Heavy mass expansion, light-by-light scattering and the anomalous magnetic moment of the muon

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Contributions from light-by-light scattering to $(g_\mu - 2)/2$, the anomalous magnetic moment of the muon, are mediated by the exchange of charged fermions or scalar bosons. Assuming large masses M for the virtual particles and employing the technique of large mass expansion, analytical results are obtained for virtual fermions and scalars in the form of a series in $(m_\mu/M)^2$. This series is well convergent even for the case $M = m_\mu$. For virtual fermions, the expansion confirms published analytical formulae. For virtual scalars, the result can be used to evaluate the contribution from charged pions. In this case our result confirms already available numerical evaluations, however, it is significantly more precise.

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High precision measurements of the muon anomalous magnetic moment, $a_\mu = (g_\mu - 2)/2$, are used for stringent quantitative tests of the theories suggested for describing particle interactions. Presently the world average of the muon anomalous magnetic moment has a relative precision of 0.7 parts per million (ppm) [1, 2]

$$a_\mu(\mathrm{exp}) = 11659203(8) \times 10^{-10}\,. \qquad (1)$$

and a further reduction of the experimental error by a factor two is within reach.

In the standard model (SM) the theoretical value of a_μ is given by a sum of three contributions: $a_\mu(\mathrm{SM}) = a_\mu(\mathrm{QED}) + a_\mu(\mathrm{weak}) + a_\mu(\mathrm{had})$. The contribution from QED, $a_\mu(\mathrm{QED})$, which includes those from virtual leptons, and the one from weak interactions, $a_\mu(\mathrm{weak})$, can be uniquely evaluated in perturbation theory with the results: $a_\mu(\mathrm{QED}) = 11658470.57(0.29) \times 10^{-10}$ [3] and $a_\mu(\mathrm{weak}) = 15.1(0.4) \times 10^{-10}$ [4]. The hadronic piece $a_\mu(\mathrm{had})$, however, is sensitive to long distance effects and cannot be evaluated in a perturbative framework. Using the experimental value from Eq. (1) one expects for the remaining hadronic contribution $a_\mu(\mathrm{had}) = a_\mu(\mathrm{exp}) - a_\mu(\mathrm{QED}) - a_\mu(\mathrm{weak}) = 717(8) \times 10^{-10}$. To test the standard model and to search for "new physics", this value must be reproduced by a precise evaluation of the effects to be discussed in the following.

The dominant contributions to $a_\mu(\mathrm{had})$ are those from the hadronic vacuum polarisation in lowest order (one particle irreducible part) with most recent results $a_\mu(\mathrm{had,LO}) = (702 \pm 6 \pm 14) \times 10^{-10}$ [5], $a_\mu(\mathrm{had,LO}) = 692(6) \times 10^{-10}$ [6], $a_\mu(\mathrm{had,LO}) = 683.6(8.6) \times 10^{-10}$ [7], $a_\mu(\mathrm{had,LO}) = (683 \pm 5.9 \pm 2.0) \times 10^{-10}$ [8] and the range between $a_\mu(\mathrm{had,LO}) = (684.7 \pm 6.0_{\mathrm{exp}} \pm 3.6_{\mathrm{rad}}) \times 10^{-10}$ and $a_\mu(\mathrm{had,LO}) = (709.0 \pm 5.1_{\mathrm{exp}} \pm 1.2_{\mathrm{rad}} \pm 2.8_{\mathrm{SU(2)}}) \times 10^{-10}$ for analyses based on e^+e^- and τ data sets respectively [9].

The next-to-leading order receives one contribution from the reiteration of the hadronic vacuum polarisation which is known to be negative and can be calculated unambiguously and with sufficient precision, $a_\mu(\mathrm{had,NLO;pol}) = -10.1(0.6) \times 10^{-10}$ [10]. Its sign is strictly fixed: the kernel is negative definite [11] and the spectral density of the two-point correlator of the hadronic electromagnetic current is positive. Another NLO contribution originates from light-by-light scattering which cannot be calculated on the basis of data or first principles. The evaluations of this term have changed dramatically during last years [4, 12–15], with $a_\mu(\mathrm{had,NLO;lbl}) = 8.6(3.2) \times 10^{-10}$ as the present conservative estimate based on the evaluation of hadronic formfactors and resonance contributions [1, 15]. It is remarkable that the approach of ref. [14] based on the concept of quark-hadron duality with an effective quark mass accounting for the QCD long distance effects leads to $a_\mu(\mathrm{had,NLO,lbl})_{\mathrm{duality}} = 14(3) \times 10^{-10}$ in good agreement with the evaluations based on models for the hadronic formfactors.

One important part of light-by-light-scattering amplitudes originates from neutral, low mass intermediate resonances, dominantly π^0 and, less important, η, another from charged pion loops. This second term is the main subject of the present paper.

In view of the low mass of the muon and pion, as compared to the characteristic scale of the pion form factor, $m_\rho^2 \approx 0.6\ \mathrm{GeV}^2$, pions can be treated as pointlike. The corresponding problem of loops of (pointlike) fermions has been solved in analytical form, with the final result expressed in the form of a complicated expression of polylogarithms depending on (m_f^2/m_μ^2) [17].

In the present paper, a different approach is adopted, which makes use of the fact that $m_\mu^2/m_\pi^2 \approx 0.6$ is significantly smaller than 1. Using the heavy mass expansion [16], the result is obtained in the form of a power series, which can be systematically constructed up to arbitrary orders in $(m_\mu/m_\pi)^2$. The corresponding expansion can also be constructed for heavy fermions, leptons as well as quarks. The fermionic contribution to light-by-light scattering is given by the 6 diagrams depicted in Fig. 1 (with permutations of legs omitted). The analytical value for this contribution is known in QED for arbitrary values of m and M since long [17]. Also the

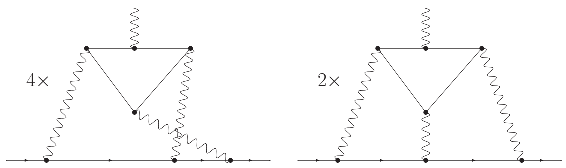

FIG. 1: QED-type light-by-light diagrams

two leading terms in the m^2/M^2 expansion can be found in this reference. Below we reproduce these terms and give three more terms of the expansion. To evaluate the contribution of charged pion loops we consider the pion as an elementary pointlike particle and use the Lagrangian

$$L_{\rm sQED} = |D_\mu \pi|^2 - m_\pi^2 |\pi|^2 \ , \quad D_\mu = \partial_\mu - ieA_\mu \qquad (2)$$

to describe its interaction with photons. This is well justified since the integral is convergent in the high energy region, and the internal structure of the pion is not yet resolved for momenta of order m_μ or m_π.

The 21 diagrams for the light-by-light contribution as derived from scalar QED are displayed in Fig. 2 (permutations of legs omitted). Numerical results for this contribution were already reported in refs. [12, 18]. Analytical expressions, however, are presented in this work for the first time. In view of the fact that this contribution in indeed comparable with the expected experimental error, an independent confirmation seems furthermore, highly desirable.

To compute the contributions of different diagrams we first project onto the relevant form factor of the anomalous magnetic moment [19]. Subsequently we employ the well-known method of asymptotic expansion in the small mass ratio m/M [16]. As an example let us consider the large mass expansion of the generic diagram depicted in Fig. 3. To obtain the asymptotic expansion for a given diagram one has to compute the sum of different contributions which are simpler than the original Feynman integral. These different contributions can be classified according to the so called "hard" subgraphs of the original diagram Γ. These are defined as subgraphs containing all heavy lines such that the corresponding co-subgraph remains one-particle irreducible. The last step of the heavy mass expansion consists of the simple Taylor expansion of the "hard" subgraphs in small masses and external momenta. The generic prototype diagram without the

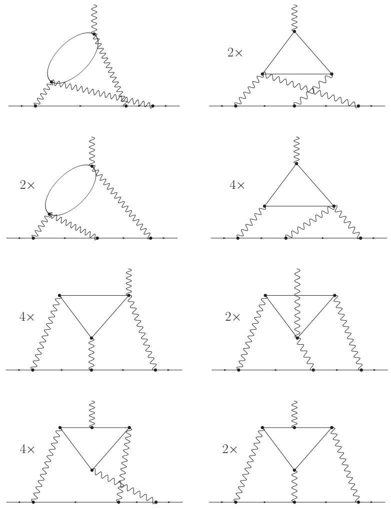

FIG. 2: Scalar QED light-by-light diagrams

external photon line shown in Fig. 3 leads to four different "hard" subgraphs: 1. the graph Γ itself; 2. the subgraph formed by the lines (1,2,3); 3. the subgraph formed by the lines (1,2,3,4,5,7); 4. the subgraph formed by the lines (1,2,3,5,6,8). We now discuss these pieces separately.

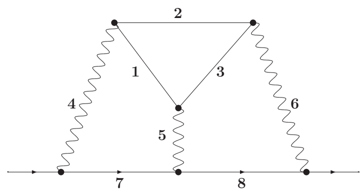

FIG. 3: A prototype diagram. Lines 1, 2 and 3 have the mass of pion, lines 7 and 8 — the mass of muon while lines 4, 5 and 6 are masseless.

1. In this case we perform a Taylor expansion of the diagram in Fig. 3 in the external muon momentum p and the mass m. The resulting tensors $p_{\mu_1}p_{\mu_2}\dots p_{\mu_n}$ can be treated with the use of the combinatoric formula

$$\begin{aligned} p_{\mu_1}p_{\mu_2}\dots p_{\mu_{2n}} &\rightarrow \frac{\Gamma(d/2)}{2^n\Gamma(d/2+n)}(p^2)^n g_{s,(n)} \\ p_{\mu_1}p_{\mu_2}\dots p_{\mu_{2n+1}} &\rightarrow 0 \end{aligned} \qquad (3)$$

where $g_{s,(n)}$ is the totally symmetric product of n metric tensors $g_{\mu\nu}$, d is the space time dimension within dimension regularization. The resulting scalar one-scale 3-loop bubbles where computed using the package MATAD [20].

2. Here we perform the Taylor expansion in the momenta external to the subgraph formed by lines (1,2,3). The resulting symmetric tensors built from the loop momentum of the "hard" one-loop tadpole integrals are expressed by the squared "hard" momentum with the

help of Eq. (3). The remaining scalar one-loop tadpoles are expressed in terms of Γ-functions. Two-loop scalar on-shell integrals are evaluated with the package ONSHELL2 [21].

3. For this subgraph and similarly for subgraph (4) we perform a Taylor expansion in the external momenta and the muon mass m. The resulting tensor one-loop on-shell integrals over soft momenta can be reduced to scalar integrals for which simple explicit results exist (e.g. [22]). Another way to handle the tensor structures that has been used for a check of the calculation is the reduction of the tensor integrals for a hard subgraph (explicit formulae are given in [23]). The remaining one-loop on-shell and two-loop single scale tadpole integrals can be finally reduced to Γ-functions.

The steps described above were automized with the Feynman diagram analyser DIANA [24] and the computer algebra system FORM [25].

To discuss our results we introduce the standard normalization for the light-by-light contribution

$$a_\mu(\mathrm{lbl}) = \left(\frac{\alpha}{\pi}\right)^3 a_\mu(\gamma\gamma)\,. \tag{4}$$

For the fermionic light-by-light contribution we find

$$\begin{aligned}
a_\mu(\gamma\gamma;\mathrm{QED}) &= \frac{m^2}{M^2}\left(\frac{3}{2}\zeta_3 - \frac{19}{16}\right)\\
&+\frac{m^4}{M^4}\left(\frac{13}{18}\zeta_3 - \frac{161}{1620}\zeta_2 - \frac{831931}{972000} - \frac{161}{3240}L^2 - \frac{16189}{97200}L\right)\\
&+\frac{m^6}{M^6}\left(\frac{17}{36}\zeta_3 - \frac{13}{224}\zeta_2 - \frac{1840256147}{3556224000}\right.\\
&\left.-\frac{4381}{120960}L^2 - \frac{24761}{317520}L\right)\\
&+\frac{m^8}{M^8}\left(\frac{7}{20}\zeta_3 - \frac{2047}{54000}\zeta_2 - \frac{453410778211}{1200225600000}\right.\\
&\left.-\frac{5207}{189000}L^2 - \frac{41940853}{952560000}L\right)\\
&+\frac{m^{10}}{M^{10}}\left(\frac{5}{18}\zeta_3 - \frac{1187}{44550}\zeta_2 - \frac{86251554753071}{287550049248000}\right.\\
&\left.-\frac{328337}{14968800}L^2 - \frac{640572781}{23051952000}L\right) + O\left(\frac{m^{12}}{M^{12}}\right)
\end{aligned} \tag{5}$$

where $L = \ln(M^2/m^2)$, m and M denoting muon and fermion mass respectively, $\zeta_2 = \zeta(2) = \pi^2/6$, $\zeta_3 = \zeta(3)$. The first two terms of this expansion coincide with the result given explicitly in ref. [17], the other terms are new [26].

The expansion in scalar QED, relevant for the charged pion contribution, has the form

$$\begin{aligned}
a_\mu(\gamma\gamma;\mathrm{sQED}) &= \frac{m^2}{M^2}\left(\frac{1}{4}\zeta_3 - \frac{37}{96}\right)\\
&+\frac{m^4}{M^4}\left(\frac{1}{8}\zeta_3 + \frac{67}{6480}\zeta_2 - \frac{282319}{1944000} + \frac{67}{12960}L^2 + \frac{7553}{388800}L\right)\\
&+\frac{m^6}{M^6}\left(\frac{19}{216}\zeta_3 + \frac{157}{36288}\zeta_2 - \frac{767572853}{7112448000}\right.\\
&\left.+\frac{1943}{725760}L^2 + \frac{51103}{7620480}L\right)\\
&+\frac{m^8}{M^8}\left(\frac{11}{160}\zeta_3 + \frac{943}{432000}\zeta_2 - \frac{3172827071}{37507050000}\right.\\
&\left.+\frac{8957}{6048000}L^2 + \frac{22434967}{7620480000}L\right)\\
&+\frac{m^{10}}{M^{10}}\left(\frac{17}{300}\zeta_3 + \frac{139}{111375}\zeta_2 - \frac{999168445440307}{14377502462400000}\right.\\
&\left.+\frac{128437}{149688000}L^2 + \frac{1033765301}{691558560000}L\right) + O\left(\frac{m^{12}}{M^{12}}\right).
\end{aligned} \tag{6}$$

These results where obtained in a general covariant gauge, thus providing additional checks at different steps of the calculation.

Let us discuss some general features of both series. The leading term does not contain logarithms of the mass ratio and represents therefore a pure hard contribution. It is obtained from the direct expansion of the graph in the small parameters of the problem: the muon momentum p and the muon mass m. The integral is infrared finite and no soft subgraphs appear in this order. It is represented by a one-scale integral with the scale given by M. In the next order of m^2/M^2 soft subgraphs appear and, consequently, logarithms of the mass ratio. This is expected on general grounds: A gauge invariant effective action proportional to $(F_{\mu\nu})^4/M^4$ — the Euler-Heisenberg Lagrangian — can be constructed which is induced by heavy particles. In all orders of the expansion the maximal power of the logarithm is two, a consequence of the singularities of the Feynman diagrams, which is reflected in the structure of the expansion.

From Eqs. (5,6) it is apparent that the result is sensitive to loop momenta of order M. Otherwise the leading order would behave as $1/M^4$ and could be easily obtained by using the local Euler-Heisenberg Lagrangian.

Numerically we get for the fermionic contribution ($x = m^2/M^2$)

$$\begin{aligned}
&a_\mu(\gamma\gamma;\mathrm{QED}) = 0.6156x\\
&+(-0.1512 + 0.1666\ln(x) - 0.0497\ln^2(x))x^2\\
&+(-0.0453 + 0.0780\ln(x) - 0.0362\ln^2(x))x^3\\
&+(-0.0194 + 0.0440\ln(x) - 0.0276\ln^2(x))x^4\\
&+(-0.0099 + 0.0278\ln(x) - 0.0219\ln^2(x))x^5 + O(x^5).
\end{aligned} \tag{7}$$

The series seems to converge even at the point $x = 1$ where the sum of five terms $a_\mu(m = M) = 0.616 - 0.151 - 0.045 - 0.019 - 0.010\ldots$ leads to 0.39, which has to be compared with the exact result $a_\mu(m = M)|_{\mathrm{exact}} = 0.3710\ldots$ [17, 27].

In scalar QED the expansion has the following numerical form

$$a_\mu(\gamma\gamma;\mathrm{sQED}) = -0.0849x$$

$$
\begin{aligned}
&+(0.0220 - 0.0194\ln(x) + 0.0052\ln^2(x))x^2 \\
&+(0.0049 - 0.0067\ln(x) + 0.0027\ln^2(x))x^3 \qquad (8)\\
&+(0.0016 - 0.0029\ln(x) + 0.0015\ln^2(x))x^4 \\
&+(0.0007 - 0.0015\ln(x) + 0.0009\ln^2(x))x^5 + O(x^5).
\end{aligned}
$$

At the point $m = M$ one finds $a_\mu(\gamma\gamma;\mathrm{sQED}) = -0.0849 + 0.0220 + 0.0049 + 0.0016 + 0.0007 = -0.0557$ with reasonably decreasing individual terms. In this case the exact result is unknown, it could, however be calculated with the technique of ref. [17]. The difference between the contributions of fermions and scalars is apparent: for scalars the result is negative and almost an order of magnitude smaller in absolute value. The rate of convergence is quite similar in both cases.

In the present case of interest the actual value of the expansion parameter is given by the ratio $(m_\mu/m_\pi)^2 = (106/139.6)^2 = 0.577$. This leads to a rapidly converging series $a_\mu(\gamma\gamma;\mathrm{sQED}) = -0.0490\,(1-0.233-0.037-0.008-0.002) = -0.0353$ with an estimated accuracy better than 0.5%. This value should be compared with the numerical evaluation of ref. [18] where two methods were used with the results $a_\mu(\gamma\gamma;\mathrm{sQED})|_{\mathrm{num},1} = -0.0437(36)$ and $a_\mu(\gamma\gamma;\mathrm{sQED})|_{\mathrm{num},2} = -0.0383(20)$. The slight disagreement between these two values is presumably caused by, quoting the authors of ref. [18], "a delicateness of cancellation when separate contributions are put together". The present approach is based on an analytical evaluation that guarantees the absence of error accumulation. The accuracy of the result is determined by the convergence rate of the series in Eq. (8) which is quite high for physical values of muon and pion masses.

To summarize: A systematic expansion of the light-by-light contribution to the anomalous magnetic moment of the muon has been presented which is valid for virtual fermions and bosons with masses above or equal m_μ. The formulae are sufficiently accurate for the physical applications of interest.

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