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Five-loop vacuum polarization in pQCD: $\mathcal{O}(m_q^2 \alpha_s^4 n_f^2)$ contribution

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Abstract

We present the analytical calculation of the contribution of order $m_q^2 \alpha_s^4 n_f^2$ to the absorptive part of the vacuum polarization function of vector currents. This term constitutes an important gauge-invariant part of the full $\mathcal{O}(m_q^2 \alpha_s^4)$ correction to the total cross-section of e^+e^- annihilation into hadrons. The results are compared to predictions following from various optimization schemes.

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1 Introduction

In QCD the correlator of two currents is a central object from which important physical consequences can be deduced (for a detailed review see, e.g. [1]). In particular, important physical observables like the cross-section of e^+e^- annihilation into hadrons and the decay rate of the Z boson are related to the vector and axial-vector current correlators. Furthermore, total decay rates of CP even or CP odd Higgs bosons can be obtained by considering the scalar and pseudo-scalar current densities, respectively.

From the theoretical viewpoint the two-point correlators are especially suited for evaluations in the framework of perturbative QCD (pQCD) [2]. Indeed, due to the simple kinematics (only one external momentum) even multiloop calculations can be analytically performed. Non-perturbative contributions can be effectively controlled through the the operator product expansion [3, 4]. As a consequence, the results for practically all physically interesting correlators (vector, axial-vector, scalar and pseudo-scalar) are available up to order α_s^2 taking into account the *full* quark mass dependence [5, 6, 7].

In many important cases (with the Z decay rate a prominent example) the external momentum is much larger than the masses of the (active) quarks involved. This justifies to neglect these masses in a first approximation which significantly simplifies the calculation. As a result, in massless QCD the vector and scalar correlators are analytically known to α_s^3 [8, 9, 10]. The residual quarks mass effects can be taken into account via the expansion in quark masses. At present this has been done for the quadratic and quartic terms to the same α_s^3 order [11, 12, 13].

During the past years, in particular through the analysis of Z decays at LEP and of τ decays, an enormous reduction of the experimental uncertainty (down to $\mathcal{O}(10^{-3})$) and in $R(s)$ has been achieved with the perspective of a further reduction by a factor of four at a future linear collider [14].

Inclusion of the $\mathcal{O}(\alpha_s^3)$ corrections [8] is mandatory already now. Quark mass effects as well as corrections specific to the axial current [15, 16, 17] must be included for the case of Z-decays. The remaining theoretical uncertainty from uncalculated higher orders is at present comparable to the experimental one [1]. Thus, the full calculation of the next contributions, those of $\mathcal{O}(\alpha_s^4)$, to $R(s)$ is on agenda.

In massless approximation $R(s)$ is conveniently written as

$$\begin{aligned} R(s) &= \sum_f Q_f^2\, 3 \Bigg(1 + a_s + a_s^2(1.98571 - 0.115295\, n_f) + \\ &\quad a_s^3(-6.63694 - 1.20013\, n_f - 0.00517836\, n_f^2) + a_s^4\, r_0^{V,4} \Bigg) \dots, \end{aligned} \tag{1}$$

where $a_s = \alpha_s(\mu^2 = s)/\pi$ and the standard $\overline{\rm MS}$ renormalization prescription [18] is understood. The a_s^4 term can be further decomposed as a polynomial in n_f, namely

$$r_0^{V,4} = r_{0,0}^{V,4} + r_{0,1}^{V,4}\, n_f + r_{0,2}^{V,4}\, n_f^2 + r_{0,3}^{V,4}\, n_f^3. \tag{2}$$

The term of order $\alpha_s^4\, n_f^3$ (and, in fact, all terms of order $\alpha_s(\alpha_s\, n_f)^n$) have been obtained earlier by summing the renormalon chains [19]. These are technically very simple to compute[2], numerically small ($r_{0,3}^{V,4} = 0.02152$) and not directly sensitive to the nonabelian character of QCD, in contrast to the terms of order $\alpha_s^4 n_f^2$. The first nontrivial contribution — the subleading $\mathcal{O}(n_f^2)$ piece in (2) — have been only recently analytically computed in [20] with the result[3]

$$r_{0,2}^{V,4} = -0.7974. \tag{3}$$

Taken by itself eq. (3) is not of much use for the phenomenology for obvious reasons. However, it gives a strong extra support to the well-known prediction of the full $\mathcal{O}(\alpha_s^4)$ contribution of the Kataev and Starshenko[21] (for a detailed discussion see [22]). The prediction is often used to estimate the theoretical uncertainty in $R(s)$ due to higher order not yet computed perturbative contributions.

Another point of concern are the quark mass effects at order α_s^4. Fortunately, simple estimates (see e.g. [11]) show that, say, the quadratic mass effects are completely under control[4] at the scale of the Z-boson mass since terms of order $\mathcal{O}(\alpha_s^3)$ are available and small. This holds true for the vector correlator with the leading term proportional $\alpha_s m_b^2$ as well as for the axial vector correlator, where the the leading in α_s term is present already at the Born level and thus proportional to m_b^2. Quarks mass effect are getting progressively more important at lower energies. From the conceptual viewpoint it would be important to test the evolution of the strong coupling as predicted by the beta function and the standard QCD matching procedure through a determination of a_s from essentially the same observable, however, at lower energy. The region from several GeV above charm threshold (corresponding to the maximal energy of BEPC around 5.0 GeV) to just below the B meson threshold at around 10.5 GeV corresponding to the "off resonance" measurements of CESR or B-meson factory is particularly suited for this purpose. As a consequence of the favorable error propagation,

$$\delta a_s(s) = \frac{a_s^2(s)}{a_s^2(M_Z^2)}\delta a_s(M_Z^2),$$

the accuracy in the measurement (compared to 91 GeV) may decrease by factor of about 3 or even 4 at 10 and 5.6 GeV respectively, to achieve comparable precision in Λ_{QCD}.

[2] At least for a fixed number of loops.
[3] For brevity we display it in the numerical form.
[4] *Provided* one uses the properly chosen parameterization in terms of the running quark mass.

Technically speaking, the evaluation of the leading in n_f term of order $O(m_q^2\alpha_s^4 n_f^3)$ is again rather simple while the subleading term of order $O(m_q^2\alpha_s^4 n_f^2)$ presents a problem comparable to that in the massless limit. In this work we describe the corresponding calculation and its results. We limit the discussion to the vector current correlator, which is relevant for hadron production through the electromagnetic current. We also compare our results with predictions following from various optimization schemes. Our basic conclusion is once again that the FAC/PMS optimization predictions are remarkably close to reality. They provides a quantitative argument in favour of the corresponding full prediction.

2 Generalities

To fix notation we start considering two-point correlator of vector currents and the corresponding vacuum polarization function ($j_\mu^v = \overline{Q}\gamma_\mu Q$; Q is a quark field with mass m, all other $n_f - 1$ quarks are assumed to be massless)

$$\Pi_{\mu\nu}(q) \quad = \quad i\int \mathrm{d}x e^{iqx}\langle 0|T[\ j_\mu^v(x) j_\nu^v(0)\ ]|0\rangle = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2). \qquad (4)$$

The physical observable $R(s)$ is related to $\Pi(q^2)$ by

$$R(s) = 12\pi\Im\,\Pi(q^2 + i\epsilon). \qquad (5)$$

It is convenient to decompose $R(s)$ into the massless and the quadratic terms

$$R(s) \quad = \quad 3\left\{r_0^V + \frac{m^2}{s}r_2^V\right\} + \ldots = 3\left\{\sum_{i\geq 0} a_s^i\left(r_0^{V,i} + \frac{m^2}{s}r_2^{V,i}\right)\right\} + \ldots.$$

Here we have set the normalization scale $\mu^2 = s$; $a_s = \alpha_s(s)/\pi$ and dots stand for corrections proportional to m^4 and higher powers of the quark mass.

For the calculation of $r_0^{V,4}$ we had to deal with divergent parts of five-loop diagrams and finite parts of the four-loop ones (see, e.g the corresponding discussion in [9, 23]). In fact, as was first discovered in Ref. [11], the quadratic mass correction $r_2^{V,4}$ can be obtained with the help of renormalization group methods *exclusively* from the four-loop function Π_2 defined as ($a_s = \alpha_s(\mu^2)$)

$$\Pi = \frac{3}{16\pi^2}\left(\Pi_0 + \frac{m^2}{-q^2}\Pi_2\right), \quad \Pi_2 = \sum_{i\geq 0} a_s^i k_2^{V,i}.$$

Indeed, due to an extra factor of m^2 the function Π_2 obeys an *unsubtracted* dispersion relation. As a result the following *homogeneous* RG equation holds:

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \gamma_m(a_s)m\frac{\partial}{\partial m} + \beta(a_s)a_s\frac{\partial}{\partial a_s}\right)\Pi_2 = 0. \qquad (6)$$

Equivalently, since Π_2 is independent of m,

$$\frac{\partial}{\partial L}\Pi_2 = \left(-2\gamma_m(a_s) - \beta(a_s)a_s\frac{\partial}{\partial a_s}\right)\Pi_2, \tag{7}$$

where $L = \ln\frac{\mu^2}{-q^2}$. Once the rhs of eq. (7) is known, one can find the function Π_2 up to a constant contribution which has no effect on $R(s)$. Thus, to compute, say, the $\mathcal{O}(\frac{m^2}{s}a_s^3)$ correction to $R(s)$ one needs to know

- The three-loop QCD β function function (starting at $\mathcal{O}(\alpha_s^2)$) and the quark mass anomalous dimension (starting at $\mathcal{O}(\alpha_s)$): available from [24, 25], each up to order α_s^3.
- The very polarization operator Π_2 at *three* loops (that is of order α_s^2): was computed also long ago in [26].

Altogether, this implies that the $\mathcal{O}(\frac{m^2}{s}a_s^3)$ term in $R(s)$ *could* have been been computed as early as in 1986. But in reality this was done only by 5 years later [12].

From the above discussion it is clear which ingredients are necessary to compute quadratic mass terms in $R(s)$ at $\mathcal{O}(a_s^4)$:

1. The four -loop QCD β function function and quark mass anomalous dimension: available from [27, 28, 29].
2. The polarization operator Π_2 at $four$ loops, including its *constant* part.

3 Results

Using the new technique described in [30, 31, 20] and the parallel version of FORM [32, 33], we have computed the leading and subleading (in n_f) contributions of order α_s^3 to the polarization function Π_2 ($\mathcal{O}(\alpha_s^2)$ results are known from [26])

$$\begin{aligned}\Pi_2 = -8 \quad &- \quad \frac{64}{3}\,a_s + a_s^2\left\{\frac{95}{9}\,n_f + -\frac{18923}{54} - \frac{784}{27}\,\zeta_3 + \frac{4180}{27}\,\zeta_5\right\}\\ &+ \quad a_s^3\left\{\left[-\frac{5161}{1458} - \frac{8}{27}\,\zeta_3\right]\,n_f^2 + \left[\frac{62893}{162} + \frac{424}{27}\,\zeta_3^2 - \frac{4150}{243}\,\zeta_3\right.\right.\\ &\qquad\left.\left. + \frac{20}{3}\,\zeta_4 - \frac{28880}{243}\,\zeta_5\right]\,n_f + k_{2,0}^{[V]3}\right\}\end{aligned} \tag{8}$$

or, numerically,

$$\Pi_2 = -8-21.333\,a_s+a_s^2\,(10.56\,n_f-224.80)+a_s^3\,(-3.896\,n_f^2+274.37\,n_f+k_{2,0}^{V,3}), \tag{9}$$

where we have set $a_s = \alpha_s(-q^2)$, $\mu^2 = -q^2$.

The straightforward use of eqs. (5,7) yields ($a_s = \alpha_s(s), \mu^2 = s$)

$$
\begin{aligned}
r_2^V = 12\frac{\alpha_s}{\pi} \quad &+ \quad a_s^2 \left\{ -\frac{13}{3}\, n_f + \frac{253}{2} \right\} \\
&+ \quad a_s^3 \left\{ \left[\frac{125}{54} - \frac{1}{9}\pi^2 \right] n_f^2 + \left[-\frac{4846}{27} + \frac{17}{3}\pi^2 - \frac{466}{27}\,\zeta_3 \right.\right. \\
&\qquad \left.\left. + \frac{1045}{27}\,\zeta_5 \right] n_f + 2442 - \frac{285}{4}\pi^2 + \frac{490}{3}\,\zeta_3 - \frac{5225}{6}\,\zeta_5 \right\} \\
&+ \quad a_s^4 \left\{ \left[-\frac{2705}{1944} + \frac{13}{108}\pi^2 \right] n_f^3 + \left[\frac{91943}{486} - \frac{1121}{108}\pi^2 + \frac{53}{9}\,\zeta_3^2 \right.\right. \\
&\qquad \left.\left. + \frac{13}{324}\,\zeta_3 - \frac{3610}{81}\,\zeta_5 \right] n_f^2 + r_{2,0}^{V,4}\, n_f + r_{2,1}^{V,4} \right\}. \qquad (10)
\end{aligned}
$$

Numerically

$$
\begin{aligned}
r_2^V = 12\, a_s \quad &+ \quad a_s^2\,(-4.3333\; n_f + 126.5) + a_s^3 \left(1.2182\, n_f^2 - 104.167\, n_f + 1032.14 \right) \\
&+ \quad a_s^4 \left(-0.20345\, n_f^3 + 49.0839\, n_f^2 + \; r_{2,1}^{V,4}\, n_f \; + r_{2,0}^{V,4} \right). \qquad (11)
\end{aligned}
$$

4 Discussion

Let us contrast eq. (8) with predictions based on the optimization methods FAC (Fastest Apparent Convergence) and PMS (Principle of Minimal Sensitivity) [34, 35, 36, 21]. As is well known in many cases FAC and PMS predictions are quite close [37]. To compare with an independent prediction we will also use so-called NNA (Naive NonAbelianization) approach of [38].

In Table 1 we list the FAC and PMS predictions for the function Π_2 (see $\leftarrow$ [40]). From the three entries corresponding to $n_f = 3, 4$ and 5 one easily restores the FAC/PMS prediction for the n_f dependence of the a_s^3 term in Π_2

$$
\begin{aligned}
k_2^{V,3}(\mathrm{FAC}) &= -4.2\, n_f^2 + 277\, n_f - 2886, &(12)\\
k_2^{V,3}(\mathrm{PMS}) &= -4.2\, n_f^2 + 282\, n_f - 2916. &(13)
\end{aligned}
$$

The comparison with (9) shows very good agreement with the calculated terms of order n_f and n_f^2. As a natural next step we will now use the available *exact* information about the coefficients $k_{2,2}^{V,3}$ and $k_{2,1}^{V,3}$ to find $k_{2,0}^{V,3}$ from FAC/PMS prediction for a selected value of $n_f = 3$. The choice seems to be natural as in many cases FAC/PMS predictions made for $n_f = 3$ are in better agreement with the exact results (see. e.g. [21, 37, 22]). We obtain (in fact, very close values are produced with $n_f = 4$ and 5)

$$
k_{2,0}^{V,3}(\mathrm{FAC}, nf = 3) = -2880, \quad k_{2,0}^{V,3}(\mathrm{PMS}, nf = 3) = -2897. \qquad (14)
$$

Now we turn to NNA. The corresponding prescription is to simply make the substitution $n_f \to -3/2\beta_0 = n_f - 33/2$ in the leading term of the n_f expansion.

The result is

$$k_2^{V,3}(\mathrm{NNA}) = -1060.69 + 128.568\, n_f - 3.896\, n_f^2. \tag{15}$$

This prediction is in obvious conflict with the result of the direct calculation of the term linear in n_f.

Table 1: Estimates for the coefficient $k_2^{V,3}$ based on FAC and PMS optimization schemes.

Method	$n_f = 3$	$n_f = 4$	$n_f = 5$
FAC	-2092	-1844	-1604
PMS	-2109	-1856	-1612

5 Conclusions

We have presented the analytical calculation of contributions of order $m_q^2 \alpha_s^4 n_f^2$ to the vacuum polarization function of vector currents. Predictions of FAC, PMS- and NNA optimization methods are tested against the exact results. Good quantitative agreement is found for the first two cases. NNA seemingly predicts only sign and order of magnitude of analytical results.

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