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Towards Order α_s^4 Accuracy in τ-decays

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Recently computed terms of orders $\mathcal{O}(\alpha_s^4 n_f^2)$ in the perturbative series for the τ decay rate, and similar (new) strange quark mass corrections, are used to discuss the validity of various optimization schemes. The results are then employed to arrive at improved predictions for the complete terms order $\mathcal{O}(\alpha_s^4)$ and $\mathcal{O}(\alpha_s^5)$ in the massless limit as well as for terms due to the strange quark mass. Phenomenological implications are presented.

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I. INTRODUCTION

The dependence of the τ-decay rate on the strong coupling α_s has been used for a determination of α_s at lower energies, with the most recent results of $0.334 \pm 0.007_{\rm exp} \pm 0.021_{\rm theo}$ and $0.3478 \pm 0.009_{\rm exp} \pm 0.019_{\rm theo}$ by the ALEPH [1] and OPAL [2] collaborations. After evolution up to higher energies these results agree remarkably well with determinations based on the hadronic Z decay rate. In view of the relatively large value of $\alpha_s(M_\tau)$ estimates for the yet unknown terms of higher orders play an important role in current determinations of α_s at low energies. This is in contrast to high energy measurements, where the uncertainty from terms of order α_s^4 of 0.001 to 0.002 [3] is somewhat less or at most comparable to the present experimental errors.

The situation is even more problematic for the determination of the strange quark mass from the Cabbibo suppressed τ decays. Perturbative QCD corrections affecting the m_s^2 term are extremely large and contributions from increasing powers of α_s are barely decreasing, which casts doubts on our ability to extract a reliable result for m_s from this (in principle) clean and straightforward measurement [4–11].

Partial results of order α_s^4 for the absorptive part of the massless vector and scalar correlators have been obtained recently [12], namely terms proportional to n_f^2, where n_f denotes the number of massless fermion species. These allow to test two popular optimization schemes — based on the principles of "minimal sensitivity" (PMS) and of fastest apparent convergence (FAC) [13–15] — which have been used to predict yet uncalulated higher order terms [16, 17].

It will be demonstrated that the predictions of both schemes (coinciding at α_s^4) for the coefficient of order $n_f^2\alpha_s^4$ are in reasonable agreement with our calculations, which are then used to predict the complete fixed order (FO) and the "contour improved" (CI) [18, 19] $\mathcal{O}(\alpha_s^4)$ contributions to the τ decay rate. Employing the four-loop QCD beta

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function in combination with improved α_s^4 terms a rough estimate even for $\mathcal{O}(\alpha_s^5)$ terms can be obtained. The results lead to fairly stable values for α_s consistent with current analysises.

Unfortunately, no essential decrease of the difference between the central values $\alpha_s(M_\tau)$ as obtained with FI and CI approaches is observed after including order α_s^4 and α_s^5 corrections, assuming for the moment that these estimates are indeed correct. On the other hand, the theoretical uncertainty assigned to $\alpha_s(M_\tau)$ *within* each method according to standard techniques does decrease significantly.

The implication of this approach for the extraction of m_s from Cabbibo suppressed decays will be investigated along the same lines. New results will be presented for the terms of order $n_f\alpha_s^3 m_s^2$ in the total rate. In this case the agreement between FAC/PMS predictions and our results is quite encouraging and naturally suggests the use of the former as a reliable prediction for the complete $\alpha_s^3 m_s^2$ term. Following an approach discussed in [17], even a rough estimate $m_s^2\alpha_s^4$ terms can be obtained from these considerations.

However, the rapid increase of the coefficients indicates that the inherent uncertainty of the present m_s determinations will not necessarily decrease with inclusion of the higher orders. As we will see, the situation is somewhat better for the spin one contribution if considered separately.

II. GENERALITIES

We start with the well-known representation [20–24] of the tau-lepton hadronic rate as the contour integral along a circle C of radius $|s| = M_\tau^2$

$$R_\tau = 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\Pi^{[2]}(s) - \frac{2}{M_\tau^2}\Pi^{[1]}(s)\right]. \tag{1}$$

Here $\Pi^{[1]}$ and $\Pi^{[2]}$ are proper flavour combinations of the polarization operators appearing in the decomposition of the correlators of vector and axial vector currents of light quarks

$$\begin{aligned}\Pi^{V/A}_{\mu\nu,ij}(q,m_i,m_j,m,\mu,\alpha_s) &= i\int dx e^{iqx}\langle T[\,j^{V/A}_{\mu,ij}(x)(j^{V/A}_{\nu,ij})^\dagger(0)\,]\rangle \\ &= g_{\mu\nu}\Pi^{[1]}_{ij,V/A}(q^2) + q_\mu q_\nu \Pi^{[2]}_{ij,V/A}(q^2)\end{aligned} \tag{2}$$

with $m^2 = \sum_{f=u,d,s} m_f^2$ and $j^{V/A}_{\mu,ij} = \bar q_i \gamma_\mu(\gamma_5) q_j$. The two (generically different) quarks with masses m_i and m_j are denoted by q_i and q_j respectively.

For the case of the τ-lepton the relevant combinations of quark flavours are $ij = ud$ and $ij = us$. The polarization functions $\Pi^{[l]}_{V/A}$, $\;l = 1,2$ are conveniently represented in the form ($Q^2 = -q^2$)

$$(Q^2)^{(l-2)}\Pi^{[l]}_{us,V/A}(q^2) = \frac{3}{16\pi^2}\Pi^{[l]}_{V/A,0}(\frac{\mu^2}{Q^2},\alpha_s) + \frac{3}{16\pi^2}\sum_{D\geq 2} Q^{-D}\Pi^{[l]}_{V/A,D}(\frac{\mu^2}{Q^2},m_s^2,\alpha_s). \tag{3}$$

Here the first term on the rhs corresponds to the massless limit while the first term in the sum stands for quadratic mass corrections. We neglect the masses of u and d quarks. Therefore in perturbative QCD $\Pi^{[l]}_V = \Pi^{[l]}_A$ and we will often omit the subscript V/A in the following. Current conservation implies: $\Pi^{[1]}_0 = \Pi^{[2]}_0$.

The full tau-lepton hadron rate R_τ can be presented as a sum of spin 1 and spin 0 parts, viz.

$$\begin{aligned}R_\tau^{(1)} &= 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^{(1)}(s) + \Pi^{[1]}(0)/s\right], \\ R_\tau^{(0)} &= 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\Pi^{(0)}(s) - \Pi^{[1]}(0)/s\right].\end{aligned} \tag{4}$$

where

$$\Pi^{(1)} = -\Pi^{[1]}/q^2, \quad \Pi^{(0)} = \Pi^{[2]} + \Pi^{[1]}/q^2. \tag{5}$$

and the contribution of the singularity at the origin (proportional to $\Pi^{[1]}(0)$) has to be included. A nonvanishing value of $\Pi^{[1]}(0)$ is a nonperturbative constant.

On the other hand, the unknown constant drops out if one considers moments

$$R_\tau^{(1,0)k,l}(s_0) = \int_0^{s_0} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau^{(1,0)}}{ds}, \tag{6}$$

with $k \geq 0$, $\ l \geq 1$. (Note that the moments introduced in [19] are related to ours as $R^{kl}_\tau = R^{(1)k,l}_\tau + R^{(0)k,l}_\tau$.)

The decay rate R_τ may be expressed as the sum of different contributions corresponding to Cabibbo suppressed or allowed decay modes, vector or axial vector contributions and the mass dimension of the corrections

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \tag{7}$$

with

$$\begin{aligned} R_V &= \frac{3}{2}|V_{ud}|^2 \left(1 + \delta_0 + \sum_{D=2,4,\dots} \delta_{V,ud,D}\right), \\ R_A &= \frac{3}{2}|V_{ud}|^2 \left(1 + \delta_0 + \sum_{D=2,4,\dots} \delta_{A,ud,D}\right), \\ R_S &= 3|V_{us}|^2 \left(1 + \delta_0 + \sum_{D=2,4,\dots} \delta_{us,D}\right). \end{aligned} \tag{8}$$

Here D indicates the mass dimension of the fractional corrections $\delta_{V/A,ij,D,}$ and $\delta_{ij,D}$ denotes the average of the vector and the axial vector contributions: $\delta_{ij,D} = (\delta_{V,ij,D} + \delta_{A,ij,D})/2$. If a decomposition into different spin/parity contributions is made or a particular pattern of moments is considered then we will use the corresponding obvious generalization of (8). For instance,

$$R^{(1)kl}_{S,V} = a_{kl}\, |V_{us}|^2 \left(1 + \delta^{kl}_0 + \sum_{D=2,4,\dots} \delta^{(1),kl}_{V,us,D}\right), \tag{9}$$

and

$$R^{(0)kl}_{S,V} = |V_{us}|^2 \left(\sum_{D=2,4,\dots} \delta^{(0),kl}_{V,us,D}\right). \tag{10}$$

Thus, in our notation we have the relation

$$\delta^{kl}_{V,us,2} = a_{kl}\delta^{(1),kl}_{V,us,2} + \delta^{(0),kl}_{V,us,2}. \tag{11}$$

The integral in eq. (1) is, obviously, insensitive, to the Q^2-independent terms in the polarization functions $\Pi^{[1]}_0$ and $\Pi^{[1]}_2$. This means that without loss of generality we may deal with the corresponding (Adler) D-functions, viz.

$$D^{[1]}_0(Q^2) \equiv -\frac{3}{4}Q^2\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi^{[1]}_0, \quad D^{[1]}_2(Q^2) \equiv -\frac{1}{2}Q^2\frac{\mathrm{d}}{\mathrm{d}Q^2}\Pi^{[1]}_2.$$

An important property of the functions $D^{[1]}_0, D^{[1]}_2$ and $\Pi^{[2]}_2$ is their scale independence, which implies that they are directly related to measurements.

The Adler functions $D^{[1]}_0$ and $D^{[1]}_2$ have been calculated with $\mathcal{O}(\alpha_s^3)$ accuracy, the polarization function $\Pi^{[2]}_2$, however, to $\mathcal{O}(\alpha_s^2)$ only (see [7] and references therein).

The (apparent) convergence of the perturbative series for $D^{[1]}_0$ is acceptable, the one for $D^{[1]}_2$ and $\Pi^{[2]}_2$ is at best marginal. This has led to significant theoretical uncertainties in extracting α_s and to a fairly unstable behaviour in extraction of m_s from τ-decays.

To improve the situation, we have computed the two leading terms in the large n_f expansion of the next order, i.e. terms of order $\mathcal{O}(n_f^3\alpha_s^4)$, $\mathcal{O}(n_f^2\alpha_s^4)$ to $D^{[1]}_0$, $D^{[1]}_2$ and $\mathcal{O}(n_f^2\alpha_s^3)$, $\mathcal{O}(n_f\alpha_s^3)$ to $\Pi^{[2]}_2$. Our results are described in the following section.

III. FIXED ORDER RESULTS IN α_s^3 AND α_s^4

Using the technique described in [12, 25, 26] and the parallel version of FORM [27, 28], the leading and subleading (in n_f) terms of the next order in the perturbative series for $D^{[1]}_0$, $D^{[1]}_2$ and $\Pi^{[2]}_2$ have been obtained in the standard

$\overline{\mathrm{MS}}$ renormalization scheme [29, 30]:

$$
\begin{aligned}
D_0^{[1]} &= 1 + a_s + a_s^2 \left\{ \left[-\frac{11}{12} + \frac{2}{3}\,\zeta_3 \right] n_f + \frac{365}{24} - 11\,\zeta_3 \right\} \\
&+ a_s^3 \left\{ \left[\frac{151}{162} - \frac{19}{27}\,\zeta_3 \right] n_f^2 + \left[-\frac{7847}{216} + \frac{262}{9}\,\zeta_3 - \frac{25}{9}\,\zeta_5 \right] n_f \right. \\
&\qquad \left. + \frac{87029}{288} - \frac{1103}{4}\,\zeta_3 + \frac{275}{6}\,\zeta_5 \right\} \\
&+ a_s^4 \left\{ \left[-\frac{6131}{5832} + \frac{203}{324}\,\zeta_3 + \frac{5}{18}\,\zeta_5 \right] n_f^3 + \left[\frac{1045381}{15552} + \frac{5}{6}\,\zeta_3^2 \right. \right. \\
&\qquad \left. \left. - \frac{40655}{864}\,\zeta_3 - \frac{260}{27}\,\zeta_5 \right] n_f^2 + d_{0,1}^{[1]4}\, n_f + d_{0,0}^{[1]4} \right\} \\
&= 1 + a_s + a_s^2 \{-0.1153\, n_f + 1.986\} \\
&+ a_s^3 \left\{0.08621\, n_f^2 - 4.216\, n_f + 18.24\right\} \\
&+ a_s^4 \left\{ -0.01009\, n_f^3 + 1.875\, n_f^2 + d_{0,1}^{[1]4}\, n_f + d_{0,0}^{[1]4} \right\},
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
D_2^{[1]} = m_s^2 \Bigg(1 &+ \frac{5}{3}\,a_s + a_s^2 \left\{ \left[-\frac{11}{8} + \frac{2}{3}\,\zeta_3 \right] n_f + \frac{5185}{144} - \frac{39}{2}\,\zeta_3 \right\} \\
&+ a_s^3 \left\{ \left[\frac{8671}{11664} - \frac{13}{27}\,\zeta_3 \right] n_f^2 + \left[-\frac{44273}{972} + \frac{3257}{81}\,\zeta_3 \right. \right. \\
&\qquad \left. \left. - \frac{5}{6}\,\zeta_4 - \frac{1265}{81}\,\zeta_5 \right] n_f + \frac{2641517}{5184} - \frac{131275}{216}\,\zeta_3 + \frac{12845}{36}\,\zeta_5 \right\} \\
&+ a_s^4 \left\{ \left[-\frac{396781}{559872} + \frac{461}{1296}\,\zeta_3 - \frac{1}{48}\,\zeta_4 + \frac{5}{18}\,\zeta_5 \right] n_f^3 + \left[\frac{61913567}{1119744} - \frac{59}{54}\,\zeta_3^2 \right. \right. \\
&\qquad \left. \left. - \frac{352549}{7776}\,\zeta_3 + \frac{67}{96}\,\zeta_4 + \frac{22859}{3888}\,\zeta_5 \right] n_f^2 + d_{2,1}^{[1]4}\, n_f + d_{2,0}^{[1]4} \right\} \Bigg) \\
= m_s^2 \Bigg(1 &+ 1.667\, a_s + a_s^2 \{-0.5736\, n_f + 12.57\} \\
&+ a_s^3 \left\{0.1646\, n_f^2 - 14.31\, n_f + 149.\right\} \\
&+ a_s^4 \left\{ -0.01563\, n_f^3 + 6.067\, n_f^2 + d_{2,1}^{[1]4}\, n_f + d_{2,0}^{[1]4} \right\} \Bigg),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\Pi_2^{[2]} = -4m_s^2 \Bigg(1 &+ \frac{7}{3}\,a_s + a_s^2 \left\{ \left[-\frac{25}{24} - \frac{2}{9}\,\zeta_3 \right] n_f + \frac{15331}{432} + \frac{359}{54}\,\zeta_3 - \frac{520}{27}\,\zeta_5 \right\} \\
&+ a_s^3 \left\{ \left[\frac{2131}{11664} + \frac{19}{81}\,\zeta_3 \right] n_f^2 + \left[-\frac{68135}{1944} - \frac{52}{27}\,\zeta_3^2 \right. \right. \\
&\qquad \left. \left. - \frac{3997}{486}\,\zeta_3 - \frac{5}{6}\,\zeta_4 + \frac{3875}{243}\,\zeta_5 \right] n_f + k_{2,0}^{[2]3} \right\} \Bigg) \\
= -4m_s^2 \Bigg(1 &+ 2.333\, a_s + a_s^2 \{-1.309\, n_f + 23.51\} \\
&+ a_s^3 \left\{ 0.4647\, n_f^2 - 32.08\, n_f + k_{2,0}^{[2]3} \right\} \Bigg).
\end{aligned}
\tag{14}
$$

Here we have used $a_s = \frac{\alpha_s(Q^2)}{\pi}, m_s = m_s(Q^2)$ and set the normalization scale $\mu^2 = Q^2$; results for generic values of μ can be easily recovered with the standard renormalization group techniques. The result for the α_s^4 terms in $D_0^{[1]}$ has been already presented in [12], the coefficients of the α_s^4 and α_s^3 terms in in $D_2^{[1]}$ and $\Pi_2^{[2]}$ respectively are new.

IV. IMPLICATIONS FOR THE α_s^4 PREDICTIONS AND PHENOMENOLOGICAL ANALYSIS

A. Massless case

FAC (Fastest Apparent Convergence) and PMS (Principle of Minimal Sensitivity) methods are both based eventually on the concept of scheme-invariant properties and the idea of the choice of an "optimal" scheme to provide better convergence of the resulting perturbative series. For both methods the optimal scheme depends on the physical observable we are dealing with. With FAC it should be a scheme which minimizes (set to zero by construction) all the terms of order α_s^2 and higher, while the PMS scheme is fixed by the requirement that the perturbative expansion for the observable is as insensitive as possible to a change in the scheme. The assumption that a renormalization scheme is in a sense optimal sets certain constraints on not yet computed higher order corrections in any other scheme. These constraints can be used to "predict" (at least roughly) the magnitude of these corrections.

For the function $D_0^{[1]}$ the result is known since long from Ref. [16] (see Table 1, column 5). From the three entries corresponding to $n_f = 3, 4$ and 5 one easily restores the FAC/PMS prediction for the n_f dependence of the α_s^4 term in $D_0^{[1]}$ (the term of order n_f^3 was fixed to its computed value):

$$d_0^{[1]4} = 127.6 - 44.2\, n_f + 3.64\, n_f^2 - 0.0100928\, n_f^3 d_0^{[1]4} = 127.58 - 44.211\, n_f + 3.6439\, n_f^2 - 0.0100928\, n_f^3. \tag{15}$$

(Note that FAC and PMS predictions happen to coincide for the α_s^4 term.)

n_f	$d_3^{\rm exact}$	$d_3^{\rm FAC}$	$d_3^{\rm PMS}$	$d_4^{\rm FAC/PMS}$	$d_5^{\rm FAC}$
3	6.371	5.604	6.39	27 ± 16	145 ± 100
4	2.758	4.671	5.26	8 ± 28	40 ± 160
5	-0.68	3.762	4.16	-8 ± 44	-3 ± 230

TABLE I: Estimates for the coefficients $d_3 = d_0^{[1]3}$ and $d_4 = d_0^{[1]4}$ in the function $D_0^{[1]}$ based on FAC and PMS optimizations. The estimate for of $d_0^{[1]4}$ employs the exact value of $d_0^{[1]3}$. The last column contains the FAC predictions for the coefficient $d_5^{\rm FAC}$ which was obtained assuming the value for $d_0^{[1]4}$ as given in the fifth column; the corresponding uncertainties have been estimated as described in the text.

It is interesting to compare the FAC/PMS predictions for the n_f dependence of coefficient $d_0^{[1]3}$ with the exact result given in eq. 12. The results of both estimates are

$$d_0^{[1]3}(FAC) = 8.54 - 1.013\, n_f + 0.0116\, n_f^2, \tag{16}$$

$$d_0^{[1]3}(PMS) = 9.93 - 1.23\, n_f + 0.0125\, n_f^2. \tag{17}$$

The comparison of the complete α_s^3 and partial α_s^4 results with FAC and PMS estimates leads to the following observations:

- Starting from α_s^3, the leading in α_s *and* n_f terms of order $\alpha_s^3 n_f^2$ and $\alpha_s^4 n_f^3$ are numerically quite small (at least for $n_f \leq 6$) and, thus, should have a negligibly small influence on the coefficients of the α_s expansion. On the other hand, the term subleading in n_f, say, of order $\alpha_s^3 n_f$ is comparable in size with the term of order $\alpha_s^3 n_f^0$. Similarly, the $\alpha_s^4 n_f^3$ term is significantly smaller than the $\alpha_s^4 n_f^2$ one, whereas the $\alpha_s^4 n_f$ and $\alpha_s^4 n_f^0$ terms are expected to be of similar magnitude.

- In general FAC/PMS methods correctly reproduce sign and order of magnitude of the higher order coefficients. The agreement is getting better when the coefficients happen to be large, as is the case for the quadratic mass corrections (see below).

- Taken separately, the FAC/PMS estimates of the coefficients of the n_f expansion could deviate rather strongly from the true result. However, for a given n_f, the deviation in the predicted value of the full $O(\alpha_s^n)$ term tends to be significantly smaller than what could be expected from summing individual terms of the n_f expansion. In addition, for the particular point $n_f = 3$ very good agreement is observed.

  To illustrate this feature, let us consider the worst case: the FAC/PMS prediction for the $\alpha_s^3 n_f$ term in the function $d_0^{[1]}$. Here the ratio of the exact result relative to the predicted one is quite large (about 4). Without knowledge of the $\alpha_s^3 n_f^0$ contribution one would expect that the uncertainty of the prediction for the full α_s^3 coefficient should be at least around

$$(d_{0,1}^{[1]3}|_{exact} - d_{0,1}^{[1]3}|_{FAC})\ n_f.$$

  For $n_f = 3, 4, 5$ this amounts to 9, 12 and 15, which should be compared to the corresponding differences of the full order $O(\alpha_s^3)$ (summed over all contributing power of n_f) coefficients, viz. $1, 2, 4$. This example demonstrates that the deviation of FAC/PMS predictions for subleading in n_f terms may well serve as conservative estimate of the accuracy in the prediction of the complete terms of orders α_s^3 and α_s^4.

These observations motivate the assumption that the prediction for the coefficient $d_0^{[1]4} (\equiv d_{0,0}^{[1]4} + d_{0,1}^{[1]4} n_f + d_{0,2}^{[1]4} n_f^2 + d_{0,3}^{[1]4} n_f^3)$ should also be correct within

$$\pm (d_0^{[1]4}|_{exact} - d_0^{[1]4}|_{FAC})\ n_f^2$$

Thus, in our phenomenological analysis of the τ-lepton decays the estimate

$$d_0^{[1]4}|_{n_f=3} = 27 \pm 16 \tag{18}$$

will be used. On the basis of this improved estimate and the four loop β function [31] one may even speculate about the α_s^5 term (whose exact evaluation is completely out of reach in the foreseeable future). Following the discussion of Kataev and Starshenko [17], one obtains

$$d_0^{[1]5}|_{n_f=3} = 145 \pm 100, \tag{19}$$

not far from the previous estimates of Ref. [17]. The variation of $d_0^{[1]4}$ by ± 16 leads to the variation of $d_0^{[1]5}$ by ± 100. For other values of n_f, the corresponding predictions can be obtained in the same way. They are listed in Table 1.

The FAC/PMS prediction (15) for the n_f dependence of the coefficient $d_0^{[1]4}$ does not take into account the available knowledge of the corresponding n_f^2 part. One can easily include this by fitting the FAC/PMS predictions for only *two* values of n_f with a *linear* function of n_f. As a result one obtains [38] (we have boxed the predicted coefficients in order to separate them clearly from the input)

$$d_0^{[1]4}(\text{FAC/PMS}, n_f = 3, 4) = \boxed{105.7 - 31.8\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3, \tag{20}$$

$$d_0^{[1]4}(\text{FAC/PMS}, n_f = 4, 5) = \boxed{107.7 - 32.3\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3, \tag{21}$$

$$d_0^{[1]4}(\text{FAC/PMS}, n_f = 3, 5) = \boxed{106.4 - 32.0\, n_f} + 1.875\, n_f^2 - 0.01009\, n_f^3. \tag{22}$$

One could now perform a self-consistency check of Table I by predicting, say, $d_0^{[1]4}$ for $n_f = 5$ from eq. (20). The result – (-7.5) – is compared successfully to the value listed in the table, viz. -8. The corresponding predictions from eqs. (21) and (22) are also in very good agreement to Table I.

An instructive example of how knowledge and inclusion of the subleading n_f term can improve FAC/PMS predictions is provided by the (exactly known) coefficient $d_0^{[1]3}$. Indeed, assuming the knowledge of $d_{0,2}^{[1]3}$ and $d_{0,1}^{[1]3}$ and using the values of d_3^{FAC} and d_3^{PMS} (at $n_f = 3$) as given by Table 1, one easily arrives at

$$d_0^{[1]3}(\text{FAC}) = \boxed{17.48} - 4.216\, n_f + 0.08621\, n_f^2 \tag{23}$$

and

$$d_0^{[1]3}(\text{PMS}) = \boxed{18.27} - 4.216\, n_f + 0.08621\, n_f^2, \tag{24}$$

which should be compared to the exact value $d_{0,0}^{[1]3} = 18.24$. Repeating the same exercise for $n_f = 4, 5$ we get 20.16 (FAC) and 20.75 (PMS) for $n_f = 4$ as well as 22.69 (FAC) and 23.08 (PMS) for $n_f = 5$ respectively.

Bearing in mind that in many cases FAC/PMS predictions made for $n_f = 3$ are in better agreement with the exact results (see, e.g. Table I and tables from Ref. [17]), we suggest eq. (20) as the best FAC/PMS prediction for the constant term and the term linear in n_f in the coefficient $d_0^{[1]4}$.

Eqs. (18,19) can be used to predict R_τ in the massless limit first in fixed order perturbation theory (FOPT)

$$R_\tau^{\rm FOPT} = \\ 3\left(1 + a_s + 5.202\, a_s^2 + 26.37\, a_s^3 + a_s^4\ (105 \pm 16) + a_s^5 (138 \pm 230 \pm 100)\right) \ . \tag{25}$$

Here $a_s = \alpha_s(M_\tau)/\pi$. The first uncertainty in the α_s^5 term comes from that of $d_0^{[1]4}$ while the second is our estimation of the error in the very coefficient $d_0^{[1]5}$. (Of course, within this approach they are strongly correlated).

Similarly, we can use "contour improved" (CI) formulae [18, 19] (assuming as reference value $\alpha_s(M_\tau) = 0.334$ [1]) to get

$$R_\tau^{CI} = 3\left(1 + 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + 1.31\, a_s^4 d_0^{[1]4} + 0.95\, a_s^5 d_0^{[1]5}\right) \tag{26}$$

or, equivalently,

$$R_\tau^{CI} = 3\left(1 + 1.364\, a_s + 2.54\, a_s^2 + 9.71\, a_s^3 + a_s^4 (35 \pm 20) + a_s^5 (138 \pm 95)\right). \tag{27}$$

Let us compare our new value for the coefficient $d_0^{[1]4}$ in (18) with the ones used in extracting $\alpha_s(M_\tau)$ from τ data by the OPAL [2] and ALEPH [1] collaborations, namely

$$d_0^{[1]4}|_{n_f=3} = 25 \pm 25 \text{ (OPAL)}, \quad 50 \pm 50 \text{ (ALEPH)}. \tag{28}$$

While OPAL's central value is basically the same as ours, their error bar is somewhat larger. In the case of ALEPH both the central value and the uncertainty assigned to it are significantly larger than our numbers. In this connection, we would like to stress that eq. (18) utilizes completely new non-trivial information given in (12): the subleading term in n_f term of order α_s^4.

It is of interest to see in detail which accuracy in the determination of $\alpha_s(M_\tau)$ one could achieve assuming eqs. (18,19). Let us introduce the quality δ_P as follows:

$$R_{\tau S=0} = \frac{\Gamma(\tau \to h_{S=0}\nu)}{\Gamma(\tau \to l\bar{\nu}\nu)} = |V_{ud}|^2 S_{EW} R_\tau, \tag{29}$$

with

$$R_\tau = 3(1 + \delta_P + \delta_{EW} + \delta_{NP}).$$

The first term here is the parton result, the second stands for pQCD effects. The non-perturbative corrections represented by δ_{NP} happens to be rather small $\delta_{NP} = -.003 \pm 0.003$ (see, e.g. [22]). Here the flavour mixing matrix element $|V_{ud}|^2 = 0.9475 \pm 0.0016$ [32]. The factor $S_{EW} = 1.0194$ is the electroweak correction which collects the large logarithmic terms [33], while $\delta_{EW} = 0.001$ is an additive electroweak correction [34]. Using for definiteness the result of ALEPH [1]

$$R_{\tau S=0} = 3.492 \pm 0.016, \tag{30}$$

one arrives at

$$\delta_P^{\rm exp} = 0.207 \pm 0.007. \tag{31}$$

To get a value for $\alpha_s(M_\tau)$ one should simply fit $\delta_P^{\rm exp}$ against $R_\tau/3 - 1$ as given by eq. (25) or by eq. (27) to get a result corresponding FOPT or CIPT (Fixed Order or Contour Improved PT).

Unfortunately, there is no unique way to assign a theoretical uncertainty $\delta\alpha_s$ to the obtained value of $\alpha_s(M_\tau)$. In the literature one finds several suggestions. Let us consider them in turn.

1. $\delta\alpha_s$ is a half of the shift in α_s induced by the last fully computed term in the PT (that is by the one of order α_s^3 at present).

2. $\delta\alpha_s$ is equal to the change in α_s caused by varying the normalization point μ around M_τ, typically within the range of 1.1–2.5 GeV.

   The corresponding results read (terms of order α_s^4 and higher in eq. (25) and in the D-function have been set to zero)

$$\alpha_s^{\rm FOPT}(M_\tau) = 0.345 \pm (0.025|0.037), \quad (32)$$
$$\alpha_s^{\rm CIPT}(M_\tau) = 0.364 \pm (0.012|0.021). \quad (33)$$

   Here the first(second) value in brackets correspond to the use of the first(second) suggestion for the error estimation. After evolution from M_τ to M_Z this corresponds to

$$\alpha_s^{\rm FOPT}(M_Z) = 0.1209 \pm (0.0024|0.0037), \quad (34)$$
$$\alpha_s^{\rm CIPT}(M_Z) = 0.1229 \pm (0.0011|0.0020). \quad (35)$$

3. $\delta\alpha_s$ is equal to the change in α_s caused by the uncertainty in the predicted (that is not yet completely known) higher order terms in the perturbative series for R_τ;

4. $\delta\alpha_s$ is a half of the difference in the $\alpha_s(M_\tau)$ as obtained within FOPT and CIPT. This difference comes from different handling of higher order terms.

In order to quantify the error estimates according to 3. and 4. we have shown in Table II the results for $\alpha_s(M_\tau)$ obtained with various choices for $d_0^{[1]4}$, $d_0^{[1]5}$ and μ. The entries with the choices ± 100 for the coefficients illustrate the large change in α_s which would result from a failure of PMS and FAC once higher order terms are included. For the plausible values of $d_0^{[1]4}$ and $d_0^{[1]5}$ we observe an significant decrease of the μ dependence after inclusion of additional terms in the α_s series.

Our final predictions for $\alpha_s^{\rm FOPT}(M_\tau)$ and $\alpha_s^{\rm CIPT}(M_\tau)$ are given in the first column of Table III, together with experimental error[39] and the combined theory uncertainty. The values of theory uncertainties are listed separately in columns 3,4 and 5. The corresponding values at the scale of M_Z are

$$\alpha_s^{\rm FOPT}(M_Z) = 0.1192 \pm 0.0007 \pm 0.002, \quad (36)$$
$$\alpha_s^{\rm CIPT}(M_Z) = 0.1219 \pm 0.001 \pm 0.0006. \quad (37)$$

$d_0^{[1]4}$	α_s^4	$d_0^{[1]5}$	α_s^5
27	0.331 ± 0.02	145	0.33 ± 0.02
	0.357 ± 0.009		0.354 ± 0.004
43	0.329 ± 0.02	245	0.324 ± 0.01
	0.353 ± 0.01		0.348 ± 0.005
11	0.333 ± 0.02	45	$0.335\pm?$
	0.361 ± 0.008		0.360 ± 0.002
100	0.324 ± 0.02	100	0.314 ± 0.01
	0.340 ± 0.01		0.338 ± 0.007
-100	$0.348\pm?$	-100	$0.349\pm?$
	0.398 ± 0.03		$0.401\pm?$

TABLE II: The predicted value of $\alpha_s(M_\tau)$ in dependence of chosen values for the coefficients $d_0^{[1]4}$, $d_0^{[1]5}$. The second and the forth columns differ in the the number of terms in the perturbative series included. The upper value of α_s is the one predicted within FOPT, the lower corresponds to CIPT. The uncertainty in the value of α_s corresponds to changing the normalization point μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$. The entry with question mark means that an equation for $\alpha_s(\mu)$ does not have a solution for some value of μ within the interval.

Thus we observe that the total uncertainty based on a combination of not yet calculated higher order terms, μ-dependence and scheme dependence is reduced, once α_s^4 terms are available. However, the difference between FOPT and CIPT results of roughly 0.02 is a remaining, at the moment irreducible uncertainty[40].

Method	$\alpha_s(M_\tau)$	$\Delta\,\delta_P^{\rm exp}$	$\Delta\,\mu$	$\Delta\,d_0^{[1]4}$	$\Delta\,d_0^{[1]5}$
FOPT	$0.330 \pm 0.006 \pm 0.02$	0.006	0.019	0.0045	0.0011
CIPT	$0.354 \pm 0.009 \pm 0.006$	0.009	0.0036	0.0042	0.0019

TABLE III: The value of $\alpha_s(M_\tau)$ obtained with $\delta_P^{\rm exp}$, $d_0^{[1]4}$ and $d_0^{[1]5}$ fixed to their central values according to eqs. (31,18,19) together with corresponding errors.

It is thus of interest to study this difference as a function of m_τ. In practice, this could be applied to sum rules for spectral functions as determined in e^+e^- annihilation.

Therefore, let us consider a hypothetical case of τ lepton with the mass equal 3 GeV. Assuming $\alpha_s(1.77\,{\rm GeV}) = 0.334$ and running this value to 3 GeV via standard 4-loop evolution equation one gets $\alpha_s(3\,{\rm GeV}) = 0.2558$ and predicts

$$\delta_P^{\rm exp} = 0.1353$$

which corresponds to eq. (25) with the α_s^4 and α_s^5 terms fixed to their central values (see eqs. (18–19)). Let us now investigate the results for α_s and the theory error that would result from $\delta_P^{\rm exp} = 0.1353$ as a starting point. The corresponding analogs of eqs. (32–35) and Table (II) are displayed below as eqs. ((38,39)) and Table (IV) correspondingly. The difference between FOPT and CIPT decreases significantly, and this remains true even after extrapolating to $\alpha_s(M_Z)$.

$$\begin{aligned}
\alpha_s^{\rm FOPT}(3{\rm GeV}) &= 0.263 \pm (0.013|0.014), & (38)\\
\alpha_s^{\rm CIPT}(3{\rm GeV} &= 0.265 \pm (0.005|0.008), & (39)\\
\alpha_s^{\rm FOPT}(M_Z) &= 0.1198 \pm (0.002|0.003), & (40)\\
\alpha_s^{\rm CIPT}(M_Z) &= 0.1203 \pm (0.009|0.0016). & (41)
\end{aligned}$$

$d_0^{[1]4}$	α_s^4	$d_0^{[1]5}$	α_s^5
27	0.256 ± 0.007	145	0.256 ± 0.003
	0.262 ± 0.004		0.261 ± 0.002
43	0.256 ± 0.007	245	0.254 ± 0.003
	0.26 ± 0.004		0.259 ± 0.002
11	0.257 ± 0.006	45	0.258 ± 0.005
	0.264 ± 0.003		0.263 ± 0.001
100	0.253 ± 0.008	100	0.249 ± 0.004
	0.255 ± 0.008		0.254 ± 0.002
-100	0.264 ± 0.01	-100	$0.277\pm$?
	0.279 ± 0.02		$0.28\pm$?

TABLE IV: The predicted value of $\alpha_s(M_\tau)$ in dependence of chosen values for the coefficients $d_0^{[1]4}$, $d_0^{[1]5}$ for a hypothetical case of $M_\tau = 3$ GeV. The third and the forth columns differ in the the number of terms in the perturbative series included. The upper value of α_s is the one predicted within FOPT, the lower corresponds CIPT. The uncertainty in the value of α_s corresponds to changing the normalization point μ as follows $\mu^2/M_\tau^2 = 0.4 - 2$. The entry with question mark means that an equation for $\alpha_s(M_\mu)$ does not have a solution for some value of μ within the interval.

The same coefficients $d_0^{[1]4}$ and $d_0^{[1]5}$ can be used to predict [16, 17] (the non-singlet part of) corrections of orders α_s^4 and α_s^5 to the R-ratio in Z decays:

$$R(n_f = 5) = 1 + a_s + 1.409\,a_s^2 - 12.77\,a_s^3 + (-97 \pm 44)\,a_s^4 + (76 \pm 230)\,a_s^5.$$

It is also of interest to display $R(s)$ for $n_f = 3$ and 4, which is accessible at e^+e^- colliders at lower energies.

$$R(n_f = 4) = 1 + a_s + 1.525\,a_s^2 - 11.52\,a_s^3 + (-112 \pm 30)\,a_s^4 + (-245 \pm 160)a_s^5.$$

$$R(n_f=3) = 1 + \ a_s + 1.640\, a_s^2 - 10.28\, a_s^3 + (-129 \pm 16)\, a_s^4 + (-635 \pm 100) a_s^5.$$

Our results are close to those of [16, 17], they employ, however, the additional information from [12, 31].

These formulae demonstrate rather good convergency for $n_f = 5$ and a reasonably good one for $n_f = 3$ and 4 if our predictions for the coefficients $d_0^{[1]4}$, $d_0^{[1]5}$ deviate from the true values within the assumed error margins.

B. Quadratic mass corrections

Let us first discuss the function $D_2^{[1]}$. The FAC/PMS predictions can be easily obtained following [37]; they are listed in Table V.

n_f	$d_3^{\rm exact}$	$d_3^{\rm FAC}$	$d_3^{\rm PMS}$	$d_4^{\rm FAC/PMS}$
3	107.5	89.86	91.17	1200 ± 400
4	94.37	79.13	80.11	950 ± 300
5	81.54	68.66	69.33	750 ± 200

TABLE V: Estimations of the coefficients $d_3 = d_2^{[1]3}$ and $d_4 = d_2^{[1]4}$ in the function $D_2^{[1]}$ based on the FAC and PMS optimizations. The estimation of $d_2^{[1]4}$ employs the exact value of $d_2^{[1]3}$.

We again restore the n_f dependence of the coefficient $d_2^{[1]4}$ as predicted by FAC/PMS:

$$d_2^{[1]4}(\text{FAC/PMS}) = 1931.44 - 281.956\, n_f + 9.0294\, n_f^2 - 0.0156289\, n_f^3 \tag{42}$$

as well as that of $d_2^{[1]3}$:

$$d_2^{[1]3}(\,\text{FAC}) = \quad 123.654 - 11.6638\, n_f + 0.133293\, n_f^2, \tag{43}$$

$$d_2^{[1]3}(\,\text{PMS}) = \quad 125.975 - 12.0028\, n_f + 0.134769\, n_f^2 \tag{44}$$

to be compared with

$$d_2^{[1]3}(\,\text{exact}) = 148.978 - 14.3097\, n_f + 0.16463\, n_f^2.$$

The comparison of estimates and exact results reveals a picture qualitatively similar to the massless case but with some modifications. A few important observations are in order.

1. All three terms of the n_f expansion of $d_2^{[1]3}$ are successfully predicted within about 20 % accuracy.

2. Unlike the massless case the agreement between FAC/PMS predictions for the coefficient $d_2^{[1]3}$ for $n_f = 3,4,5$ and the corresponding exact numbers is within the range of 15%–20%. On the other hand, the estimation of accuracy of the α_s^3 fixed n_f predictions obtained *exclusively* from the knowledge of the subleading contribution of $O(\alpha_s^3 n_f)$ is of the right order of magnitude but somewhat less. All this is probably a consequence of a significantly larger n_f independent contribution.

3. At $\mathcal{O}(\alpha_s^4)$ the exact result for the full coefficient

$$d_2^{[1]4} = d_{2,0}^{[1]4} + d_{2,1}^{[1]4} n_f + d_{2,2}^{[1]4} n_f^2 + d_{2,3}^{[1]4} n_f^3$$

is unknown, apart from its leading and subleading terms in n_f. The prediction for the subleading coefficient $d_{2,2}^{[1]4} = 9.0$ is by 50% larger than the exact value 6.07. The predicted values for $d_{2,0}^{[1]4}$ and $d_{2,1}^{[1]4}$ are *very* large. In view of this largeness, the estimate $(d_{2,2}^{[1]4}|_{exact} - d_{2,2}^{[1]4}|_{FAC})\ n_f^2$ (= 30, 50, 80 for $n_f = 3,4,5$ respectively) looks somewhat too optimistic. Therefore we assign a conservative 30% uncertainty to the fixed n_f predictions listed in the fifth column of the Table V.

n_f	$k_3^{\rm FAC}$	$k_3^{\rm PMS}$	$k_4^{\rm PMS}$
3	199.1	200 ± 60	2200 ± 1500
4	171.2	170 ± 50	1800 ± 1100
5	144.7	145 ± 40	1400 ± 900

TABLE VI: Estimates of the coefficients $k_3 = k_2^{[2]3}$ and $k_4 = k_2^{[2]4}$ in the function $\Pi_2^{[2]}$ based on the FAC and PMS optimizations. The estimate of $k_2^{[1]4}$ employs the predicted value for k_3; the corresponding uncertainties include only the ones induced by k_3.

At last, we repeat the analysis for the function $\Pi_2^{[2]}$. The results of FAC/PMS optimization methods are given in Table VI. Using the values of $k_3^{\rm PMS}$ from the table for $n_f = 3, 4, 5$ we reconstruct the corresponding full n_f dependence:

$$k_2^{[1]3} = 294.472 - 33.2429\, n_f + 0.696598\, n_f^2.$$

The comparison with the known terms of order n_f and n_f^2 $(-32.0843\, n_f,\quad 0.464663\, nf^2)$ demonstrates a remarkably good agreement for the subleading n_f contribution. The 50% error in the predicted value of the n_f^2 contribution looks natural as the corresponding coefficient is small. Following the same line of reasoning as above we have assigned a 30% uncertainty to the $O(\alpha_s^3)$ fixed n_f result.

To get a general idea about the size of the α_s^4 contribution to eq. (14) we used FAC/PMS and the predicted α_s^3 coefficient. The results are listed in the forth column of Table VI.

Let us now consider the effect of α_s^3 and α_s^4 corrections on the determination of the strange quark mass[41]. The mass correction to the R_τ depend on both functions $D_2^{[1]}$ and $\Pi_2^{[2]}$. Let us use the central ALEPH value of $\alpha_s(M_\tau) = 0.334$ when estimating the size of the perturbative corrections. For fixed order one finds the mass correction to the total rate:

$$\begin{aligned}
\delta^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1. + 5.33\, a_s + 46.0\, a_s^2 + 284\, a_s^3 \\
&+ 0.75\, a_s^3 k_2^{[2]3} + a_s^4(723. + 0.25\ d_2^{[1]4} + 9.84\ k_2^{[2]3} + 0.75\ k_2^{[2]4})) \\
&= -8\frac{m_s^2}{M_\tau^2}(1. + 0.567 + 0.520 + 0.521 \pm 0.05 + 0.593) \\
&= -8\frac{m_s^2}{M_\tau^2}(3.2 \pm 0.6),
\end{aligned} \tag{45}$$

where in the last line we have assumed the (maximal!) value of the $\mathcal{O}(\alpha_s^4)$ term as an estimate of the theoretical uncertainty (this convention will be used also below).

For the “contour improved” series one obtains

$$\begin{aligned}
\tilde{\delta}^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1.44 + 3.65\, a_s + 30.9\, a_s^2 + 72.2\, a_s^3 + 1.18\, a_s^3 k_2^{[2],3} \\
&\qquad + a_s^4(0.678\ d_2^{[1]4} + 1.06\ k_2^{[2]4})) \\
&= -8\frac{m_s^2}{M_\tau^2}(1.44 + 0.389 + 0.349 + 0.371 \pm 0.09 + 0.403) \\
&= -8\frac{m_s^2}{M_\tau^2}(2.95 \pm 0.4).
\end{aligned} \tag{46}$$

Now we consider the contributions of spin 1 and spin 0 separately. The lowest moments ($L = 0$) of the spin-dependent functions depend on a nonperturbative quantity and, thus, can not be treated perturbatively in principle [7].

For the spin one part and for $(k,l) = (0,1)$ we find

$$\begin{aligned}
\delta^{(1)01}_{us,2} &= -5\frac{m_s^2}{M_\tau^2}(1. + 4.83a + 35.7\,a_s^2 + 276.\,a_s^3 + a_s^4(1350 + d_2^{[1]4}))) \\
&= -5\frac{m_s^2}{M_\tau^2}(1. + 0.514 + 0.404 + 0.331 + 0.326) \\
&= -5\frac{m_s^2}{M_\tau^2}(2.58 \pm 0.33)
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
\tilde{\delta}^{(1)01}_{us,2} &= -5\frac{m_s^2}{M_\tau^2}(1.37 + 2.55a + 16.1\,a_s^2 + 135\,a_s^3 + 0.895\,a_s^4 d_2^{[1]4}) \\
&= -5\frac{m_s^2}{M_\tau^2}(1.37 + 0.271 + 0.182 + 0.163 + 0.137) \\
&= -5\frac{m_s^2}{M_\tau^2}(2.12 \pm 0.14).
\end{aligned} \tag{48}$$

Note that spin 1 contribution is determined by the component $\Pi^{[1]}$ alone and is known up to third order. Clearly, this series is decreasing in a reasonable way (comparable to the behaviour of $\tilde{\delta}^{00}_{us,2}$) and, at the same time, only moderately dependent on the improvement prescription with $\tilde{\delta}^{(1)01}_{us,2}/\delta^{(1)01}_{us,2} = 0.82$. On the basis of Eq. (48) this moment might well serve for a reliable m_s determination, with a sufficiently careful interpretation of the theoretical uncertainty.

The corresponding spin zero part is, per se, proportional to m_s^2 (not counting non-perturbative, so-called "condensate" contributions) and thus could be considered as ideal for a measurement of m_s. However, the behaviour of the perturbative series

$$\begin{aligned}
\delta^{(0)01}_{us,2} &= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(1. + 9.33a + 110\,a_s^2 + 1323\;a_s^3 \\
&\qquad\qquad + a_s^4(12200 + d_2^{[1]4} + 17.5k_2^{[2]3}) \\
&= \frac{m_s^2}{M_\tau^2}(1. + 0.992 + 1.24 + 1.59 + 2.16) \\
&= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(7.0 \pm 2)
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
\tilde{\delta}^{(0)01}_{us,2} &= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 11.2a + 126.\,a_s^2 + 289.\,a_s^3 + 6.63\,a_s^3 k_2^{[2]3} \\
&\qquad\qquad + a_s^4(2.71 d_2^{[1]4} + 7.76 k_2^{[2]4})) \\
&= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 1.19 + 1.42 + 1.94 + 2.6 \pm 1.31) \\
&= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(10.3 \pm 2.6)
\end{aligned} \tag{50}$$

shows a rapid growth of the coefficients. The series is not expected to provide an accurate prediction for the mass effects.

V. SUMMARY

Implications of the newly calculated $\alpha_s^4 n_f^2$ terms together with the $\alpha_s^4 n_f^3$ terms for an improved extraction of α_s from the τ decay are presented. Arguments are presented in support of predictions for the remaining terms of order $\alpha_s^4 n_f$ and $\alpha_s^4 n_f^0$ which are based on FAC or PMS optimization. The complete calculation will lead to a reduction of the theory uncertainty within the frameworks of FOPT or CIPT down to a negligible amount. However, an irreducible difference between the results from these two schemes of $\delta\alpha_s(M_\tau) \approx 0.02$ corresponding to $\delta\alpha_s(M_Z) \approx 0.002$ persists even after inclusion of $\mathcal{O}(\alpha_s^4)$ (and even $\mathcal{O}(\alpha_s^5)$ terms). Similar investigations, based on data up to higher energies (e. g. for fictitious heavy lepton of 3 GeV of for sum rules based on e^+e^- data) would lead to significantly smaller errors.

New contributions of orders $\mathcal{O}(m_s^2\alpha_s^4 n_f^2)$ and $\mathcal{O}(\alpha_s^4 n_f^2)$ to (axial)vector correlators relevant for the QCD description of the semileptonic τ decay into hadrons are obtained. The moments $R^{(0,0)}$ and $R^{(0,1)}$ are evaluated separately for spin zero and spin one final states [7]. The results are tested against predictions of FAC/PMS optimization methods. Good agreement is found. This has motivated us to take the full of FAC/PMS predictions as the basis for a new extraction of α_s and m_s from τ-decays with $\mathcal{O}(\alpha_s^4)$ accuracy. Using $\delta_P^{\rm exp} = 0.207$ we find 0.330 and 0.1192 for $\alpha_s^{\rm FOPT}(M_\tau)$ and $\alpha_s^{\rm FOPT}(M_Z)$ respectively. In the framework of contour improved evaluation these values increase to 0.354 and 0.1219 respectively

In contrast to the massless result, the PT series contributing to the m_s^2 dependent part seem to be barely convergent and the additional higher order terms seemingly do not lead to any significant improvement of the theoretical accuracy in the determination of the strange quark mass from the τ decays.

A slightly more favorable pattern of convergence is observed for the moments of the spin one contribution separately.

VI. ACKNOWLEDGMENTS

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[1] R. Barate *et al.* (The ALEPH collaboration), Z. Phys. C **76**, 15 (1997); Eur. Phys. J. C **4**, 409 (1998); Eur. Phys. J. C **11**, 599 (1999).
[2] K. Ackerstaff *et al.* (The OPAL collaboration), Eur. Phys. J. C **7**, 571 (1999).
[3] K.G. Chetyrkin, J.H. Kühn, and A. Kwiatkowski, Phys. Rep. **277**, 189 (1996).
[4] K. Maltman, Phys. Rev. D **58**, 093015 (1998).
[5] K.G. Chetyrkin, and A. Kwiatkowski, Z. Phys. C **59**, 525 1993.
[6] K.G. Chetyrkin, and A. Kwiatkowski, hep-ph/9805232.
[7] K.G. Chetyrkin, J.H. Kühn, and A.A. Pivovarov, Nucl. Phys. B **533**, 473 (1998).
[8] A. Pich and J. Prades, JHEP **9806**, 013 (1998); *ibid* **9910**, 004 (1999).
[9] J.G. Körner, F. Krajewski, and A.A. Pivovarov, Eur. Phys. J. C **20**, 259 (2001);
[10] S. Chen, M. Davier, E. Gamiz, A. Hocker, A. Pich, and J. Prades, Eur. Phys. J. C **22**, 31 (2001).
[11] K. Maltman, hep-ph/0209091.
[12] P.A. Baikov, K.G. Chetyrkin, and J.H. Kühn, Phys. Rev. Lett. **88**, 012001 (2002).
[13] P.M. Stevenson, Phys. Rev. D **23**, 2916 (1981); Phys. Lett. B **100**, 61 (1981); Nucl. Phys. B **203**, 472 (1982); Phys. Lett. B **231**, 65 (1984).
[14] G. Grunberg, Phys. Lett. B **95**, 70 (1980); Phys. Rev. D **29**, 2315 (1984).
[15] N.V. Krasnikov, Nucl. Phys. B **192**, 497 (1981);
[16] A.L. Kataev and V.V. Starshenko, Mod. Phys. Lett. A **10**, 235 (1995).
[17] A.L. Kataev and V.V. Starshenko, Nucl. Phys. Proc. Suppl. **39BC**, 312 (1995), [arXiv:hep-ph/9408395].
[18] A.A. Pivovarov, Sov. J. Nucl. Phys. 54, 676 (1991); Z. Phys. C 53, 461 (1992); Nuovo Cim. 105A, 813 (1992).
[19] F. Le Diberder and A. Pich, Phys. Lett. B **286**, 147 (1992); **289**, 165 (1992).
[20] K. Schilcher, M.D. Tran, Phys. Rev. D **29**, 570 (1984).
[21] E. Braaten, Phys. Rev. Lett. **53**, 1606 (1988).
[22] S. Narison, A. Pich, Phys. Lett. B **211**, 183 (1988).
[23] E. Braaten, Phys. Rev. D **39**, 1458 (1989).
[24] E. Braaten, S. Narison, A. Pich, Nucl. Phys. B **373**, 581 (1992).
[25] P.A. Baikov, Phys. Lett. B **385**, 404 (1996); Nucl. Inst. Meth. **A389**, 347 (1997);
P.A. Baikov and M. Steinhauser, Comp. Phys. Commun. **115**, 161 (1998).
[26] P.A. Baikov, Phys. Lett. B **474**, 385 (2000).
[27] D. Fliegner, A. Retey, and J.A. Vermaseren, hep-ph/0007221.
[28] J.A.M. Vermaseren, math-ph/0010025.
[29] G. 't Hooft, Nucl. Phys. B **61**, 455 (1973).
[30] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D **18**, 3998 (1978).
[31] T. van Ritbergen, J.A. Vermaseren, and S.A. Larin, Phys. Lett. B **400**, 379 (1997).
[32] K. Hagiwara et al., Phys. Rev. D **66**, 010001 (2002).

[33] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **61**, 1815 (1988).
[34] E. Braaten and C.S. Li, Phys. Rev. D **42**, 3888 (1990).
[35] S. Groote, J.G. Korner and A.A. Pivovarov, Mod. Phys. Lett. A **13**, 637 (1998).
[36] J.G. Körner, F. Krajewski, and A.A. Pivovarov, Phys. Rev. D **63**, 036001 (2001).
[37] K.G. Chetyrkin, B.A. Kniehl, and A. Sirlin, Phys. Lett. B **402**, 359 (1997).
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[39] According to eq. (31) the latter includes (small) uncertainties in the values of $|V_{ud}|$ and the nonperturbative correction δ_{NP} in addition to the experimental error per se as displayed in eq. (30).
[40] This is in agreement to the analysis of Ref. [35], where it has been concluded that "the resummed values of α_s from τ decay lie outside the convergence radii and can therefore not be obtained from a power series expansion. Regular perturbation series do not converge to their resummed counterparts. The experimental value of R_τ appears to be too large for a fixed order perturbation analysis to apply". See also [36].
[41] For a recent review of various attempts to extract the strange quark mass from τ data see Ref. [11].