TTP02-44

Five-loop vacuum polarization in pQCD: $O(\alpha_s^4 N_f^2)$ results*

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We give a short presentation of the first genuine QCD five-loop calculations, viz. analytical contributions of order $\alpha_s^4 n_f^2$ and $m_q^2 \alpha_s^4 n_f^2$ to the vacuum polarization function of vector currents. These corrections form an important gauge-invariant subset of the full $\mathcal{O}(\alpha_s^4)$ correction to the total cross-section of e^+e^- annihilation into hadrons.

1. Introduction

In QCD the correlator of two currents is a central object from which important physical consequences can be deduced (for a detailed review see, e.g. [1]). In particular, important physical observables like the cross-section of e^+e^- annihilation into hadrons and the decay rate of the Z boson are related to the vector and axial-vector current correlators. Furthermore, total decay rates of CP even or CP odd Higgs bosons can be obtained by considering the scalar and pseudo-scalar current densities, respectively.

From the theoretical viewpoint the two-point correlators are especially suited for pQCD evaluations. Indeed, due to a simple kinematic (only one external momentum) even multiloop calculations can be analytically performed (whereas the non-perturbative contributions can be effectively put under control with the help of the operator product expansion [2,3]). As a consequence, the results for practically all physically interesting correlators (vector, axial-vector, scalar and pseudo-scalar) are available up to order α_s^2 taking into account the *full* quark mass dependence [4,5].

In many important cases (with the Z decay rate being a prominent example) the momentum transfer is much larger than the masses of (active) quarks involved. This leads to the possibility to use the massless approximation which significantly simplifies the calculation. As a result, in massless QCD the vector and scalar correlators are analytically known to α_s^3 [6–8]. The residual quarks mass effects can be taken into account via the expansion in quark masses. At present this has been done for the quadratic and quartic terms to the same α_s^3 order [9–11].

During the past years, in particular through the analysis of Z decays at LEP and of τ decays, an enormous reduction of the experimental uncer-

*Talk presented at RADCOR/Loops and Legs 2002, Kloster Banz, Germany

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tainty (down to 3%) in $R(s)$ has been achieved with the perspective of a further reduction by a factor of four at a future linear collider [12].

Inclusion of the $\mathcal{O}(\alpha_s^3)$ corrections [6] is thus mandatory already now. Quark mass effects as well as corrections specific to the axial current [13] must be included for the case of Z-decays. The remaining theoretical uncertainty from uncalculated higher orders is at present comparable to the experimental one [1]. Thus, the full calculation of the next $\mathcal{O}(\alpha_s^4)$ contributions to $R(s)$ is on agenda.

A few words about technical aspects: Due to the well-known "optical theorem" $R(s)$ is related to the absorptive part of the vector vacuum polarization function. As was first demonstrated in [14] the absorptive part of the arbitrary (L+1)-loop p-integral (that is a massless Feynman integral depending on exactly one external momentum) is expressible in terms the corresponding (L+1)-loop UV counterterm along with some L-loop p-integrals (the former have be to known *including* their non-divergent finite parts).

The calculation of UV counterterms is simpler than that of the very integral, because in the MS scheme any UV counterterm is polynomial in momenta and masses [15]. This observation was effectively employed in Ref. [16] to simplify considerably the calculation of UV counterterms. The method was further developed and named InfraRed Rearrangement (IRR) in Ref. [17]. It essentially amounts to an appropriate transformation of the IR structure of Feynman amplitudes by setting to zero some external momenta and masses (in some cases after some differentiation is performed with respect to the latter). As a result the calculation of UV counterterms is much simplified by reducing the problem to the evaluation of p-integrals. The method of IRR was extended and generalized in Ref. [14] with the help of so-called R^*-operation. It was proven that any UV counterterm for any (L+1)-loop Feynman integral can be expressed (in purely algebraic way) in terms of pole and finite parts of some (L)-loop p-integrals.

Thus, to compute, say, α_s^3 terms in $R(s)$ one should essentially deal with two separate problems: (i) the reduction to 3-loop p-integrals and (ii) the calculation of the huge number of resulting p-integrals.

The second problem has been solved (for the 3-loop case only!) by elaborating a dedicated program MINCER [18] written in FORM [19], which implements the generic solution [20] of the integration-by-parts recurrence relations to evaluate massless three loop propagators. To generalize the algorithm to the 4-loop case (needed for $\mathcal{O}(\alpha_s^4)$ calculations) seems to be a formidable problem at present. We have used a different method based essentially on a large D-expansion (D is the space-time dimension). For more details see Refs. [21–24].

In original $\mathcal{O}(\alpha_s^3)$ calculations of $R(s)$ [6] the IRR and reduction to p-integrals was performed manually — an error-prone and time-consuming procedure difficult to implement in computer algebra. The problem was completely solved in Ref. [7]. Here a general formula was found, which explicitly expresses the result in terms of well-known renormalization constants and three-loop massless propagators, thus resolving the complicated combinatorics of ultraviolet and infrared subtractions. This formula is valid for any number of loops and is, therefore, an important ingredient of any $\mathcal{O}(\alpha_s^4)$ calculation.

The calculational effort for a full evaluation of $R(s)$ in $\mathcal{O}(\alpha_s^4)$ is enormous. As first step to this task we have computed a gauge invariant subset, namely the terms of order $\alpha_s^4 n_f^2$, where n_f denotes the number of fermion flavours. The terms of order $\alpha_s^4 n_f^3$ (and, in fact, all terms of order $\alpha_s(\alpha_s n_f)^n$) have been obtained earlier by summing the renormalon chains [25]. These are numerically small and not directly sensitive to the nonabelian character of QCD, in contrast to the terms of order $\alpha_s^4 n_f^2$.

This talk presents results for the (diagonal) vector correlator at order $\alpha_s^4 n_f^2$. We also compare our results with some of available predictions following from various "optimization" schemes.

2. Results

To fix notation we start with considering two-point correlator of vector quark currents and corresponding vacuum polarization function ($j_\mu^v =$

$\overline{Q}\gamma_\mu Q$; Q is a quark field with mass m, all other $n_f - 1$ quarks are assumed to be massless)

$$\begin{aligned}\Pi_{\mu\nu}(q) &= i\int \mathrm{d}x e^{iqx}\langle 0|T[\ j^v_\mu(x) j^v_\nu(0)\ ]|0\rangle \\ &= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi(q^2). \qquad (1)\end{aligned}$$

The physical observable $R(s)$ is related to $\Pi(q^2)$ by $R(s) = 12\pi \Im\, \Pi(q^2 + i\epsilon)$. It is convenient to decompose $R(s)$ into the massless and the quadratic in the quark mass contributions as follows:

$$\begin{aligned}R(s) &= 3\left\{r_0^V + \frac{m^2}{s} r_2^V\right\} + \ldots \\ &= 3\left\{\sum_{i\geq 0} a_s^i \left(r_0^{V,i} + \frac{m^2}{s} r_2^{V,i}\right)\right\} + \ldots .\end{aligned}$$

Here we have set the normalization scale $\mu^2 = s$; $a_s = \alpha_s(s)/\pi$ and dots stand for corrections proportional to m^4 and higher powers of the quark mass. Since the function $R(s)$ is known to order α_s^3 from works [6,7] (massless limit) and [9] ($\mathcal{O}(m^2)$ contribution) we give below only the ($\overline{\mathrm{MS}}$-renormalized [26]) $\mathcal{O}(\alpha_s^4)$ results.

$$\begin{aligned}r_0^{V,4} &= \\ &+ n_f^3\left[-\frac{6131}{5832} + \frac{11}{432}\pi^2 + \frac{203}{324}\zeta_3 - \frac{1}{54}\pi^2\zeta_3 + \frac{5}{18}\zeta_5\right] \\ &+ n_f^2\left[\frac{1045381}{15552} - \frac{593}{432}\pi^2 - \frac{40655}{864}\zeta_3 + \frac{11}{12}\pi^2\zeta_3 + \frac{5}{6}\zeta_3^2 - \frac{260}{27}\zeta_5\right] + \ldots, \qquad (2)\end{aligned}$$

$$\begin{aligned}r_2^{V,4} &= \\ &+ n_f^3\left[-\frac{2705}{1944} + \frac{13}{108}\pi^2\right] \\ &+ n_f^2\left[\frac{91943}{486} - \frac{1121}{108}\pi^2 + \frac{13}{324}\zeta_3 + \frac{53}{9}\zeta_3^2 - \frac{3610}{81}\zeta_5\right] + \ldots \qquad (3)\end{aligned}$$

Here dots stand for still unknown terms of order $\alpha_s^4 n_f$ and $\alpha_s^4 n_f^0$. Numerically r_0^V and r_2^V read

$$\begin{aligned}r_0^V &= 1 + a_s + a_s^2(1.986 - 0.1153\, n_f) \\ &+ a_s^3\left(-6.6369 - 1.2001\, n_f - 0.00518\, n_f^2\right) \\ &+ a_s^4\left(0.02152\, n_f^3 - 0.7974\, n_f^2 + n_f\, r_{0,1}^{V,4} + r_{0,0}^{V,4}\right), \qquad (4)\end{aligned}$$

$$\begin{aligned}r_2^V &= 12 a_s + a_s^2(126.5 - 4.33333\, n_f) \\ &+ a_s^3\left(1032.14 - 104.167 n_f + 1.21819 n_f^2\right) \\ &+ a_s^4\left(49.0839 n_f^2 - 0.203453 n_f^3 + n_f\, r_{2,1}^{V,4} + r_{2,0}^{V,4}\right). \qquad (5)\end{aligned}$$

The result for the a_s^4 terms in r_0^V has been already presented in [23], the coefficients of the a_s^4 terms in r_2^V are new.

For calculation of $r_0^{V,4}$ we have had to deal with divergent parts of five-loop diagrams and finite parts of the four-loop ones. In fact, as was first discovered in Ref. [9], the quadratic mass correction $r_2^{V,4}$ can be obtained with the help of renormalization group methods *exclusively* from the four-loop function Π_2 defined as ($a_s = \alpha_s(\mu^2), \mu = -q^2$)

$$\Pi = \frac{3}{16\pi^2}\left(\Pi_0 + \frac{m^2}{-q^2}\Pi_2\right), \quad \Pi_2 = \sum_{i\geq 0} a_s^i k_2^{V,i}.$$

The coefficients $k_2^{V,i}$ for $i = 1, 2$ are known from [27] while our result for $k_2^{V,3}$ reads

$$\begin{aligned}k_2^{V,3} &= \\ &+ n_f^2\left[-\frac{5161}{1458} - \frac{8}{27}\zeta_3\right] \\ &+ n_f\left[\frac{62893}{162} - \frac{4150}{243}\zeta_3 + \frac{424}{27}\zeta_3^2 + \frac{20}{3}\zeta_4 - \frac{28880}{243}\zeta_5\right] + \ldots \qquad (6)\end{aligned}$$

or, numerically,

$$\begin{aligned}\Pi_2 &= -8 - 21.333\, a_s + (10.56\, n_f - 224.80)\, a_s^2 \\ &+ a_s^3(274.37\, n_f - 3.896\, n_f^2 + k_{2,0}^{V,3}). \qquad (7)\end{aligned}$$

3. Discussion

In this section we will compare our results with predictions of FAC/PMS optimization methods. FAC (Fastest Apparent Convergence) and PMS

(Principle of Minimal Sensitivity) methods are both based eventually on the concept of scheme-invariant properties and the idea of the choice of an "optimal" scheme to provide better convergence of the resulting perturbative series [28–31].

For both methods the optimal scheme depends on the physical observable we are dealing with. With FAC it should be a scheme which minimizes (set to zero by construction) all the terms of order α_s^2 and higher, while the PMS scheme is fixed by the requirement that the perturbative expansion for the observable is as insensitive as possible to a change in the scheme. The assumption that a renormalization scheme is in a sense optimal sets certain constraints on not yet computed higher order corrections in any other scheme.

Since the massless case have been already discussed in Ref. [23] we concentrate here on the mass correction. In Table 1 we list the FAC/PMS predictions for the the function Π_2. The predictions have been obtained following the work [32]. From the three entries corresponding to $n_f = 3, 4$ and 5 one easily restores the FAC/PMS prediction for the n_f dependence of the a_s^3 term in Π_2

$$k_2^{V,3}(FAC) = -2886 + 277\, n_f - 4.2\, n_f^2, \quad (8)$$

$$k_2^{V,3}(PMS) = -2916 + 282\, n_f - 4.2\, n_f^2. \quad (9)$$

A comparison to (7) shows a very good agreement with exactly known terms of order n_f and n_f^2.

At last, encouraged by the agreement, we use the approach of [9] to deduce from Table 1 the full result for the quadratic mass correction to $R(s)$. We will do it separately for $n_f = 2, 3, 4, 5$ and 6 respectively (as FAC and PMS predictions are so close one each other we will consider for definiteness the latter case).

$$r_2^V(n_f = 2) = 12\,(\,a_s + 9.819\, a_s^2 + 69.056\, a_s^3 + 371\, a_s^4) \quad (10)$$

$$r_2^V(n_f = 3) = 12\,(\,a_s + 9.459\, a_s^2 + 60.883\, a_s^3 + 271\, a_s^4) \quad (11)$$

$$r_2^V(n_f = 4) = 12\,(\,a_s + 9.097\, a_s^2 + 52.913\, a_s^3 + 179\, a_s^4) \quad (12)$$

$$r_2^V(n_f = 5) = 12\,(\,a_s + 8.736\, a_s^2 + 45.146\, a_s^3 + 96\, a_s^4) \quad (13)$$

$$r_2^V(n_f = 6) = 12\,(\,a_s + 8.375\, a_s^2 + 37.582\, a_s^3 + 21\, a_s^4) \quad (14)$$

Table 1
Estimates for the coefficient $k_2^{V,3}$ based on FAC and PMS optimizations.

Method	$n_f = 3$	$n_f = 4$	$n_f = 5$
FAC	-2092	-1844	-1604
PMS	-2109	-1856	-1612

4. Conclusions

We have presented the results of first genuine QCD five-loop calculations viz. analytical contributions of order $\alpha_s^4 n_f^2$ and $m_q^2 \alpha_s^4 n_f^2$ to the vacuum polarization function of vector currents. The results are tested against predictions of FAC/PMS optimization methods. A good agreement is found.

Our results have direct relevance for the theoretical description of the τ decay rate, including its strange quark mas dependence. For a detailed discussion of the issue see Ref. [33].

The present calculation does not include all possible topological types of diagrams appearing in order α_s^4. Our experience shows that these new topologies can be solved in the same way, given sufficient computer resources. Work in this direction is under way.

The authors are grateful to M. Steinhauser for the carefull reading of the manuscript and very useful comments. This work was supported by the DFG-Forschergruppe *"Quantenfeldtheorie, Computeralgebra und Monte-Carlo-Simulation"* (contract FOR 264/2-1), by INTAS (grant 00-00313), by RFBR (grant 01-02-16171), by Volkswagen Foundation and by the European Union under contract HPRN-CT-2000-00149.

REFERENCES

1. K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, Phys. Rep. 277 (1996) 189.

2. M.A.Shifman, A.I. Vainshtein, V.I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.
3. K.G. Wilson, Phys. Rev. 179 (1969) 1499; D3 (1971) 1818.
4. K.G. Chetyrkin, J.H. Kühn, and M. Steinhauser, Phys. Lett. B 371 (1996) 93; Nucl. Phys. B 482 (1996) 213; Nucl. Phys. B 505 (1997) 40.
5. K.G. Chetyrkin, R. Harlander, and M. Steinhauser, Phys. Rev. D 58 (1998) 014012.
6. S.G. Gorishny, A.L. Kataev and S.A. Larin, Phys. Lett. B 259 (1991) 144;
L.R. Surguladze and M.A. Samuel, Phys. Rev. Lett. 66 (1991) 560; (E) ibid., 2416.
7. K.G. Chetyrkin, Phys. Lett. B 391 (1997) 402.
8. K.G. Chetyrkin, Phys. Lett. B 390 (1997) 309.
9. K.G. Chetyrkin, J.H. Kühn, Phys. Lett. B 248 (1990) 359.
10. K.G. Chetyrkin and J.H. Kuhn, Phys. Lett. B 406 (1997) 102.
11. K.G. Chetyrkin, R.V. Harlander, J.H. Kühn, Nucl. Phys. B 586 (2000) 56.
12. TESLA, Technical Design Report, DESY 2001-011; ECFA 2001-209; Part III, *Physics at an e^+e^- Linear Collider*, R.-D. Heuer, D. Miller, F. Richard, P. Zerwas, eds.
13. B. A. Kniehl, J. H. Kühn, Phys. Lett. B 224 (1990) 229; Nucl. Phys. B 329 (1990) 547.
14. K.G. Chetyrkin and V. A. Smirnov, Phys. Lett. B 144 (1984) 41
15. J.C. Collins, Nucl. Phys. B 92 (1975) 477.
16. A.A. Vladimirov, Teor. Mat. Fiz. 43 (1980) 210.
17. K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Nucl. Phys. B 174 (1980) 345.
18. S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, Preprint NIKHEF-H/91-18 (1991).
19. J.A.M. Vermaseren, Symbolic Manipulation with FORM, Version 2, CAN, Amsterdam, 1991.
20. K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* B192 (1981) 159.
21. P. A. Baikov, Phys. Lett. B385, (1996) 404; Nucl. Instr. & Methods A389 (1997) 347; P. A. Baikov, M. Steinhauser, Comput. Phys. Commun. 115 (1998) 161.
22. P. A. Baikov, Phys. Lett. B474 (2000) 385-388;
23. P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. 88, (2002) 012001 .
24. P.A. Baikov, talk at this workshop.
25. M. Beneke, Nucl. Phys. B 405 (1993) 424.
26. W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D **18**, 3998 (1978).
27. S.G. Gorishny, A.L. Kataev, S.A. Larin, Nuovo Cimento 92 (1986) 117.
28. P. M. Stevenson, Phys. Rev. D 23, 2916 (1981); Phys. Lett. B 100, 61 (1981); Nucl. Phys. B 203, 472 (1982); Phys. Lett. B 231, 65 (1984).
29. G. Grunberg, Phys. Lett. B 95, 70 (1980); Phys. Rev. D 29, 2315 (1984).
30. N.V. Krasnikov, Nucl. Phys. B 192, 497 (1981);
31. A. L. Kataev and V. V. Starshenko, Mod. Phys. Lett. A 10, 235 (1995).
32. K G. Chetyrkin, B. A. Kniehl and A. Sirlin, Phys. Lett. B 402 (1997) 359.
33. P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, "Towards Order α_s^4 Accuracy in τ-decays", in preparation.