constants of integration k_i is made by imposing that the solution satisfies Eq. (312).

In terms of the variable x, defined in Eq. (25), the solution reads:



$$= \mu_0^{(4-D)}\int\{d^Dk\}\frac{1}{[k^2+a]\,[(Q-k)^2+a]}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-\epsilon}\sum_{i=-1}^{2}\epsilon^i B_i+\mathcal{O}\left(\epsilon^3\right), \quad (317)$$

where:

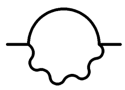
$$B_{-1} = 1, \quad (318)$$

$$B_0 = 2-2\left[\frac{1}{2}-\frac{1}{(1-x)}\right]H(0,x), \quad (319)$$

$$B_1 = 4-4\left[\frac{1}{2}-\frac{1}{(1-x)}\right]\Bigg\{-\frac{\zeta(2)}{2}+H(0,x)+\frac{1}{2}H(0,0,x) -H(-1,0,x)\Bigg\}, \quad (320)$$

$$B_2 = 4+4\left[\frac{1}{2}-\frac{1}{(1-x)}\right]\Bigg\{\frac{\zeta(2)}{2}\Big[1+\frac{1}{2}H(0,x)-H(-1,x)\Big]+\frac{\zeta(3)}{2} -\frac{1}{2}\Big[H(0,x)+\frac{1}{2}H(0,0,x)-H(-1,0,x)+\frac{1}{4}H(0,0,0,x) -\frac{1}{2}H(-1,0,0,x)-H(0,-1,0,x)-H(-1,-1,0,x)\Big]\Bigg\}. \quad (321)$$

B.0.3 Bubble on the mass-shell



$$= \mu_0^{(4-D)}\int\{d^Dk\}\frac{1}{k^2\,[(Q-k)^2+a]}\Bigg|_{Q^2=-a}$$

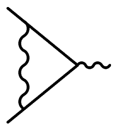
$$= -\frac{(1-\epsilon)}{a(1-2\epsilon)}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-\epsilon}\sum_{i=-1}^{2}\epsilon^i E_i+\mathcal{O}\left(\epsilon^3\right), \quad (322)$$

where:

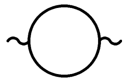
$$
\begin{aligned}
E_{-1} &= 1\,, && (323)\\
E_0 &= 2\,, && (324)\\
E_1 &= 4\,, && (325)\\
E_2 &= 8\,. && (326)
\end{aligned}
$$

B.0.4 Scalar Vertex at 1 loop

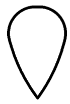


$$
= \mu_0^{(4-D)} \int \frac{d^D k}{(2\pi)^{(D-2)}} \frac{1}{k^2\,[(p_1-k)^2+a]\,[(p_2+k)^2+a]}
$$

$$
= \frac{(1-2\epsilon)}{\epsilon}\,\frac{1}{(Q^2+4a)}
$$



$$
- \frac{(1-\epsilon)}{a\,\epsilon}\,\frac{1}{(Q^2+4a)}
$$



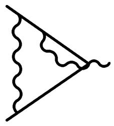
$$
= \left(\frac{a}{\mu_0^2}\right)^{-\epsilon} \sum_{i=-1}^{2} \epsilon^i F_i + \mathcal{O}\left(\epsilon^3\right), \qquad (327)
$$

where:

$$
\begin{aligned}
aF_{-1} &= -\frac{1}{2}\left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right] H(0,x)\,, && (328)\\
aF_0 &= \left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\left\{\frac{\zeta(2)}{2} - \frac{1}{2}H(0,0,x) + H(-1,0,x)\right\}, && (329)\\
aF_1 &= -2\left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\left\{-\frac{\zeta(3)}{2} - \frac{\zeta(2)}{2}\left[\frac{1}{2}H(0,x) - H(-1,x)\right]\right.\\
&\quad +\frac{1}{4}H(0,0,0,x) + H(-1,-1,0,x) - \frac{1}{2}H(-1,0,0,x)\\
&\quad \left.-\frac{1}{2}H(0,-1,0,x)\right\}, && (330)\\
aF_2 &= \left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\left\{-\frac{9\zeta^2(2)}{20} + \zeta(3)\,[H(0,x) - 2H(-1,x)]\right.\\
&\quad +\zeta(2)\left[\frac{1}{2}H(0,0,x) + 2H(-1,-1,x) - H(-1,0,x) - H(0,-1,x)\right]\\
&\quad -\frac{1}{2}H(0,0,0,0,x) + 4H(-1,-1,-1,0,x) - 2H(-1,-1,0,0,x)\\
&\quad -2H(-1,0,-1,0,x) - 2H(0,-1,-1,0,x) + 4H(-1,0,0,0,x)\\
&\quad \left.+H(0,-1,0,0,x) + H(0,0,-1,0,x)\right\}. && (331)
\end{aligned}
$$

C Reducible 2-loop diagrams

In this appendix we give the expressions of the reducible diagrams of Figs. (4–6).



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_9\mathcal{D}_{10}\mathcal{D}_{14}} \quad (332)$$

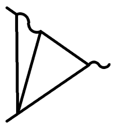
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(1)} + \mathcal{O}(\epsilon)\,, \quad (333)$$

where:

$$aE_{-2}^{(1)} = \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,x)\,, \quad (334)$$

$$aE_{-1}^{(1)} = -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) - 2H(0,x) - H(0,0,x) \\ +2H(-1,0,x)] \quad (335)$$

$$aE_{0}^{(1)} = -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [2\zeta(2) + 2\zeta(3) - (4+3\zeta(2))H(0,x) \\ -2\zeta(2)H(-1,x) - 2H(0,0,x) + 4H(-1,0,x) - 5H(0,0,0,x) \\ +2H(0,-1,0,x) + 2H(-1,0,0,x) - 4H(-1,-1,0,x)]\,. \quad (336)$$

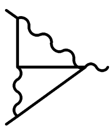


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_4\mathcal{D}_6\mathcal{D}_7\mathcal{D}_{14}\mathcal{D}_{15}} \quad (337)$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} E_0^{(2)} + \mathcal{O}(\epsilon)\,, \quad (338)$$

where:

$$aE_0^{(2)} = -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \left\{\frac{\zeta^2(2)}{5} + 2\zeta(3)H(0,x) - \zeta(2)H(0,0,x) \\ -2\zeta(2)H(1,0,x) - 4H(0,0,-1,0,x) + 2H(0,0,1,0,x) \\ -4H(0,1,0,0,x) - 2H(1,0,0,0,x)\right\}. \quad (339)$$

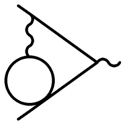


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2\mathcal{D}_4\mathcal{D}_6\mathcal{D}_8\mathcal{D}_{11}} \quad (340)$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} E_0^{(3)} + \mathcal{O}(\epsilon)\,, \quad (341)$$

where:

$$aE_0^{(3)} = \left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] \left\{\frac{27\zeta^2(2)}{10} + \zeta(3)H(0,x) + 3\zeta(2)H(0,0,x) \\ -6\zeta(2)H(0,-1,x) + 2H(0,0,0,0,x) + 2H(0,1,0,0,x) \\ -2H(0,-1,0,0,x)\right\}. \quad (342)$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_7\mathcal{D}_8\mathcal{D}_9\mathcal{D}_{10}} \tag{343}$$

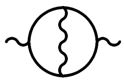
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(4)} + \mathcal{O}(\epsilon)\,, \tag{344}$$

where:

$$aE_{-2}^{(4)} = \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,x)\,, \tag{345}$$

$$aE_{-1}^{(4)} = -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) - H(0,0,x) + 2H(-1,0,x)] \tag{346}$$

$$\begin{aligned} aE_0^{(4)} = {} & \frac{1}{(1+x)}\left[1 - \frac{1}{(1+x)}\right][6\zeta(2) + 2H(0,0,x)] - \left[\frac{1}{(1-x)}\right. \\ & \left. -\frac{1}{(1+x)}\right][\zeta(3) - 2H(0,x) - \zeta(2)H(-1,x) - H(0,0,0,x) \\ & -2H(-1,-1,0,x) + H(-1,0,0,x) + H(0,-1,0,x)]\,. \end{aligned} \tag{347}$$

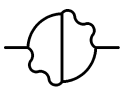


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2\mathcal{D}_6\mathcal{D}_8\mathcal{D}_{12}\mathcal{D}_{16}} \tag{348}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} E_0^{(5)} + \mathcal{O}(\epsilon)\,, \tag{349}$$

where:

$$\begin{aligned} aE_0^{(5)} = {} & \frac{2}{(1-x)}\left[1 - \frac{1}{(1-x)}\right][3\zeta(3) + 4H(-1,0,0,x) - 4H(0,-1,0,x) \\ & +2H(0,1,0,x) - 2H(1,0,0,x)]\,. \end{aligned} \tag{350}$$

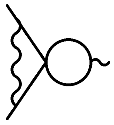


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_5\mathcal{D}_7\mathcal{D}_8\mathcal{D}_{10}} \tag{351}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} E_0^{(6)} + \mathcal{O}(\epsilon)\,, \tag{352}$$

where:

$$aE_0^{(6)} = 6\zeta(2)\ln 2 - \frac{3}{2}\zeta(3)\,. \tag{353}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_7 \mathcal{D}_9 \mathcal{D}_{10} \mathcal{D}_{13}} \tag{354}$$

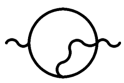
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(7)} + \mathcal{O}(\epsilon)\,, \tag{355}$$

where:

$$aE_{-2}^{(7)} = \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,x)\,, \tag{356}$$

$$\begin{aligned} aE_{-1}^{(7)} = & -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) - 2H(0,x) - H(0,0,x) \\ & +2H(-1,0,x)] + \frac{2}{(1-x)}\left[1 - \frac{1}{(1-x)}\right] H(0,0,x)\,, \end{aligned} \tag{357}$$

$$\begin{aligned} aE_{0}^{(7)} = & -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [2\zeta(2) + 2\zeta(3) - (4-\zeta(2))H(0,x) \\ & -2\zeta(2)H(-1,x) + 2H(0,0,x) - 4H(-1,0,x) - H(0,0,0,x) \\ & -4H(-1,-1,0,x) + 2H(0,-1,0,x) + 2H(-1,0,0,x) \\ & +\frac{2}{(1-x)}\left[1 - \frac{1}{(1-x)}\right] [\zeta(2)H(0,x) - 2H(0,0,x) + 3H(0,0,0,x) \\ & -4H(-1,0,0,x) - 2H(0,-1,0,x)]\,. \end{aligned} \tag{358}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{12} \mathcal{D}_{16}} \tag{359}$$

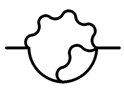
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(8)} + \mathcal{O}(\epsilon)\,, \tag{360}$$

where:

$$E_{-2}^{(8)} = \frac{1}{2}\,, \tag{361}$$

$$E_{-1}^{(8)} = \frac{5}{2} - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, \tag{362}$$

$$\begin{aligned} E_{0}^{(8)} = & \left[1 - \frac{2}{(1-x)}\right] [\zeta(2) - 5H(0,x) + 2H(-1,0,x)] \\ & +\frac{2}{(1-x)^2} H(0,0,x)\,. \end{aligned} \tag{363}$$

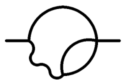


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_{10} \mathcal{D}_{15}} \tag{364}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} E_0^{(9)} + \mathcal{O}\left(\epsilon\right), \tag{365}$$

where:

$$E_0^{(9)} = \frac{19}{2} - 2\zeta(2)\,. \tag{366}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2\mathcal{D}_6\mathcal{D}_8\mathcal{D}_{11}} \tag{367}$$

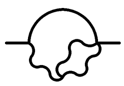
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(10)} + \mathcal{O}\left(\epsilon\right), \tag{368}$$

where:

$$E_{-2}^{(10)} = \frac{1}{2}\,, \tag{369}$$

$$E_{-1}^{(10)} = \frac{5}{2}\,, \tag{370}$$

$$E_0^{(10)} = \frac{19}{2} - 4\zeta(2)\,, \tag{371}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_{10}} \tag{372}$$

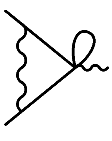
$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(11)} + \mathcal{O}\left(\epsilon\right), \tag{373}$$

where:

$$E_{-2}^{(11)} = \frac{1}{2}\,, \tag{374}$$

$$E_{-1}^{(11)} = \frac{5}{2}\,, \tag{375}$$

$$E_0^{(11)} = \frac{19}{2} - 4\zeta(2)\,, \tag{376}$$

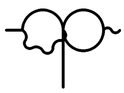


$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_7\mathcal{D}_9\mathcal{D}_{10}} \tag{377}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(12)} + \mathcal{O}\left(\epsilon\right), \tag{378}$$

where:

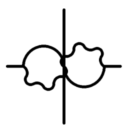
$$\begin{aligned}
E^{(12)}_{-2} &= -\frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] H(0,x)\,, &(379)\\
E^{(12)}_{-1} &= \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) - H(0,x) - H(0,0,x) \\
&\quad +2H(-1,0,x)]\,, &(380)\\
E^{(12)}_{0} &= \frac{1}{2}\left[\frac{1}{(1-x)} - \frac{1}{(1+x)}\right] [\zeta(2) + 2\zeta(3) - (1-\zeta(2))H(0,x) \\
&\quad -\zeta(2)H(-1,x) - H(0,0,x) + 4H(-1,0,x) - H(0,0,0,x) \\
&\quad -4H(-1,-1,0,x) + 2H(0,-1,0,x) + 2H(-1,0,0,x)]\,. &(381)
\end{aligned}$$



$$\begin{aligned}
&= \mu_0^{2(4-D)} \int \{d^Dk_1\}\{d^Dk_2\} \frac{1}{\mathcal{D}_1\mathcal{D}_7\mathcal{D}_{10}\mathcal{D}_{13}} &(382)\\
&= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E^{(13)}_i + \mathcal{O}(\epsilon)\,, &(383)
\end{aligned}$$

where:

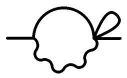
$$\begin{aligned}
E^{(13)}_{-2} &= 1\,, &(384)\\
E^{(13)}_{-1} &= 4 - \left[1 - \frac{2}{(1-x)}\right] H(0,x)\,, &(385)\\
E^{(13)}_{0} &= 12 + \left[1 - \frac{2}{(1-x)}\right][\zeta(2) - 4H(0,x) - H(0,0,x) \\
&\quad +2H(-1,0,x)]\,. &(386)
\end{aligned}$$



$$\begin{aligned}
&= \mu_0^{2(4-D)} \int \{d^Dk_1\}\{d^Dk_2\} \frac{1}{\mathcal{D}_2\mathcal{D}_4\mathcal{D}_6\mathcal{D}_{11}} &(387)\\
&= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E^{(14)}_i + \mathcal{O}(\epsilon)\,, &(388)
\end{aligned}$$

where:

$$\begin{aligned}
E^{(14)}_{-2} &= 1\,, &(389)\\
E^{(14)}_{-1} &= 4\,, &(390)\\
E^{(14)}_{0} &= 12\,. &(391)
\end{aligned}$$



$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_1 \mathcal{D}_7 \mathcal{D}_{10}} \tag{392}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(15)} + \mathcal{O}(\epsilon)\,, \tag{393}$$

where:

$$\frac{E_{-2}^{(15)}}{a} = -1\,, \tag{394}$$

$$\frac{E_{-1}^{(15)}}{a} = -3\,, \tag{395}$$

$$\frac{E_{0}^{(15)}}{a} = -7\,. \tag{396}$$

$$= \mu_0^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{\mathcal{D}_2 \mathcal{D}_6 \mathcal{D}_{8}} \tag{397}$$

$$= \left(\frac{a}{\mu_0^2}\right)^{-2\epsilon} \sum_{i=-2}^{0} \epsilon^i E_i^{(16)} + \mathcal{O}(\epsilon)\,, \tag{398}$$

where:

$$\frac{E_{-2}^{(16)}}{a} = -1\,, \tag{399}$$

$$\frac{E_{-1}^{(16)}}{a} = -3\,, \tag{400}$$

$$\frac{E_{0}^{(16)}}{a} = -7\,. \tag{401}$$

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