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Two-Loop Form Factors in QED

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Abstract

We evaluate the on shell form factors of the electron for arbitrary momentum transfer and finite electron mass, at two loops in QED, by integrating the corresponding dispersion relations, which involve the imaginary parts known since a long time. The infrared divergences are parameterized in terms of a fictitious small photon mass. The result is expressed in terms of Harmonic Polylogarithms of maximum weight 4. The expansions for small and large momentum transfer are also given.

Key words: Feynman diagrams, Multi-loop calculation, Dispersion Relations
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1 Introduction

The imaginary parts of the on shell form factors of the electron at two loops in QED (the relevant graphs are shown for completeness in Fig.1) were calculated analytically long ago [1] in terms of Nielsen's polylogarithms [2] of maximum weight $w = 3$. The real parts however are still missing, as the integration of the dispersion relations for the form factors could not be carried out in closed form within the family of Nielsen's polylogarithms only.

We show in this paper that the previous calculation can be completed within the family of the Harmonic Polylogarithms (HPL's) introduced in [3]. The result involves HPL's of maximum weight $w = 4$.

The plan of the paper is as follows. In Section 2 we recall, from [1], the proper dispersion relations satisfied by the two form factors. In Section 3 we list the results of the analytic integration for arbitrary momentum transfer $t = -Q^2$ and finite electron mass m. For positive Q^2 all the terms are real; as the expression is exact, it can be continued to t timelike and above threshold, where the form factors develop their imaginary parts. In Section 4 we give the expansions of the form factors for $Q^2 \to \infty$, and in Section 5 the expansions for $Q^2 \to 0$. In the Appendix A we recall the definition and the main properties of the HPL's, while in Appendix B we derive the analytic formulas which were used in the calculation.

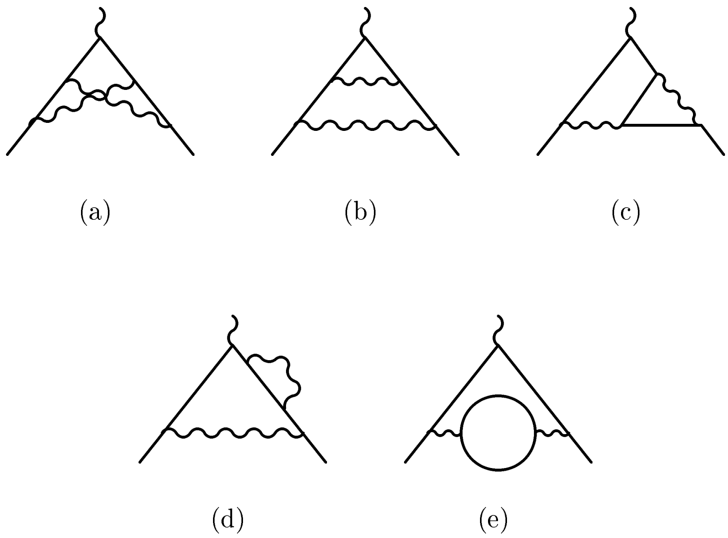

Figure 1: The Vertex Graphs at two loops in QED (multiplicities understood).

2 Dispersion Relations

The imaginary parts of the two on-shell vertex form factors of the electron at two loop in QED, $\mathrm{Im}F_1^{(2)}(t)$, $\mathrm{Im}F_2^{(2)}(t)$ were evaluated in [1] for arbitrary value of the momentum transfer t and finite electron mass, within the Pauli-Villars regularization scheme for the ultraviolet divergences (needed for the renormalization of the inserted one loop subgraphs) and by using a small fictitious mass λ for the regularization of the infrared divergences. In the imaginary parts t is above the threshold $4m^2$, where m is the electron mass; by introducing the dimensionless variable x through

$$t = m^2\frac{(1+x)^2}{x} \ , \qquad x = \frac{\sqrt{t}-\sqrt{t-4m^2}}{\sqrt{t}+\sqrt{t-4m^2}} \ , \tag{2.1}$$

all the imaginary parts are can be written in terms of x, $\log x$, $\log(1-x)$, $\log(1+x)$ and Nielsen polylogarithms [2] of maximum weight 3 and of argument $x, -x, x^2$.

As discussed in Eq.(1.30) of [1], the corresponding real parts are then given by properly subtracted dispersion relations, which for spacelike momentum transfer $t = -Q^2$ read

$$
\begin{aligned}
F_1^{(2)}(-Q^2) &= -\frac{4m^2 Q^2}{Q^2+4m^2} F_1^{(2)'}(0) \\
&- \frac{Q^4}{Q^2+4m^2} \int_{4m^2}^{\infty} \frac{dt'}{t'+Q^2} \frac{t'-4m^2}{t'^2} \frac{1}{\pi} \mathrm{Im}\, F_1^{(2)}(t') \ , \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
F_2^{(2)}(-Q^2) &= \frac{4m^2}{Q^2+4m^2} F_2^{(2)}(0) \\
&+ \frac{Q^2}{Q^2+4m^2} \int_{4m^2}^{\infty} \frac{dt'}{t'+Q^2} \frac{t'-4m^2}{t'} \frac{1}{\pi} \mathrm{Im}\, F_2^{(2)}(t') \ . \qquad (2.3)
\end{aligned}
$$

Indeed, the above imaginary parts are singular at $t = 4m^2$, due to the infrared singularities of the 2-photon ladder graph showing up in the $\lambda \to 0$ limit – hence the subtraction at $Q^2 = -4m^2$ and the factor $(t'-4m^2)$ for making the dispersive integrals convergent at threshold. Note that the charge slope $F_1^{(2)'}(0)$ and the magnetic anomaly $F_2^{(2)}(0)$ are needed for writing the above dispersion relations (which therefore cannot be used, in that form, for their calculation). Let us further observe that Eq.(2.2) gives $F_1^{(2)}(0) = 0$, the usual renormalization condition of the charge form factor.

Similarly to Eq.(2.1), in the spacelike case we put

$$
Q^2 = m^2 \frac{(1-y)^2}{y} \ , \qquad y = \frac{\sqrt{Q^2+4m^2}-\sqrt{Q^2}}{\sqrt{Q^2+4m^2}+\sqrt{Q^2}} \ ; \qquad (2.4)
$$

with the change of integration variable $t' = m^2(1+x')^2/x'$, the dispersive integrals occurring in Eq.s(2.2,2.3) are seen to be of the form

$$
\int_{4m^2}^{\infty} \frac{dt'}{t'+Q^2} f(t') = \int_0^1 dx' \left(\frac{1}{x'} - \frac{1}{x'+y} - \frac{1}{x'+1/y} \right) \phi(x') \ . \qquad (2.5)
$$

When the actual explicit expressions of the imaginary parts in terms of Nielsen's polylogarithms are used within the dispersive integrals Eq.(2.2,2.3), it is found that the resulting integrals cannot be evaluated in terms of that same class of functions. The Nielsen's polylogarithms, however, are special cases of the Harmonic Polylogarithms [3], HPL's; when the more general formalism of the HPL's is used, one finds that not only the imaginary parts, but also the real parts given by the dispersive integrals can all be expressed in terms of HPL's of argument y and 1 (the HPL's of argument 1 are expressed in turn as combinations of a few mathematical constants like $\zeta(3)$, the Riemann ζ-function of index 3, π^2 and $\ln 2$).

We give in the following paragraphs the explicit results of the integration. The relevant formulae are shortly discussed in the Appendix.

3 Results

We present in this section the results of the analytic integration of Eq.(2.2) and Eq.(2.3) as a function of $Q^2 = -t$. As already said, λ is the "small photon mass" used for the parameterization of the infrared divergences. All the polylogarithms depend on Q^2 through the variable y defined in Eq.(2.4). Let us recall that $y(Q^2 = \infty) = 0$, $y(Q^2 = 0) = 1$; when the momentum transfer t is positive (timelike) and varies from 0 to the physical threshold $t = 4m^2$, $Q^2 = -(t + i\epsilon)$ varies from $-i\epsilon$ to $Q^2 = -4m^2 - i\epsilon$; correspondingly, $y = e^{i\phi}$, with $\phi = 0$ at $t = 0$ and $\phi = \pi$ at $t = 4m^2$ (and the form factors are still real despite the complex value of y). When t is above the threshold, it is convenient to write $y = -x + i\epsilon$, where x is the variable defined in Eq.(2.1), so that as t varies from $4m^2$ to $+\infty$, x varies correspondingly from 1 to 0. For

$y = -x + i\epsilon$ the polylogarithms with rightmost index equal to 0 develop an imaginary part which generates the imaginary parts of the form factors (see [3,4] and the Appendix A for more details).

The explicit results are

$$
\begin{aligned}
F_1^{(2)}(-Q^2) \;=\; & \ln^2\left(\frac{\lambda}{m}\right)\Biggl\{\frac{m^2}{Q^2}\Bigl[H(0,0;y)\Bigr] + \frac{Q^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigl[H(0;y)\Bigr] + \frac{m^2}{Q^2+4m^2}\Bigl[-H(0,0;y)\Bigr] \\
& \qquad + \frac{m^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigl[2H(0;y)\Bigr] + \frac{1}{2} + H(0,0;y)\Biggr\} \\
& + \ln\left(\frac{\lambda}{m}\right)\Biggl\{\frac{m^2}{Q^2}\Bigl[-\frac{1}{12}\pi^2 H(0;y) + 2H(0,0;y) - 2H(-1,0,0;y) - H(0,-1,0;y) \\
& \qquad + \frac{3}{2}H(0,0,0;y)\Bigr] \\
& \qquad + \frac{Q^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigl[-\frac{1}{12}\pi^2 + \frac{7}{4}H(0;y) - H(-1,0;y) + \frac{1}{2}H(0,0;y)\Bigr] \\
& \qquad + \frac{m^2}{Q^2+4m^2}\Bigl[\frac{1}{12}\pi^2 H(0;y) - H(0,0;y) + 2H(-1,0,0;y) \\
& \qquad\qquad + H(0,-1,0;y) - \frac{3}{2}H(0,0,0;y)\Bigr] \\
& \qquad + \frac{m^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigl[-\frac{1}{6}\pi^2 + 4H(0;y) - 2H(-1,0;y) + H(0,0;y)\Bigr] \\
& \qquad + 1 - \frac{1}{12}\pi^2 H(0;y) + \frac{3}{2}H(0,0;y) - 2H(-1,0,0;y) - H(0,-1,0;y) \\
& \qquad + \frac{3}{2}H(0,0,0;y)\Biggr\} \\
& + \frac{m^2}{Q^2}\Bigl[\frac{61}{1440}\pi^4 - \zeta(3) - \frac{1}{4}\pi^2 H(0;y) + \frac{1}{2}\zeta(3)H(0;y) + \zeta(3)H(1;y) \\
& \qquad + \frac{1}{12}\pi^2 H(-1,0;y) + 2H(0,0;y) + \frac{7}{24}\pi^2 H(0,0;y) + \frac{1}{6}\pi^2 H(1,0;y) \\
& \qquad - 2H(-1,0,0;y) + H(0,-1,0;y) - \frac{1}{2}H(0,0,0;y) - 2H(0,1,0;y) \\
& \qquad + 2H(-1,-1,0,0;y) + H(-1,0,-1,0;y) - \frac{3}{2}H(-1,0,0,0;y) \\
& \qquad - H(0,-1,0,0;y) - \frac{3}{2}H(0,0,-1,0;y) + \frac{13}{4}H(0,0,0,0;y) \\
& \qquad + H(0,0,1,0;y) - 2H(1,0,-1,0;y) + 2H(1,0,0,0;y) + 2H(1,0,1,0;y)\Bigr] \\
& + \frac{Q^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigl[-\frac{1}{32}\pi^2 - \frac{1}{12}\pi^4 + \frac{1627}{864}H(0;y) + \frac{29}{72}\pi^2 H(0;y) \\
& \qquad - 2\zeta(3)H(0;y) + \frac{61}{8}H(-1,0;y) + \frac{1}{2}\pi^2 H(0,-1;y) - \frac{93}{16}H(0,0;y) \\
& \qquad - \frac{1}{3}\pi^2 H(0,0;y) - 4H(1,0;y) + \frac{29}{12}H(0,0,0;y) + H(0,-1,0,0;y) \\
& \qquad + 3H(0,0,-1,0;y) - \frac{5}{2}H(0,0,0,0;y) - 2H(0,0,1,0;y) \\
& \qquad + H(0,1,0,0;y)\Bigr] \\
& + \frac{m^6}{(Q^2+4m^2)^2\sqrt{Q^2(Q^2+4m^2)}}\Bigl[\frac{19}{8}\pi^4 + 7\pi^2 H(0;y) - 84\zeta(3)H(0;y)
\end{aligned}
$$

$$
\begin{aligned}
& -15\pi^2 H(0,-1;y) + \frac{11}{2}\pi^2 H(0,0;y) \\
& +12\pi^2 H(1,0;y) + 42H(0,0,0;y) \\
& +126H(0,-1,0,0;y) - 192H(0,0,-1,0;y) \\
& +3H(0,0,0,0;y) + 48H(0,0,1,0;y) \\
& -180H(0,1,0,0;y) + 72H(1,0,0,0;y)\Bigg] \\
& + \frac{m^4}{(Q^2+4m^2)^2}\Bigg[-\frac{41}{9}\pi^2 + 42\zeta(3) + \frac{15}{2}\pi^2 H(-1;y) + \frac{1}{4}\pi^2 H(0;y) + \frac{494}{9}H(0,0;y) \\
& -63H(-1,0,0;y) + 96H(0,-1,0;y) + \frac{33}{2}H(0,0,0;y) \\
& -24H(0,1,0;y) + 90H(1,0,0;y)\Bigg] \\
& + \frac{m^4}{(Q^2+4m^2)\ \sqrt{Q^2(Q^2+4m^2)}}\Bigg[4\pi^2 - \frac{49}{80}\pi^4 + \frac{178}{9}H(0;y) - \frac{7}{4}\pi^2 H(0;y) \\
& +35\zeta(3)H(0;y) + 96H(-1,0;y) + 4\pi^2 H(0,-1;y) \\
& -60H(0,0;y) - \frac{25}{12}\pi^2 H(0,0;y) - 24H(1,0;y) \\
& -5\pi^2 H(1,0;y) - \frac{21}{2}H(0,0,0;y) \\
& -38H(0,-1,0,0;y) + 37H(0,0,-1,0;y) \\
& +H(0,0,0,0;y) - 4H(0,0,1,0;y) \\
& +35H(0,1,0,0;y) - 30H(1,0,0,0;y)\Bigg] \\
& + \frac{m^2}{Q^2+4m^2}\Bigg[-\frac{49}{9} + 3\pi^2\ln 2 - \frac{17}{12}\pi^2 - \frac{61}{1440}\pi^4 - \frac{11}{2}\zeta(3) - \frac{15}{4}\pi^2 H(-1;y) \\
& +\frac{35}{24}\pi^2 H(0;y) - \frac{1}{2}\zeta(3)H(0;y) - \zeta(3)H(1;y) - \frac{1}{12}\pi^2 H(-1,0;y) \\
& -\frac{71}{3}H(0,0;y) - \frac{7}{24}\pi^2 H(0,0;y) - \frac{1}{6}\pi^2 H(1,0;y) + \frac{7}{2}H(-1,0,0;y) \\
& -15H(0,-1,0;y) + 4H(0,0,0;y) + 5H(0,1,0;y) - \frac{17}{2}H(1,0,0;y) \\
& -2H(-1,-1,0,0;y) - H(-1,0,-1,0;y) + \frac{3}{2}H(-1,0,0,0;y) \\
& +H(0,-1,0,0;y) + \frac{3}{2}H(0,0,-1,0;y) - \frac{13}{4}H(0,0,0,0;y) \\
& -H(0,0,1,0;y) + 2H(1,0,-1,0;y) - 2H(1,0,0,0;y) \\
& -2H(1,0,1,0;y)\Bigg] \\
& + \frac{m^2}{\sqrt{Q^2(Q^2+4m^2)}}\Bigg[-\frac{2}{3}\pi^2 - \frac{11}{90}\pi^4 + \frac{19}{108}H(0;y) - \frac{23}{72}\pi^2 H(0;y) - 7H(0;y)\zeta(3) \\
& -4H(-1,0;y) + \frac{1}{2}\pi^2 H(0,-1;y) + H(0,0;y) - \frac{1}{2}\pi^2 H(0,0;y) \\
& -2H(1,0;y) + \frac{1}{3}\pi^2 H(1,0;y) - \frac{23}{12}H(0,0,0;y) + 5H(0,-1,0,0;y) \\
& +3H(0,0,-1,0;y) - \frac{11}{2}H(0,0,0,0;y) - 4H(0,0,1,0;y)
\end{aligned}
$$

$$
\begin{aligned}
& -H(0,1,0,0;y)+2H(1,0,0,0;y)\Big] \\
& + \frac{1171}{216} - \frac{1}{2}\pi^2\ln 2 + \frac{7}{16}\pi^2 + \frac{61}{1440}\pi^4 - \frac{9}{4}\zeta(3) + \frac{1}{2}\pi^2 H(-1;y) - \frac{5}{16}\pi^2 H(0;y) \\
& + \frac{1}{2}\zeta(3)H(0;y) + \zeta(3)H(1;y) + \frac{1}{12}\pi^2 H(-1,0;y) + \frac{533}{72}H(0,0;y) + \frac{7}{24}\pi^2 H(0,0;y) \\
& + \frac{1}{6}\pi^2 H(1,0;y) - \frac{1}{2}H(-1,0,0;y) + \frac{17}{4}H(0,-1,0;y) - \frac{9}{8}H(0,0,0;y) - \frac{5}{2}H(0,1,0;y) \\
& + 4H(1,0,0;y) + 2H(-1,-1,0,0;y) + H(-1,0,-1,0;y) - \frac{3}{2}H(-1,0,0,0;y) \\
& - H(0,-1,0,0;y) - \frac{3}{2}H(0,0,-1,0;y) + \frac{13}{4}H(0,0,0,0;y) + H(0,0,1,0;y) \\
& - 2H(1,0,-1,0;y) + 2H(1,0,0,0;y) + 2H(1,0,1,0;y)\ ; \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
F_2^{(2)}(-Q^2) \quad = \quad & + \ln\left(\frac{\lambda}{m}\right)\left\{\frac{m^2}{Q^2}\Big[H(0,0;y)\Big] + \frac{m^2}{Q^2+4m^2}\Big[H(0,0;y)\Big] + \frac{m^2}{\sqrt{Q^2(Q^2+4m^2)}}\Big[H(0;y)\Big]\right\} \\
& + \frac{m^4}{Q^2\ \sqrt{Q^2(Q^2+4m^2)}}\Big[- \frac{23}{480}\pi^4 + \frac{3}{4}\pi^2 H(0,-1;y) - \frac{7}{24}\pi^2 H(0,0;y) \\
& \qquad + \frac{1}{2}H(0,-1,0,0;y) + H(0,0,-1,0;y) - \frac{3}{4}H(0,0,0,0;y)\Big] \\
& + \frac{m^2}{Q^2}\Big[- \frac{5}{2}\zeta(3) + \frac{3}{8}\pi^2 H(-1;y) - \frac{17}{48}\pi^2 H(0;y) - \frac{3}{2}H(0,0;y) + \frac{5}{4}H(-1,0,0;y) \\
& \qquad - \frac{13}{8}H(0,0,0;y) - H(0,1,0;y) - H(1,0,0;y)\Big] \\
& + \frac{m^6}{(Q^2+4m^2)^2\ \sqrt{Q^2(Q^2+4m^2)}}\Big[- \frac{19}{8}\pi^4 - 7\pi^2 H(0;y) + 84\zeta(3)H(0;y) \\
& \qquad + 15\pi^2 H(0,-1;y) - \frac{11}{2}\pi^2 H(0,0;y) \\
& \qquad - 12\pi^2 H(1,0;y) - 42H(0,0,0;y) \\
& \qquad - 126H(0,-1,0,0;y) + 192H(0,0,-1,0;y) \\
& \qquad - 3H(0,0,0,0;y) - 48H(0,0,1,0;y) \\
& \qquad + 180H(0,1,0,0;y) - 72H(1,0,0,0;y)\Big] \\
& + \frac{m^4}{(Q^2+4m^2)^2}\Big[\frac{13}{3}\pi^2 - 42\zeta(3) - \frac{15}{2}\pi^2 H(-1;y) - \frac{1}{4}\pi^2 H(0;y) - \frac{166}{3}H(0,0;y) \\
& \qquad + 63H(-1,0,0;y) - 96H(0,-1,0;y) - \frac{33}{2}H(0,0,0;y) \\
& \qquad + 24H(0,1,0;y) - 90H(1,0,0;y)\Big] \\
& + \frac{m^4}{(Q^2+4m^2)\ \sqrt{Q^2(Q^2+4m^2)}}\Big[- 4\pi^2 + \frac{199}{480}\pi^4 - \frac{62}{3}H(0;y) + \frac{7}{6}\pi^2 H(0;y) \\
& \qquad - 28\zeta(3)H(0;y) - 96H(-1,0;y) - \frac{11}{4}\pi^2 H(0,-1;y) \\
& \qquad + 60H(0,0;y) + \frac{13}{8}\pi^2 H(0,0;y) + 24H(1,0;y) \\
& \qquad + 4\pi^2 H(1,0;y) + 7H(0,0,0;y) + \frac{55}{2}H(0,-1,0,0;y)
\end{aligned}
$$

$$
\begin{aligned}
& -21H(0,0,-1,0;y) - \frac{5}{4}H(0,0,0,0;y) \\
& \left. -20H(0,1,0,0;y) + 24H(1,0,0,0;y) \right] \\
& + \frac{m^2}{Q^2+4m^2}\left[\frac{17}{3} - 2\pi^2\ln 2 + \frac{1}{6}\pi^2 + \frac{7}{2}\zeta(3) + \frac{17}{8}\pi^2 H(-1;y) - \frac{47}{48}\pi^2 H(0;y) \right. \\
& \quad + \frac{95}{6}H(0,0;y) - \frac{1}{4}H(-1,0,0;y) + 3H(0,-1,0;y) - \frac{19}{8}H(0,0,0;y) \\
& \quad \left. -H(0,1,0;y) - H(1,0,0;y) \right] \\
& + \frac{m^2}{\sqrt{Q^2(Q^2+4m^2)}}\left[\frac{3}{8}\pi^2 + \frac{235}{72}H(0;y) + \frac{11}{12}\pi^2 H(0;y) + \frac{41}{2}H(-1,0;y) \right. \\
& \quad \left. -\frac{57}{4}H(0,0;y) - 8H(1,0;y) + \frac{11}{2}H(0,0,0;y) \right] . \qquad (3.7)
\end{aligned}
$$

4 Behaviour for $Q^2 \to \infty$

We present the behaviour of the two Form Factors for $Q^2 \to \infty$ (corresponding to $y \to 0$), writing for short $L = \ln(Q^2/m^2)$. The analytic continuation to large time-like t can be carried out with the replacements $Q^2 = -(t + i\epsilon)$, $L = \ln(t/m^2) - i\pi$ – so that for instance $L^2 = \ln^2(t/m^2) - \pi^2 - 2i\pi\ln(t/m^2)$.

$$
\begin{aligned}
F_1^{(2)}(-Q^2) \quad = \quad & \ln^2\left(\frac{\lambda}{m}\right)\left\{\frac{1}{2} - L + \frac{1}{2}L^2 \right. \\
& \quad + \left(\frac{m^2}{Q^2}\right)\left(-2 + 2L\right) \\
& \quad + \left(\frac{m^2}{Q^2}\right)^2\left(5 - 5L + 2L^2\right) \\
& \quad + \left(\frac{m^2}{Q^2}\right)^3\left(-\frac{50}{3} + \frac{68}{3}L - 8L^2\right) \\
& \quad \left. + \left(\frac{m^2}{Q^2}\right)^4\left(\frac{196}{3} - \frac{183}{2}L + 32L^2\right)\right\} \\
& + \ln\left(\frac{\lambda}{m}\right)\left\{1 - \frac{1}{12}\pi^2 - \frac{7}{4}L + \frac{1}{12}\pi^2 L + L^2 - \frac{1}{4}L^3 \right. \\
& \quad + \left(\frac{m^2}{Q^2}\right)\left(-\frac{5}{2} + \frac{1}{6}\pi^2 + \frac{7}{2}L - 2L^2\right) \\
& \quad + \left(\frac{m^2}{Q^2}\right)^2\left(6 - \frac{5}{12}\pi^2 - \frac{55}{4}L + \frac{1}{3}\pi^2 L + \frac{29}{4}L^2 - L^3\right) \\
& \quad + \left(\frac{m^2}{Q^2}\right)^3\left(-\frac{212}{9} + \frac{17}{9}\pi^2 + \frac{500}{9}L - \frac{4}{3}\pi^2 L - \frac{97}{3}L^2 + 4L^3\right) \\
& \quad \left. + \left(\frac{m^2}{Q^2}\right)^4\left(\frac{13813}{144} - \frac{61}{8}\pi^2 - \frac{11101}{48}L + \frac{16}{3}\pi^2 L + \frac{1079}{8}L^2 - 16L^3\right)\right\} \\
& + \frac{1171}{216} - \frac{1}{2}\pi^2\ln 2 + \frac{13}{32}\pi^2 - \frac{59}{1440}\pi^4 - \frac{9}{4}\zeta(3) - \frac{1627}{864}L - \frac{13}{144}\pi^2 L + \frac{3}{2}\zeta(3)L \\
& + \frac{229}{288}L^2 - \frac{1}{48}\pi^2 L^2 - \frac{31}{144}L^3 + \frac{1}{32}L^4
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{m^2}{Q^2}\right)\Bigg(-\frac{5113}{432}+3\pi^2\ln 2-\frac{209}{144}\pi^2+\frac{2}{45}\pi^4-\frac{5}{2}\zeta(3)+\frac{283}{48}L-\frac{5}{12}\pi^2 L\\
&\qquad +3\zeta(3)L-\frac{61}{16}L^2+\frac{1}{12}\pi^2L^2+\frac{17}{24}L^3-\frac{1}{48}L^4\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^2\Bigg(\frac{11899}{144}-12\pi^2\ln 2-\frac{101}{144}\pi^2-\frac{503}{720}\pi^4+64\zeta(3)-\frac{3977}{216}L+\frac{175}{36}\pi^2L\\
&\qquad -39\zeta(3)L+\frac{3541}{144}L^2-\frac{23}{24}\pi^2L^2-\frac{43}{18}L^3+\frac{5}{12}L^4\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^3\Bigg(-\frac{272405}{432}+48\pi^2\ln 2+\frac{9929}{216}\pi^2+\frac{227}{36}\pi^4-\frac{1481}{3}\zeta(3)-\frac{1655}{108}L\\
&\qquad -\frac{1061}{36}\pi^2L+304\zeta(3)L-\frac{425}{3}L^2+\frac{17}{2}\pi^2L^2+\frac{133}{9}L^3-\frac{19}{12}L^4\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^4\Bigg(\frac{82660981}{20736}-192\pi^2\ln 2-\frac{689557}{1728}\pi^2-\frac{15409}{360}\pi^4+3056\zeta(3)\\
&\qquad +\frac{446243}{864}L+\frac{3995}{24}\pi^2L-1922\zeta(3)L+\frac{435125}{576}L^2-\frac{673}{12}\pi^2L^2\\
&\qquad -\frac{155}{6}L^3+\frac{143}{24}L^4\Bigg)\\
&+\mathcal{O}\left(\left(\frac{m^2}{Q^2}\right)^5\right)\ . \qquad (4.8)
\end{aligned}
$$

The above formula matches with the sum of Eq.s(2.23) and (2.34-2.35a) of [5], which corresponds to the real part of $F_1^{(2)}(t)$ for large timelike t expanded in m^2/t up to the zeroth order.

$$
\begin{aligned}
F_2^{(2)}(-Q^2) \;=\; & \ln\left(\frac{\lambda}{m}\right)\Bigg\{\left(\frac{m^2}{Q^2}\right)\Bigg(-L+L^2\Bigg)\\
&\qquad +\left(\frac{m^2}{Q^2}\right)^2\Bigg(2+6L-2L^2\Bigg)\\
&\qquad +\left(\frac{m^2}{Q^2}\right)^3\Bigg(11-20L+8L^2\Bigg)\\
&\qquad +\left(\frac{m^2}{Q^2}\right)^4\Bigg(-\frac{134}{3}+\frac{232}{3}L-32L^2\Bigg)\Bigg\}\\
&+\left(\frac{m^2}{Q^2}\right)\Bigg(\frac{17}{3}-2\pi^2\ln 2+\frac{13}{24}\pi^2+\zeta(3)-\frac{235}{72}L+\frac{5}{12}\pi^2L+\frac{1}{24}L^2-\frac{1}{4}L^3\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^2\Bigg(-\frac{1609}{36}+8\pi^2\ln 2+\frac{9}{4}\pi^2+\frac{11}{30}\pi^4-56\zeta(3)+\frac{463}{36}L-3\pi^2L\\
&\qquad +28\zeta(3)L-\frac{205}{12}L^2+\frac{2}{3}\pi^2L^2+\frac{11}{6}L^3-\frac{1}{12}L^4\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^3\Bigg(\frac{33121}{72}-32\pi^2\ln 2-\frac{165}{4}\pi^2-\frac{143}{30}\pi^4+448\zeta(3)+\frac{19}{12}L\\
&\qquad +\frac{125}{6}\pi^2L-252\zeta(3)L+\frac{479}{4}L^2-\frac{22}{3}\pi^2L^2-\frac{71}{6}L^3+\frac{1}{4}L^4\Bigg)\\
&+\left(\frac{m^2}{Q^2}\right)^4\Bigg(-\frac{87919}{27}+128\pi^2\ln 2+\frac{3154}{9}\pi^2+\frac{359}{10}\pi^4-2828\zeta(3)-\frac{6589}{18}L\\
&\qquad -\frac{386}{3}\pi^2L+1680\zeta(3)L-\frac{4085}{6}L^2+51\pi^2L^2+65L^3-\frac{1}{2}L^4\Bigg)
\end{aligned}
$$

$$+\mathcal{O}\left(\left(\frac{m^2}{Q^2}\right)^5\right) . \tag{4.9}$$

5 Behaviour for $Q^2 \to 0$

We present the behaviour of the two Form Factors around $Q^2 = 0$, corresponding to $y = 1$.

$$\begin{aligned}
F_1^{(2)}(-Q^2) \;=\;& \ln^2\left(\frac{\lambda}{m}\right)\left\{\left(\frac{Q^2}{m^2}\right)^2\left(\frac{1}{18}\right)+\left(\frac{Q^2}{m^2}\right)^3\left(-\frac{1}{60}\right)\right\} \\
&+\ln\left(\frac{\lambda}{m}\right)\left\{\left(\frac{Q^2}{m^2}\right)^2\left(\frac{1}{24}\right)+\left(\frac{Q^2}{m^2}\right)^3\left(-\frac{31}{1440}\right)\right\} \\
&+\left(\frac{Q^2}{m^2}\right)\left(\frac{4819}{5184}-\frac{1}{2}\pi^2\ln 2+\frac{49}{432}\pi^2+\frac{3}{4}\zeta(3)\right) \\
&+\left(\frac{Q^2}{m^2}\right)^2\left(-\frac{1889}{6480}+\frac{11}{60}\pi^2\ln 2-\frac{8731}{172800}\pi^2-\frac{11}{40}\zeta(3)\right) \\
&+\left(\frac{Q^2}{m^2}\right)^3\left(\frac{163249}{1814400}-\frac{113}{1680}\pi^2\ln 2+\frac{723901}{33868800}\pi^2+\frac{113}{1120}\zeta(3)\right) \\
&+\mathcal{O}\left(\left(\frac{Q^2}{m^2}\right)^4\right) .
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
F_2^{(2)}(-Q^2) \;=\;& \ln\left(\frac{\lambda}{m}\right)\left\{\left(\frac{Q^2}{m^2}\right)\left(\frac{1}{6}\right)+\left(\frac{Q^2}{m^2}\right)^2\left(-\frac{19}{360}\right)+\left(\frac{Q^2}{m^2}\right)^3\left(\frac{73}{5040}\right)\right\} \\
&+\frac{197}{144}-\frac{1}{2}\pi^2\ln 2+\frac{1}{12}\pi^2+\frac{3}{4}\zeta(3) \\
&+\left(\frac{Q^2}{m^2}\right)\left(-\frac{1751}{2160}+\frac{23}{60}\pi^2\ln 2-\frac{13}{120}\pi^2-\frac{23}{40}\zeta(3)\right) \\
&+\left(\frac{Q^2}{m^2}\right)^2\left(\frac{4357}{15120}-\frac{5}{28}\pi^2\ln 2+\frac{187}{3150}\pi^2+\frac{15}{56}\zeta(3)\right) \\
&+\left(\frac{Q^2}{m^2}\right)^3\left(-\frac{111619}{1209600}+\frac{29}{420}\pi^2\ln 2-\frac{140951}{5644800}\pi^2-\frac{29}{280}\zeta(3)\right) \\
&+\mathcal{O}\left(\left(\frac{Q^2}{m^2}\right)^4\right) .
\end{aligned} \tag{5.11}$$

From the above expansions one can easily recover some familiar results, such as the renormalization condition $F_1^{(2)}(0) = 0$ and the two loop values for the slope $F_1^{(2)\prime}(0)$ (the coefficient of the first term in Q^2 in Eq.(5.10) with an overall minus sign) and of the electron anomaly $F_2^{(2)}(0)$.

6 Acknowledgments

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A The Harmonic Polylogarithms, HPL's

We recall for convenience of the reader the definition of the HPL's [3]. The HPL's form a family of functions depending on an argument, say x, and on a set of indices, say $a_i, i = 1,..,w$ or $\vec{a}$ in more compact notation, where each of the a_i can take one of the three values $1, 0, -1$ and whose number w is called the weight of the HPL. At weight $w = 1$ there are 3 HPL's, defined as

$$\begin{aligned} H(1;x) &= -\ln(1-x) \ , \\ H(0;x) &= \ln x \ , \\ H(-1;x) &= \ln(1+x) \ , \end{aligned} \tag{A-1}$$

whose derivatives can be written as

$$\frac{d}{dx} H(a;x) = f(a;x) \ , \tag{A-2}$$

where the index a can take one of the three values $(1, 0, -1)$ and the rational factors $f(a;x)$ are given by

$$\begin{aligned} f(1;x) &= \frac{1}{1-x} \ , \\ f(0;x) &= \frac{1}{x} \ , \\ f(-1;x) &= \frac{1}{1+x} \ . \end{aligned} \tag{A-3}$$

At weight $w > 1$, if all the w indices are equal to 0 let us indicate them by $\vec{0}_w$ and define correspondingly

$$H(\vec{0}_w;x) = \frac{1}{w!} \ln^w x \ ; \tag{A-4}$$

in all the other cases (*i.e.* when the indices are not all equal to zero), let us indicate any set of w indices by $(a, \vec{b})$, where a can take one of the three values $(1, 0, -1)$ and $\vec{b}$ stands for the set of the other $w - 1$ indices, and define correspondingly

$$H(a, \vec{b}; x) = \int_0^x dx' f(a;x') H(\vec{b};x') \ . \tag{A-5}$$

Note that in full generality (*i.e.* also when all the indices are equal to 0) one has

$$\frac{d}{dx} H(a, \vec{b}; x) = f(a;x) H(\vec{b};x) \ , \tag{A-6}$$

which can also be written as the equivalent indefinite integration formula

$$\int^x dx' f(a;x') \ H(\vec{b};x') = A + \ H(a, \vec{b}; x) \ , \tag{A-7}$$

where A is an integration constant.

Further, the product of two HPL's of the a same argument x and weights p, q can be expressed as a combination of HPL's of argument x and weight $r = p + q$, according to the product identity

$$H(\vec{p};x) H(\vec{q};x) \ = \sum_{\vec{r} = \vec{p} \uplus \vec{q}} H(\vec{r};x) \ , \tag{A-8}$$

where $\vec{p}, \vec{q}$ stand for the p and q components of the indices of the two HPL's, while $\vec{p} \uplus \vec{q}$ represents all possible mergers of $\vec{p}$ and $\vec{q}$ into the vector $\vec{r}$ with r components, in which the relative orders of the elements of $\vec{p}$ and $\vec{q}$ are preserved. The simplest cases of the above identities are

$$\begin{aligned} H(a;x) \ H(b;x) &= H(a,b;x) + \ H(b,a;x) \ , \\ H(a;x) \ H(b,c;x) &= H(a,b,c;x) + \ H(b,a,c;x) + \ H(b,c,a;x) \ ; \end{aligned} \tag{A-9}$$

more complicated cases are immediately established recursively (all the above formulae can indeed easily be checked by differentiating, repeatedly when needed, with respect to x). All the 3^w HPL's of weight w are linearly independent; Eq.(A-9) can however be used for replacing an HPL of weight w with products of HPL's of lower weight (but such that the sum of their weights is equal to w) and other HPL's of weight w.

In the analytic continuation from the argument y in the range $(0,1)$ to $y = -x + i\epsilon$, where x is again in the range $(0,1)$, one finds that any HPL with rightmost index equal to 0 does develop an imaginary part. It is therefore convenient to exploit Eq.(A-8) for expressing those HPL's in terms of HPL's having either no 0's on the right or only 0's as indices. One has for instance

$$H(-1,0;y) = H(-1;y)H(0;y) - H(0,-1;y) \ , \tag{A-10}$$

and similar formulae for more general cases. From the very definition one has further $H(0;-x+i\epsilon) = H(0;x) + i\pi$, and $H(-1;-x+i\epsilon) = H(-1;-x) = -H(1;x)$, $H(0,-1;-x+i\epsilon) = -H(0,1;x)$, (those last two functions have no imaginary part in that range of values of x), so that finally one obtains

$$H(-1,0;-x+i\epsilon) = -H(0;x)H(1;x) + H(0,1;x) + i\pi H(1;x) \ . \tag{A-11}$$

For a more complete discussion see [3,4].

Of particular interest are the values of the HPL's of argument equal to 1. They can be expressed as combination with rational factors of a limited number of mathematical constants, such as for instance Riemann ζ-functions, π^2 and $\ln 2$.

B The definite integrals occurring in the calculation

The imaginary parts of the form factors at two loops contain Nielsen's polylogarithms of maximum weight 3 and arguments $x', -x', x'^2$, which can be expressed in terms of HPL's of argument x'. Indeed, one has $Li_2(x) = H(0,1;x)$, $Li_2(-x) = -H(0,-1;x)$, $Li_3(x) = H(0,0,1;x)$, $Li_3(-x) = -H(0,0,-1;x)$, $S_{12}(x) = H(0,1,1;x)$, $S_{12}(-x) = H(0,-1,-1;x)$ and

$$S_{12}(x^2) = 2H(0,1,1;x) - 2H(0,1,-1;x) - 2H(0,-1,1;x) + 2H(0,-1,-1;x).$$

After the insertion of the explicit analytic expressions of the imaginary parts in Eq.s(2.2,2.3) and full partial fractioning in the integration variable x', the integrands are found to consist of powers of the rational factors x', $1/(1+x')$, $1/(1-x')$, which depend on x' only, and $1/(x'+y)$, $1/(x'+1/y)$, which depend as well on y, times HPL's of argument x' and maximum weight $w = 3$.

The integrals with the factors depending on x' only can be evaluated by parts or by means of the formulas defining the HPL's; they give end-point values of HPL's of argument 1 and maximum weight $w = 4$ (some care may be needed in properly grouping terms whose end-point values are otherwise separately divergent).

For the analytic evaluation of the integrals involving $1/(x'+y)$ and $1/(x'+1/y)$, along the lines of Section 7 of [3], we introduce two families of related functions depending, as the HPL's, on a set of w indices and the argument y, defined for $w = 1$ as

$$F(-1;y) \ := \ \int_0^1 \frac{dx'}{x'+y} = H(-1;y) - H(0;y) \ , \tag{B-1}$$

$$G(-1;y) \ := \ \int_0^1 \frac{dx'}{x'+\frac{1}{y}} = H(-1;y) \ , \tag{B-2}$$

and for $w > 1$ as

$$F(-1,\vec{b};y) \ := \ \int_0^1 \frac{dx'}{x'+y} H(\vec{b};x') \ , \tag{B-3}$$

$$G(-1,\vec{b};y) \ := \ \int_0^1 \frac{dx'}{x'+\frac{1}{y}} H(\vec{b};x') \ . \tag{B-4}$$

Note that the first index of the above functions is frozen to the value -1 (a convention suggested by the $y=1$ limiting value).

It is easy to see, proceeding by induction on w, that the functions $F(-1,\vec{a};y)$ and $G(-1,\vec{a};y)$ with a total of w indices are just homogeneous combination of HPL's functions of weight w of argument y and of their values at 1.

For $w=1$, the result is obvious from the definitions Eq.s(B-1,B-2). Next, assume that the identities are established for $F(-1,\vec{b};y)$ up to a certain weight $w-1\geq 1$ and consider the function of weight w

$$F(-1,a,\vec{b};y)=\int_0^1 \frac{dx'}{x'+y}H(a,\vec{b};x')\ . \tag{B-5}$$

Its values at $y=0,1$, according to the definition of the HPL's Eq.(A-5) are

$$\begin{aligned} F(-1,a,\vec{b};1) &= H(-1,a,\vec{b};1)\ , \\ F(-1,a,\vec{b};0) &= H(0,a,\vec{b};1)\ . \end{aligned} \tag{B-6}$$

By differentiating Eq.(B-5) with respect to y and integrating by parts on x' one obtains

$$\frac{\partial}{\partial y}F(-1,a,\vec{b};y) = f(-1;y)H(a,\vec{b};1)-\int_0^1 \frac{dx'}{x'+y}\ f(a;x')\ H(\vec{b};x')\ . \tag{B-7}$$

(We have assumed $H(a,\vec{b};0)=0$, which is the most frequent case; for $H(a,\vec{b};0)\neq 0$ see the remarks below.) In the case $a=0$, after partial fractioning in x', one has

$$\begin{aligned} \frac{\partial}{\partial y}F(-1,0,\vec{b};y) &= f(-1;y)H(0,\vec{b};1)-f(0;y)\int_0^1 dx'\left(f(0;x')-\frac{1}{x'+y}\right)\ H(\vec{b};x') \\ &= f(-1;y)H(0,\vec{b};1)-f(0;y)\Big(H(0,\vec{b};1)-F(-1,\vec{b};y)\Big)\ , \end{aligned} \tag{B-8}$$

where Eq.(A-5) and Eq.(B-3) have been used to carry out the x'-integration.

For $a=1,-1$ one obtains similarly

$$\frac{\partial}{\partial y}F(-1,1,\vec{b};y) = -f(-1;y)F(-1,\vec{b};y)\ , \tag{B-9}$$

$$\frac{\partial}{\partial y}F(-1,-1,\vec{b};y) = f(-1;y)H(-1,\vec{b};1)+f(1;y)\Big(H(-1,\vec{b};1)-F(-1,\vec{b};y)\Big) \tag{B-10}$$

In the *r.h.s.* of the previous equations appears the function $F(-1,\vec{b};y)$ of weight w; when proceeding by induction in w one can substitute the already obtained identities expressing it in terms of H's of weight $w-1$ and argument y. A final quadrature in y, carried out according to Eq.(A-5), gives $F(-1,a,\vec{b};y)$ in terms of HPL's of weight w up to an additive constant, which can be fixed by one of the Eq.s(B-6).

It is not difficult, following the above lines, to work out all the formulas needed for the calculation. As an example, we give one of the identities of weight $w=4$,

$$\begin{aligned} F(-1,1,0,0;x) &= +\frac{1}{6}H(-1;x)H(0;x)H(0;x)H(0;x)+2H(-1;x)H(0,-1;1)H(0;x) \\ &\quad -H(-1;x)H(0,0,-1;x)+H(-1;1)H(0,0,-1;1)+H(-1;1)H(0,0,1;1) \\ &\quad -\frac{1}{2}H(0,-1;x)H(0;x)H(0;x)+\frac{1}{2}H(0,-1;x)H(0,-1;x) \\ &\quad -2H(0,-1;1)H(0,-1;x)+\frac{3}{2}H(0,-1;1)H(0,-1;1)+H(0,0,-1;x)H(0;x) \\ &\quad -H(0,0,0,-1;x)+H(0,0,0,-1;1)-H(0,0,1,-1;1)\ . \end{aligned} \tag{B-11}$$

Essentially the same procedure applies to the other family of functions $G(-1,\vec{a};x)$ as well.

For particular values of the arguments, the end-point values occurring in the derivation may give divergent contributions when taken separately; in those cases, one can parameterize the end-point singularities by integrating on x' from δ to $1-\eta$ and then take the $\delta,\eta\to 0$ limit in the final result (which is of course finite).

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