Three Loop Top Quark Contributions to the ρ Parameter

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Abstract

We present results for the three-loop top quark contributions to the ρ parameter in the limit of large top quark mass. The simultaneous dependence on the mass of the Higgs boson M_H and the mass of the top quark m_t is obtained from expansions in the range of M_H around m_t and in the limit $M_H \gg m_t$. In combination with the previous result for $M_H = 0$ the dependence of the ρ parameter on the leading Yukawa contributions, i.e. on m_t and M_H, is well under control for all mass values of practical importance. The effects lead to a shift in the W mass in the order of 5 MeV and are relevant for precision measurements at TESLA.

1 Introduction

The standard model predicts a strong influence of virtual heavy top quarks in the low energy limit on four-fermion processes through their virtual contributions to the W and Z boson self energies [1]. This effect allows the indirect determination of the mass of the Higgs boson and is generally essential for electroweak precision tests.

The bulk of the heavy top quark corrections to the W and Z self energies can be collected in a deviation of the so called ρ parameter from the tree level result $\rho_{\text{tree}} = 1$. The ρ parameter is usually defined by the ratio of the neutral and charged current coupling constants at zero momentum transfer [1]:

$$\rho = \frac{J_{NC}(0)}{J_{CC}(0)}, \tag{1}$$

where $J_{CC}(0)$ is given by the Fermi coupling constant G_F determined from the μ-decay rate whereas $J_{NC}(0)$ is measured by neutrino scattering on electrons or hadrons.

In leading order contributions from the Yukawa coupling of the top quark to ρ are parameterized in terms of $\Delta\rho$. This contributions stem from the transversal parts of the self energies of the W- and Z-boson:

$$\rho = \frac{1}{1-\Delta\rho} \approx \frac{1}{c^2}\,\frac{M_W^2 - \Pi_T^{WW}(0)}{M_Z^2 - \Pi_T^{ZZ}(0)}, \tag{2}$$

where $c^2 = M_W^2/M_Z^2$ represents the cosine of the weak mixing angle, defined in the standard on-shell scheme. Corrections from vertex and box diagrams always involve extra powers of the weak coupling constant g_{weak}^2 and are thus suppressed by powers of $(g_{\rm weak}/g_{\rm Yukawa}^{\rm top})^2$, i.e. $(M_W/m_t)^2$.

In this context $\Pi_T^{WW}(0)$ and $\Pi_T^{ZZ}(0)$ can be taken as the bare one-particle irreducible amplitudes. The divergencies are absorbed by the Higgs mass, top mass, and W boson mass renormalization, i.e. by expressing the bare m_t, M_H, and M_W in terms of the renormalized quantities. In the following we will first express the final answer in terms of the $\overline{\rm MS}$ top mass m_t, the on-shell Higgs mass M_H, and the low energy Fermi constant G_F obtained from μ-decay. The latter is introduced to absorb the vector boson mass M_W. In a last step the transformation from the $\overline{\rm MS}$ top quark mass m_t to the on-shell top quark mass M_t will be performed.

The first two-loop results for $\Delta\rho$ in the limits $M_H \to 0$ and $M_H \gg m_t$ for large m_t can be found in [2], for arbitrary M_H in [3]. The result for $\Delta\rho$ of order $G_F m_t^2 \alpha_s^2$ was obtained in [4]. By including terms of $\mathcal{O}(M_W^2/m_t^2)$ this has even been extended to predictions for Δr in three-loop approximation.

The present paper is concerned with mixed QCD and electroweak contributions of order $(G_F m_t^2)^2\alpha_s$ and purely electroweak contributions of order $(G_F m_t^2)^3$. It extends the result of [5] where the special case $M_H = 0$ has been considered. Using the technique of heavy mass expansion for the limit $M_H \gg m_t$ and considering sufficiently many terms leads to stable predictions down to Higgs Mass values as low as twice m_t. $\Delta\rho$ is furthermore evaluated for M_H around m_t, and with the help of expansions, a stable approximation is provided in the region from close to zero up to twice m_t. Combining the results from the different regions, the three-loop corrections to the ρ parameter are well under control.

An outline of the calculation, including a short discussion of the renormalization, is given in section 2. In section 4 the transformation of the top quark mass from the $\overline{\rm MS}$ scheme to the on-shell top mass is treated in detail. Our results[1] at order G_F^3 and $\alpha_s G_F^2$ are presented in sections 3 and 5 in the $\overline{\rm MS}$ and on-shell definition for the top quark mass respectively.

2 Treatment of diagrams and renormalization

The computation of the three-loop contributions to $\Delta\rho$ at orders G_F^3 and $\alpha_s G_F^2$ for $M_H \approx m_t$ and $M_H \gg m_t$ is similar to the one discussed in [5] for $M_H = 0$. The same Feynman diagrams contribute: W and Z boson self energies up to three loops, top quark self energies up to two loops, one-loop Higgs boson self energies, and Higgs tadpoles up to two loops. All diagrams were generated using the program `QGRAF` [6].

In comparison to [5] a non-vanishing Higgs mass introduces a second mass scale into the problem. In order to calculate such diagrams we consider two different ranges of the Higgs mass: First, in the range of M_H around m_t we perform a straightforward Taylor

[1] Only numerical results are presented for the values of the coefficients of the various expansions. Their representations in terms of fractions and transcendental numbers can be obtained from the authors.

expansion in the mass difference $M_H - m_t$ around the point $M_H = m_t$. In the second region, i.e. for $M_H \gg m_t$, we use asymptotic expansions [7] and made use of the program `EXP` [8]. In both cases the resulting diagrams contained at most one non-vanishing mass scale.

From equation (2) the vector boson self energies can be taken at zero external momentum. After the application of the expansion procedures the resulting integrals are of tadpole type with one mass scale. Their computation can be performed using the program package `MATAD` [9] written in `FORM` [10]. Apparently the Higgs tadpoles, which are needed to renormalize the vacuum expectation value of the Higgs boson, can be treated using the same program package. To renormalize the mass of the Higgs boson, the computation of on-shell Higgs boson self energy diagrams is necessary. The corresponding renormalization constant is needed to one-loop order only and the relevant integrals are straightforward. The remaining two-loop top quark mass renormalization is, however, more involved, in particular in the on-shell scheme. Details on this issue are given in sections 3 and 4.

The whole calculation was performed using dimensional regularization and anticommuting γ_5. This prescription preserves the Ward–Takahashi identities which relate the self energies of the gauge bosons to the non-diagonal self energies of the gauge-to-goldstone and the goldstone boson self energies. The validity of these identities was verified by explicit calculations.

3 The ρ-parameter in the $\overline{\mathrm{MS}}$ scheme

In this section we present our results for the $\overline{\mathrm{MS}}$ definition of the top quark mass. The renormalization constant of the top quark mass can be obtained by calculating the relevant one- and two-loop self energy diagrams in the limit of vanishing external momentum. After the application of the expansions mentioned in section 2, the remaining integrals were computed using the program package `MATAD`.

The radiative corrections to $\Delta\rho$ depend on the $\overline{\mathrm{MS}}$ top mass $m_t \equiv m_t(m_t)$, the on-shell Higgs mass M_H, the Fermi constant G_F and the strong coupling constant α_s and are written as follows:

$$\begin{aligned}\Delta\rho &= x_t\,\Delta\rho^{(x_t)} + x_t^2\,\Delta\rho^{(x_t^2)} + \frac{\alpha_s}{\pi}\,x_t\,\Delta\rho^{(\alpha_s x_t)} + \left(\frac{\alpha_s}{\pi}\right)^2 x_t\,\Delta\rho^{(\alpha_s^2 x_t)} \\ &\quad + \frac{\alpha_s}{\pi}\,x_t^2\,\Delta\rho^{(\alpha_s x_t^2)} + x_t^3\,\Delta\rho^{(x_t^3)} + \cdots\,, \end{aligned} \tag{3}$$

with

$$x_t = \frac{G_F m_t^2}{8\sqrt{2}\pi^2} \approx 3.0\cdot 10^{-3}\,.$$

Here and later on we always set the $\overline{\mathrm{MS}}$ renormalization scale μ equal to m_t

In this paper we evaluate the coefficients $\Delta\rho^{(x_t^3)}$ and $\Delta\rho^{(\alpha_s x_t^2)}$. Numerically they are given by:

$M_H = 0$:

$$\Delta\rho^{(x_t^3)} = 47.73\,, \tag{4}$$
$$\Delta\rho^{(\alpha_s x_t^2)} = 76.91\,. \tag{5}$$

$M_H \approx m_t$: In this case we have computed five terms in the expansion in δ defined by the relation $M_H = m_t(1+\delta)$:

$$\Delta\rho^{(x_t^3)} = -40.30 - 138.26\,\delta - 32.39\,\delta^2 + 10.32\,\delta^3 - 3.27\,\delta^4 - 0.01\,\delta^5 + \mathcal{O}(\delta^6)\,, \tag{6}$$
$$\Delta\rho^{(\alpha_s x_t^2)} = 93.07 + 21.27\,\delta + 0.53\,\delta^2 - 2.29\,\delta^3 + 1.14\,\delta^4 - 0.51\,\delta^5 + \mathcal{O}(\delta^6)\,. \tag{7}$$

$M_H \gg m_t$: The application of the hard mass procedure leads to an expansion in the ratio $y \equiv 4m_t^2/M_H^2$:

$$\begin{aligned}
\Delta\rho^{(x_t^3)} &= \frac{1}{y}(-12.80 + 24.75\,\log y) \\
&- 192.86 - 394.05\,\log y - 275.02\,\log^2 y - 24.25\,\log^3 y \\
&+ y\,\left(-337.88 + 408.59\,\log y - 233.35\,\log^2 y + 108.04\,\log^3 y\right) \\
&+ y^2\,\left(82.56 - 378.31\,\log y + 26.14\,\log^2 y + 48.60\,\log^3 y\right) \\
&+ y^3\,\left(-82.58 - 291.03\,\log y - 11.37\,\log^2 y + 56.87\,\log^3 y\right) \qquad (8) \\
&+ y^4\,\left(-44.06 - 366.54\,\log y - 3.27\,\log^2 y + 59.62\,\log^3 y\right) \\
&+ y^5\,\left(-42.49 - 409.57\,\log y - 4.44\,\log^2 y + 64.70\,\log^3 y\right) \\
&+ \mathcal{O}(y^6), \\
\Delta\rho^{(\alpha_s x_t^2)} &= 216.07 + 131.38\,\log y + 57.07\,\log^2 y + 9.00\,\log^3 y \\
&+ y\,\left(-59.99 - 66.29\,\log y + 18.39\,\log^2 y - 19.22\,\log^3 y\right) \\
&+ y^2\,\left(-66.31 + 21.04\,\log y - 37.76\,\log^2 y - 5.57\,\log^3 y\right) \\
&+ y^3\,\left(-9.73 - 28.56\,\log y - 29.38\,\log^2 y - 5.46\,\log^3 y\right) \qquad (9) \\
&+ y^4\,\left(-4.48 - 37.23\,\log y - 30.13\,\log^2 y - 5.25\,\log^3 y\right) \\
&+ y^5\,\left(-2.49 - 41.12\,\log y - 31.92\,\log^2 y - 5.14\,\log^3 y\right) \\
&+ \mathcal{O}(y^6).
\end{aligned}$$

In Fig. 1 and 2 these results are plotted as functions of the ratio of the Higgs mass M_H and the top mass m_t. The left scale gives the coefficient of the contribution under consideration. The right scale includes the prefactor for $m_t = 165$ GeV, so the full numerical effect on $\Delta\rho$ can be directly read off. In Fig. 2 the numerical value $\alpha_s(m_t) = 0.109$ was used for the right scale.

The solid lines represent the sum of all terms displayed above. The dashed curves represent successive approximations. From Figs. 1 and 2 we infer that the series given in equations (6) to (9) provides a satisfactory approximation to the full result. In fact, a fairly smooth transition between the two approximations is observed for $M_H \approx 2.5m_t$.

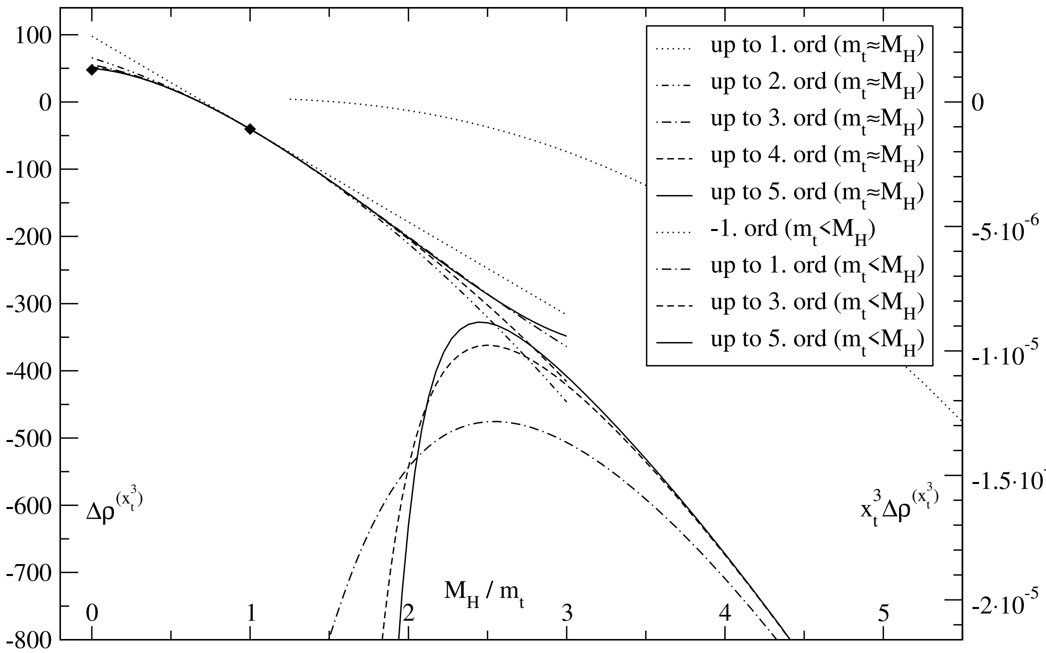

Figure 1: Contributions of order x_t^3 to $\Delta\rho$ in the $\overline{\text{MS}}$ definition of the top quark mass. The black squares indicate the points where the exact result is known.

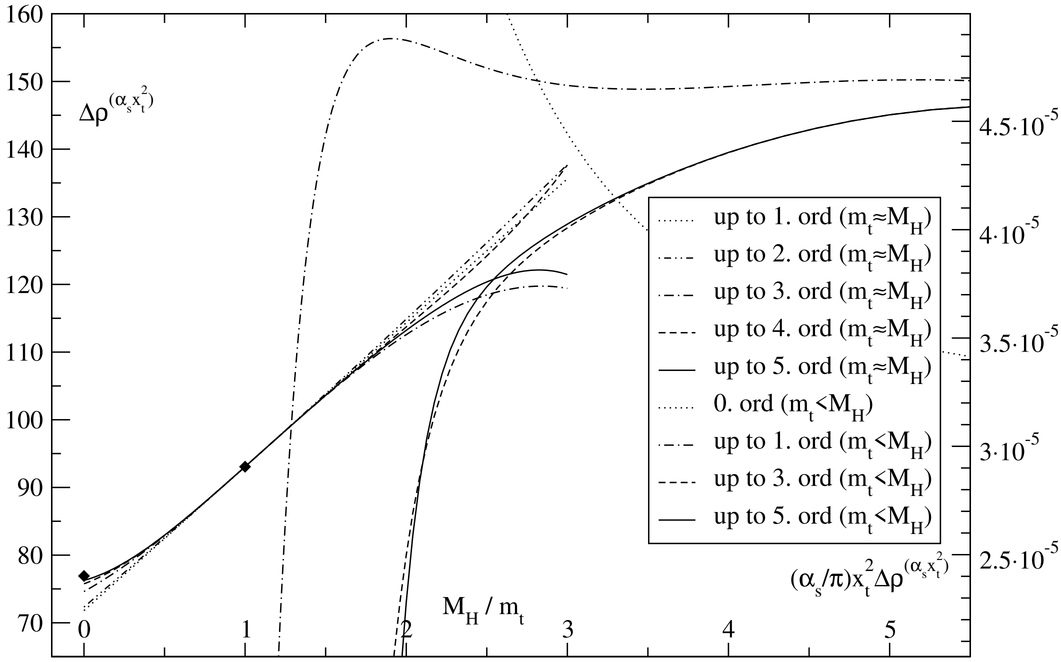

Figure 2: Contributions of order $\alpha_s x_t^2$ to $\Delta\rho$ in the $\overline{\text{MS}}$ definition of the top quark mass. The black squares indicate the points where the exact result is known.

In addition the expansion seems to provide a good approximation even down to $M_H = 0$, where it can be compared to the exact result.

Note, that the dominant term of the three-loop electroweak corrections $\Delta\rho^{(x_t^3)}$ in the limit of large Higgs masses is proportional to M_H^2. This is in agreement with the *screening theorem* [11] stating that the highest power correction depending on the mass of the Higgs boson is at least one power of M_H^2 below the expectation based on naive power counting. The terms $\sim M_H^2$ stem directly from the large mass expansion of the three-loop W and Z self energies. However, the absolute size is small when the prefactor x_t^3 is included.

The smooth behavior of the two expansions in Figs. 1 and 2 together with the agreement with the result for $M_H = 0$ leads to the assumption that the three-loop corrections are well under control in the $\overline{\rm MS}$ scheme.

4 Transition from $\overline{\rm MS}$ to on-shell top quark mass

In a final step the result for the on-shell definition of the top quark mass are presented. This requires the ratio between the $\overline{\rm MS}$ renormalized top quark mass m_t and the on-shell mass M_t in two-loop approximation, again in the limit of large top quark mass. Let us parameterize the ratio between $\overline{\rm MS}$ in the following way

$$\frac{m_t}{M_t} = 1 + \frac{\alpha_s}{\pi}C^{(\alpha_s)} + X_t C^{(X_t)} + \left(\frac{\alpha_s}{\pi}\right)^2 C^{(\alpha_s^2)} + \frac{\alpha_s}{\pi}X_t C^{(\alpha_s X_t)} + X_t^2 C^{(X_t^2)} + \ldots\,, \qquad (10)$$

using the expansion parameter

$$X_t = \frac{G_F M_t^2}{8\sqrt{2}\pi^2} \approx 3.2 \cdot 10^{-3}\,,$$

which depends on the on-shell top quark mass M_t.

The results for $C^{(\alpha_s)}$, $C^{(X_t)}$ (e.g. [12] and [13] respectively), and $C^{(\alpha_s^2)}$ [14] are well known. Results for both $C^{(\alpha_s X_t)}$ and $C^{(X_t^2)}$ were obtained from the large mass and the mass difference expansion and are needed for the present discussion.

The corresponding two-loop top quark self energy diagrams, already discussed in section 3 for vanishing external momentum, have to be recalculated for external momentum on mass shell. In general the relevant diagrams can be divided into two classes: those containing at least one virtual Higgs boson and those without Higgs bosons. The latter have already been evaluated for the computation of $\Delta\rho$ in the case $M_H = 0$ [5]. However, the former class of diagrams needs additional attention.

Let us first consider the limit $M_H \gg m_t$. Asymptotic expansions reduce the two-loop on-shell integrals containing two mass scales to combinations of one scale integrals. These one scale integrals include: tadpole integrals up to two loops, one-loop on-shell integrals, and one and two-loop integrals containing a small (compared to the internal Higgs mass) external momentum. Each of these integrals is either straightforward to calculate or can be treated using `MATAD` and `MINCER` [15]. Thus one obtains the on-shell top quark self mass in terms of the $\overline{\mathrm{MS}}$ mass in two-loop approximation for the region $M_H \gg m_t$. The result reads as follows:

$$
\begin{aligned}
C^{(X_t^2)} &= \frac{1}{Y}(1.61 - 18.00 \log Y) \\
&+ 8.09 - 2.10 \log Y - 1.50 \log^2 Y \\
&+ Y\left(-4.50 + 6.05 \log Y - 1.73 \log^2 Y\right) \\
&+ Y^2\left(14.60 - 9.74 \log Y + 1.69 \log^2 Y\right) \\
&+ Y^3\left(2.02 - 2.34 \log Y + 0.64 \log^2 Y\right) \\
&+ Y^4\left(0.56 - 0.94 \log Y + 0.32 \log^2 Y\right) \\
&+ Y^5\left(0.22 - 0.48 \log Y + 0.17 \log^2 Y\right) + \mathcal{O}(Y^6)\,,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
C^{(\alpha_s X_t)} &= 7.09 - 3.66 \log Y + 1.50 \log^2 Y \\
&+ Y\left(-3.78 + 2.93 \log Y - 0.36 \log^2 Y\right) \\
&+ Y^2\left(-0.89 + 0.99 \log Y - 0.23 \log^2 Y\right) \\
&+ Y^3\left(-0.28 + 0.42 \log Y - 0.15 \log^2 Y\right) \\
&+ Y^4\left(-0.11 + 0.21 \log Y - 0.11 \log^2 Y\right) \\
&+ Y^5\left(0.22 - 0.48 \log Y + 0.17 \log^2 Y\right) + \mathcal{O}(Y^6)\,,
\end{aligned}
\tag{12}
$$

where $Y = 4M_t^2/M_H^2$ is used.

For M_H around m_t two-loop on-shell diagrams composed of several massive and massless lines enter. Let us start with the case $M_H = m_t$ which involves two-loop on-shell diagrams with internal propagators which are either massless or of mass m_t. The program packages `ONSHELL2` [16] and `TARCER` [17], based on the recurrence relations [18], are specifically tailored to this case. In the next step the Taylor expansion around this point is performed and five terms of order $\alpha_s x_t$ and x_t^2 are evaluated. It should be mentioned, that in order x_t^2 there are several diagrams with threshold at M_H^2. This means that in order to expand these diagrams the threshold expansion should be applied [7, 19]. By explicit calculation, however, we find that the first five coefficients are given by the naive Taylor series. The results read as follows:

$$
C^{(X_t^2)} = 33.50 - 22.91\,\delta + 13.96\,\delta^2 - 2.30\,\delta^3 + 2.71\,\delta^4 - 3.46\,\delta^5 + \mathcal{O}(\delta^6)\,, \tag{13}
$$

$$
C^{(\alpha_s X_t)} = -22.15 + 34.76\,\delta - 6.83\,\delta^2 + 2.64\,\delta^3 - 1.32\,\delta^4 + 0.19\,\delta^5 + \mathcal{O}(\delta^6)\,, \tag{14}
$$

using δ defined by $M_H = M_t(1+\delta)$.

For completeness we also include the result for $M_H = 0$:

$$
C^{(X_t^2)} = 83.58 \quad \text{and} \quad C^{(\alpha_s X_t)} = -17.53\,. \tag{15}
$$

The results are shown in Fig. 3 and 4. A smooth transition between the two regions of large M_H and $M_H \approx M_t$ is observed. Furthermore, the expansion around $M_H \approx M_t$, when extrapolated down to $M_H = 0$, leads to a remarkably good agreement with the value

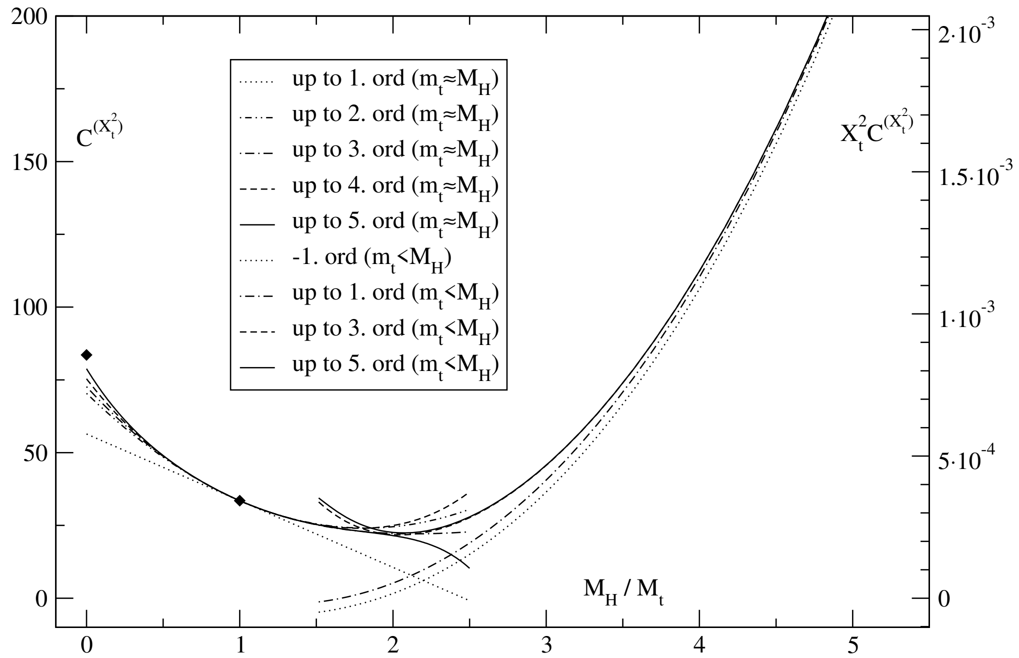

Figure 3: Contributions of order x_t^2 to the relation between the $\overline{\text{MS}}$ and on-shell top quark mass. The black squares indicate the points where the exact result is known.

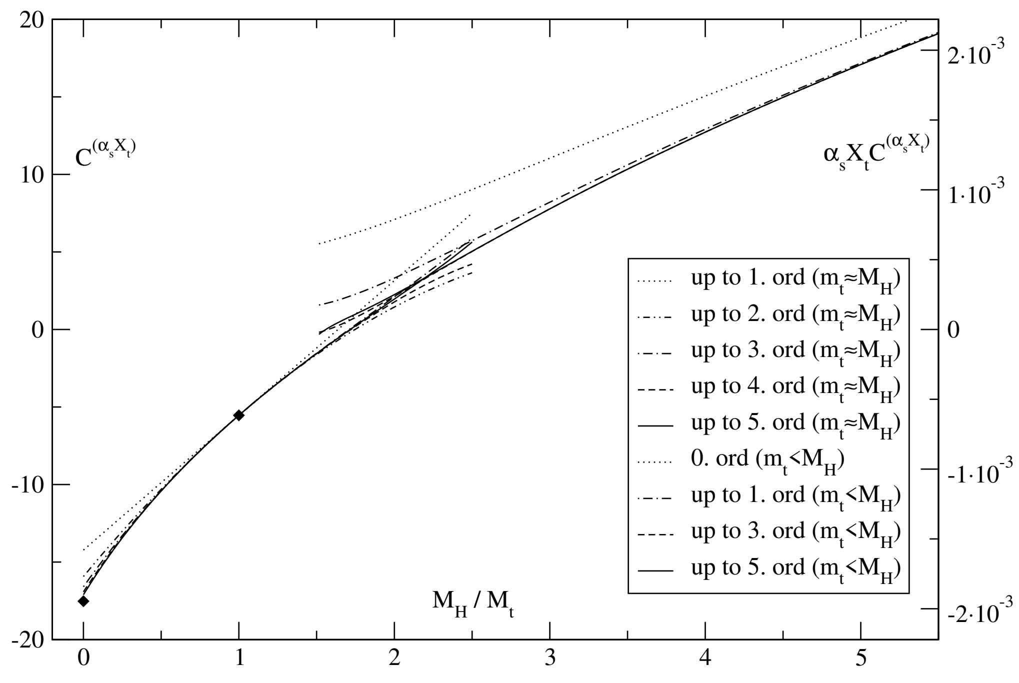

Figure 4: Contributions of order $\alpha_s x_t$ to the relation between the $\overline{\text{MS}}$ and on-shell top quark mass. The black squares indicate the points where the exact result is known.

obtained from an analytical calculation at $M_H = 0$, a further and independent test of the approximation procedure.

The result for the ratio between $\overline{\mathrm{MS}}$ mass and on-shell mass — as well as the analytical expressions for the coefficients given in sections 3 and 5 — is available from the authors upon request.

5 $\Delta\rho$ expressed through the on-shell top quark mass

Combining the results of sections 3 and 4 we compute $\Delta\rho$ in terms of the on-shell top quark mass M_t, the on-shell Higgs mass M_H and the low energy Fermi constant G_F. Our calculation reproduces the one- and two-loop results for the ρ parameter at orders G_F, G_F^2 and $G_F\alpha_s$ [1, 2, 3].

The result for the on-shell definition of the top quark mass is parameterized in the same way as the $\overline{\mathrm{MS}}$ result in section 3. In order to distinguish the two schemes we use again the expansion parameter X_t, depending on the on-shell top quark mass M_t.

For completeness we also include the results for $M_H = 0$, computed in [5]:

$$\begin{aligned} \Delta\rho^{(X_t^3)} &= 249.74 \ , && (16)\\ \Delta\rho^{(\alpha_s X_t^2)} &= 2.94 \ . && (17) \end{aligned}$$

For the expansion around $M_H = M_t$ we obtain:

$$\begin{aligned} \Delta\rho^{(X_t^3)} &= 95.92 - 111.98\delta + 8.099\delta^2 + 9.36\delta^3 + 7.27\delta^4 + -15.60\,\delta^5 + \mathcal{O}(\delta^6), && (18)\\ \Delta\rho^{(\alpha_s X_t^2)} &= 157.295 + 112.00\delta - 24.73\delta^2 + 7.39\delta^3 - 3.52\delta^4 + 2.06\,\delta^5 + \mathcal{O}(\delta^6), && (19) \end{aligned}$$

using δ defined by $M_H = M_t(1+\delta)$.

In the limit $M_H \gg M_t$ we obtain

$$\begin{aligned} \Delta\rho^{(X_t^3)} &= \frac{1}{Y}(-3.17 - 83.25\,\log Y)\\ &- 189.93 - 231.48\log Y - 142.06\,\log^2 Y + 2.75\,\log^3 Y\\ &+ Y\,\left(-332.34 + 77.71\,\log Y - 68.67\,\log^2 Y + 51.79\,\log^3 Y\right)\\ &+ Y^2\,\left(227.55 - 510.55\,\log Y + 87.77\,\log^2 Y + 6.41\,\log^3 Y\right)\\ &+ Y^3\,\left(-58.40 - 329.18\,\log Y + 20.42\,\log^2 Y + 14.54\,\log^3 Y\right) && (20)\\ &+ Y^4\,\left(-36.14 - 381.88\,\log Y + 18.63\,\log^2 Y + 15.04\,\log^3 Y\right)\\ &+ Y^5\,\left(-39.08 - 416.36\,\log Y + 13.76\,\log^2 Y + 17.19\,\log^3 Y\right)\\ &+ \mathcal{O}(Y^6),\\ \Delta\rho^{(\alpha_s X_t^2)} &= 79.73 - 47.77\,\log Y + 42.07\,\log^2 Y + 9.00\,\log^3 Y\\ &+ Y\,\left(225.16 - 179.74\,\log Y + 70.22\,\log^2 Y - 19.22\,\log^3 Y\right)\\ &+ Y^2\,\left(-76.07 + 25.33\,\log Y - 9.17\,\log^2 Y - 5.57\,\log^3 Y\right) && (21) \end{aligned}$$

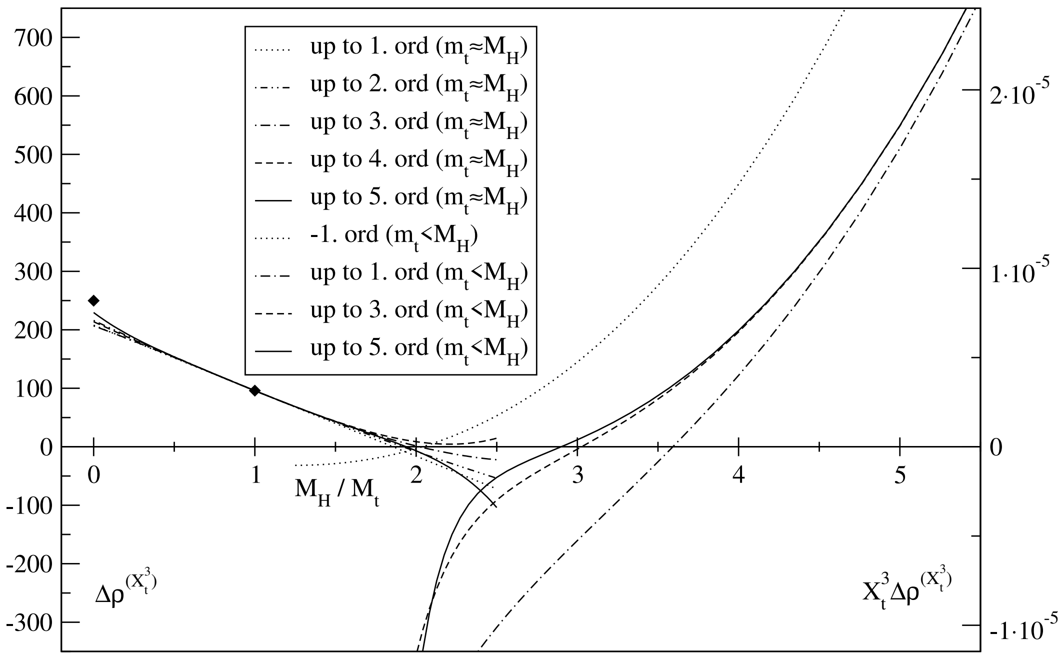

Figure 5: Contributions of order X_t^3 to $\Delta\rho$ in the on-shell definition of the top quark mass. The black squares indicate the points where the exact result is known.

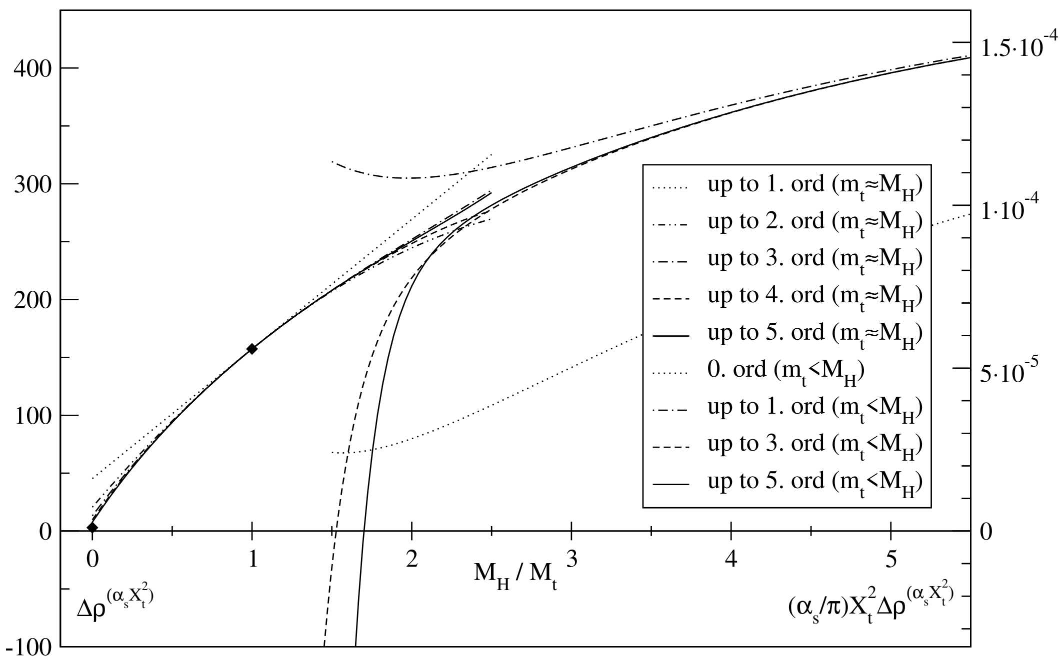

Figure 6: Contributions of order $\alpha_s X_t^2$ to $\Delta\rho$ in the on-shell definition of the top quark mass. The black squares indicate the points where the exact result is known.

$$
\begin{aligned}
&+ \quad Y^3 \left(-10.10 - 24.69 \log Y - 0.30 \log^2 Y - 5.46 \log^3 Y\right) \\
&+ \quad Y^4 \left(-4.52 - 32.85 \log Y + 0.72 \log^2 Y - 5.25 \log^3 Y\right) \\
&+ \quad Y^5 \left(-2.55 - 36.61 \log Y + 1.06 \log^2 Y - 5.14 \log^3 Y\right) \\
&+ \quad \mathcal{O}(Y^6)\,,
\end{aligned}
$$

with

$$
Y \equiv \frac{4M_t^2}{M_H^2}\,. \tag{22}
$$

The results in the on-shell definition of the top quark mass are shown in Fig. 5 and 6. Again, the solid line corresponds to the numerical values including all terms of our expansion. The left and right scales give the size of the coefficient and the full effect on $\Delta\rho$ respectively.

The $\alpha_s X_t^2$ contributions are sizeable and monotonously increasing over the full range of the Higgs mass suggesting a smoothly increasing behavior qualitatively similar to the well-known X_t^2 correction (displayed in Fig. 7). For $M_H = 0$ the result seems to be accidentally small, but increases rapidly for larger Higgs mass values. The X_t^3 term exhibits a minimum in the range between $M_H = 2M_t$ and $M_H = 3M_t$ and appears to be very small or even negative in this region.

To compare the size of the three-loop contributions to the known one- and two-loop terms we display their individual effects on $\Delta\rho$ in Fig. 7, including the corrections of order $\alpha_s^2 X_t$ computed in [4]. We start with the sum of the contributions of order X_t and $\alpha_s X_t$ which are independent of the Higgs mass and proceed by adding the particular orders to $\Delta\rho$. To identify the individual contributions for $M_H = 0$ we continued the value horizontally to the negative M_H/M_t region.

Let us first consider the $\alpha_s X_t^2$ and X_t^3 terms for the case $M_H = 0$, discussed already in [5]. Inclusion of the $\alpha_s X_t^2$ term leads to a change in $\Delta\rho$ of $1.0 \cdot 10^{-6}$, which is not visible in Fig. 7. The X_t^3 term increases $\Delta\rho$ by roughly $8.2 \cdot 10^{-6}$, again an extremely small effect. The picture alters drastically for larger values of M_H. The order $\alpha_s X_t^2$ correction is sizeable in the region of $M_H \approx M_t$ and above. It amounts to approximately 1/3 of the two-loop contribution $\Delta\rho^{(X_t^2)}$, but appears with the opposite sign. In contrast, the pure electroweak three-loop contribution $\Delta\rho^{(X_t^3)}$ yields an enhancement of about 1% compared to the order X_t^2 terms and is therefore negligible for M_H values of present interest. These corrections to $\Delta\rho$ can be used to examine the resulting shifts of observables like the W mass and the effective weak mixing angle, predicted for fixed values of M_Z, α, and G_F through the equations

$$
M_W^2 = \frac{\rho M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}\rho G_F M_Z^2}\left(\frac{1}{1-\Delta\alpha} + \ldots\right)}\right) \tag{23}
$$

and

$$
\sin^2\theta_{\text{eff}} = 1 - \frac{M_W^2}{\rho M_Z^2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha}{\sqrt{2}\rho G_F M_Z^2}\left(\frac{1}{1-\Delta\alpha} + \ldots\right)}\right), \tag{24}
$$

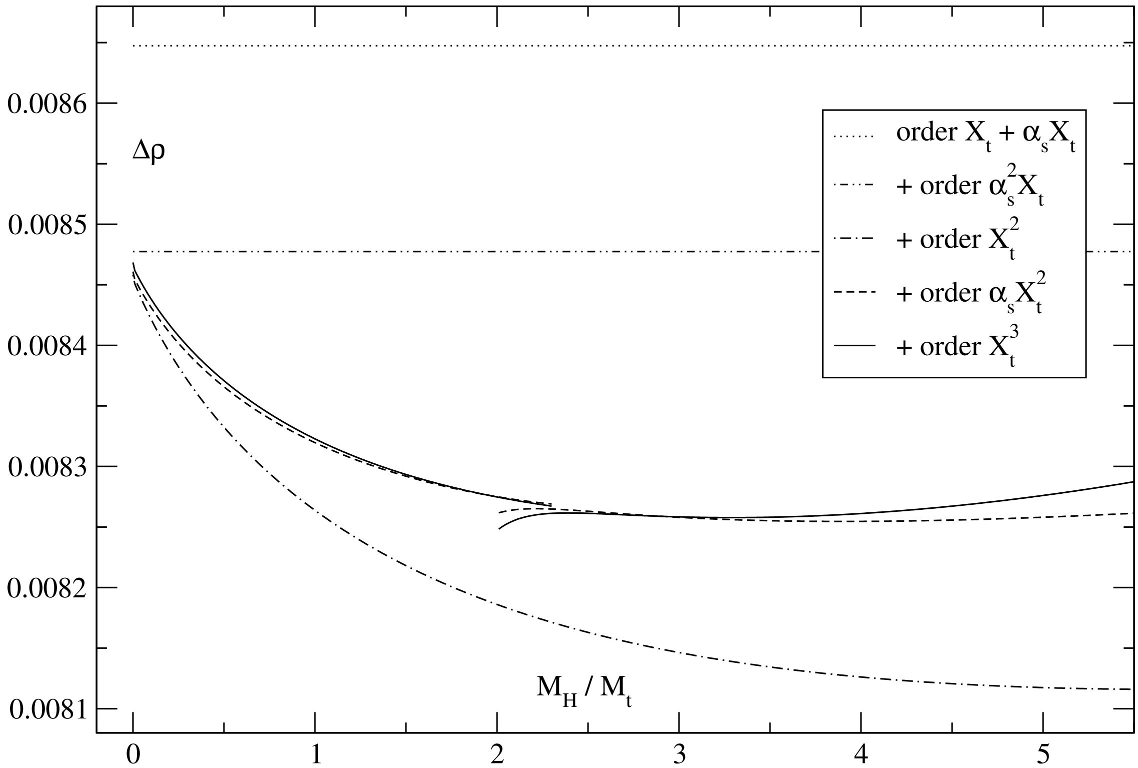

Figure 7: Effect of several contributions to $\Delta\rho$ in the on-shell definition of the top quark mass.

where $\Delta\alpha$ is the shift of α due to photon vacuum polarization effects and the dots are nonleading remainder terms. In Fig. 8 the resulting shifts are shown for the different orders of $\Delta\rho$ in dependence of the Higgs mass (We do not display the shift of order $\alpha_s X_t$ which amounts to -52.8 MeV). This shift should be compared to $\delta M_W = 6$ MeV and $\delta \sin^2\theta_{\rm eff} = 1.3 \cdot 10^{-5}$ which is the precision anticipated for TESLA operating at the Z-resonance [20].

Alternatively we also present the shift in M_t which would lead to a comparable shift in $\Delta\rho$:

$$\frac{\delta M_t}{M_t} \approx \frac{1}{2}\,\frac{\delta\Delta\rho}{3X_t}\,. \tag{25}$$

For $\delta\Delta\rho = 5\cdot 10^{-5}$ this leads to a shift of m_t by about 500 MeV, which has to be compared to $\delta m_t \approx \mathcal{O}(100\text{MeV})$ anticipated for measurements at a future linear collider.

For fixed M_t, M_W, and M_Z our result for $\Delta\rho$ has to be compensated by a shift in the prediction for M_H:

$$\frac{\delta M_H}{M_H} \approx \frac{12\sqrt{2}}{11 G_F}\,\frac{\cos^2\theta_W}{\sin^2\theta_W}\,\delta\Delta\rho \approx 670\;\delta\Delta\rho \tag{26}$$

For M_H between 120 GeV and 200 GeV the term $\alpha_s X_t^2$ leads to an upward shift between

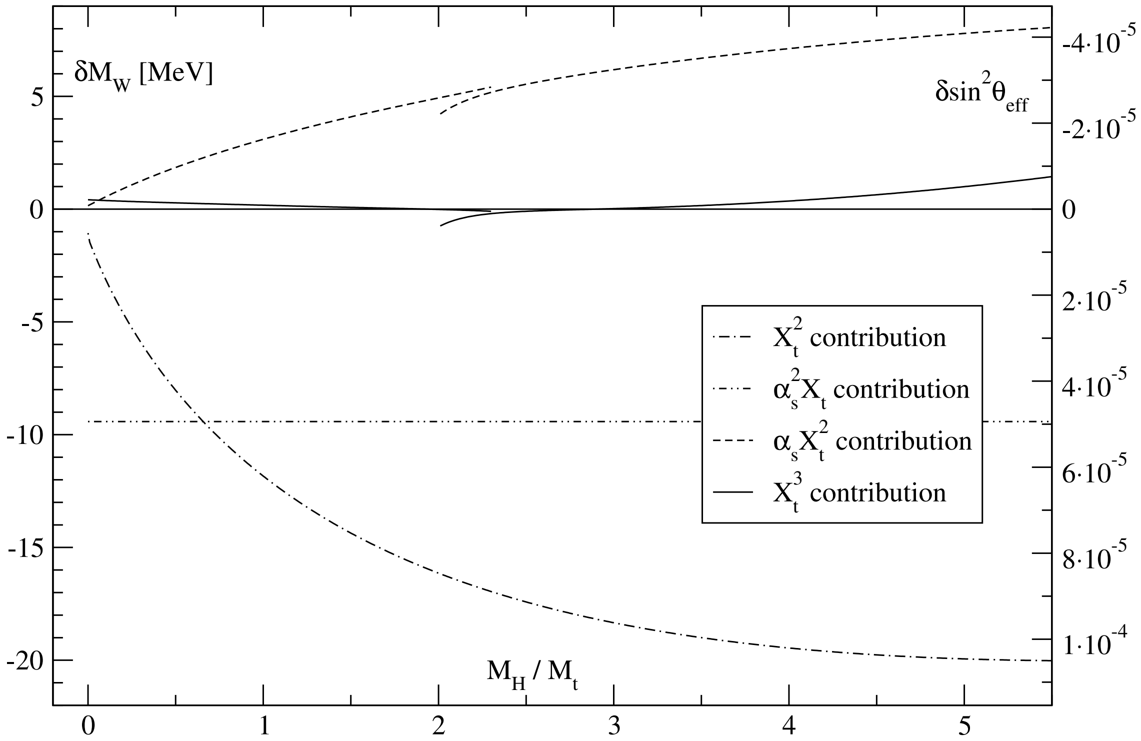

Figure 8: Shifts in M_W and in $\sin^2\theta_W$ from the various corrections to $\Delta\rho$.

4.5 and 7.5 GeV, the X_t^3 term to further positive shift of less than 0.5 GeV.

The $X_t^2\alpha_s$ term is, furthermore, comparable or larger than the non logarithmically enhanced light fermion electroweak two-loop corrections [21] and significantly larger than the purely bosonic two-loop contribution. The X_t^3 is comparable in size to the bosonic two-loop effects evaluated in [22].

Summary: Combining the results from the extreme regions $M_H = 0$, $M_H \approx M_t$, and $M_H \gg M_t$ a prediction for the X_t^3 and $\alpha_s X_t^2$ contributions to the ρ parameter has been obtained, which covers in a reliable way the full region of M_H. The magnitude of these terms is comparable to the non-enhanced two-loop corrections.

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