Anomalous Magnetic Moment Interactions between Fermions

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Abstract

The anomalous magnetic moments of both fermions are included in the fully relativistic eight-component bound state equation. When these moments are of order unity or larger, a discrepancy with the standard sixteen-component formalism appears in the hyperfine mixing of P and D states for unequal masses at the leading order $Z^4\alpha^4$.

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1 Introduction

Ever since the original Dirac equation was applied to the hydrogen atom, it has been improved by additional interactions. For the high-Z members of the hydrogen iso-electric sequence, vacuum polarization, hyperfine interaction and radiative corrections are particularly important. Recoil corrections are important at low Z; the present paper presents such corrections for the anomalous magnetic moment interactions, including the magnetic hyperfine operator.

Recoil effects are of course largest for positronium (e^-e^+), but there the traditional Dirac equation fails. To the order α^4, one may instead use the nonrelativistic reduction of the 16-component Dirac-Breit equation. That equation was originally formulated by Breit for the helium atom; its application to the isolated two-fermion system is explained in [1]. Today, the expression "Breit operators" is normally used for the spin-orbit, tensor etc operators which appear in the nonrelativistic expansion; anomalous magnetic moments are easily included. A major progress in recent years has been the "nonrelativistic quantum electrodynamics" (NRQED), which is able to extend the nonrelativistic reduction to the order $(Z\alpha)^6$. The small anomalous magnetic moments of electrons and muons, $\kappa_e \approx \kappa_\mu \approx \alpha/2\pi$, are automatically included in NRQED. To the order $(Z\alpha)^6$, the NRQED method is presently more successful than the fully relativistic Bethe-Salpeter equation. This has in fact been interpreted to signal the end of relativistic quantum mechanics as a whole, including the Dirac equation for hydrogen.

On the other hand, it has been shown that the Dirac-Breit equation can be reduced to an 8-component equation which is still relativistic and in fact much simpler [2]. The equation is Dirac-like, with seemingly four components for the electron and two components for the nuclear spin. In reality, however, it contains strictly relativistic two-body kinematics: At large separation where the potential $V(r)$ vanishes, the following "asymptotic" equation describes the relative motion, with $\boldsymbol{p}_1 = -\boldsymbol{p}_2 \equiv \boldsymbol{p} = -i\nabla$ (units $\hbar = c = 1$), at total cms energy E:

$$\boldsymbol{p}^2\psi_{as} = k^2\psi_{as}, \quad k^2 = [(E^2 - m_1^2 - m_2^2)^2 - 4m_1^2m_2^2]/4E^2. \tag{1}$$

It applies for arbitrary particle spins. For equal spins, k^2 is split symmetrically in m_1 and m_2,

$$k^2 = \epsilon^2 - \mu^2, \quad \epsilon = (E^2 - m_1^2 - m_2^2)/2E, \quad \mu = m_1 m_2/E, \tag{2}$$

where ϵ and μ are the reduced energy and reduced mass, respectively. For two spin-$\frac{1}{2}$ particles, a factorized form of (1) exists in which the potential $V(r)$ is simply added:

$$(\epsilon - V - \mu\beta - \gamma^5 \boldsymbol{\sigma}_1 \boldsymbol{p} - H_{hf,D} - H_{an} - \ldots)\psi_1 = 0. \tag{3}$$

Here $\boldsymbol{\sigma}_1$ is a set of 2×2 Pauli matrices; a second set appears in the "Dirac" hyperfine operator,

$$H_{hf,D} = i\boldsymbol{\sigma}^\times V \boldsymbol{p}/2E, \quad \boldsymbol{\sigma}^\times = \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2. \tag{4}$$

β and γ^5 are Dirac matrices, they are related to the covariant γ^μ by $\beta = \gamma^0$, $\gamma^5 \boldsymbol{\sigma}_1 = \boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$. Although they are usually written as 4×4-matrices, they are really 2×2: β, γ^5 and $i\gamma^5\beta$ form a third SU_2 algebra. The V in (3) is the electric potential, viz. the sum of Coulomb and vacuum polarization potentials. For point particles, the Coulomb potential is $q_1 q_2/r = -Z\alpha/r \equiv -\alpha_Z/r$, with $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. H_{an} is the desired contribution of the anomalous magnetic moments; the dots in (3) indicate additional operators, which lead to radiative corrections.

In the original derivation [2] from the 16-component Dirac-Breit equation, the radial variable $r_{\rm DB}$ which occurs both in the cms kinetic energy operator and in the Coulomb potential, $V_{\rm DB} = -\alpha_Z/r_{\rm DB}$, appeared in the combination $2EV_{\rm DB} + V_{\rm DB}^2$, which was transformed into $2EV(r)$ by the "quasidistance", $r = r_{\rm DB}^2(r_{\rm DB} + \alpha_Z/2E)^{-1}$. At the same time, the transformation eliminated the typical Breit retardation operators. The non-hermiticity of the resulting hyperfine operator (4) was first regarded as a serious defect [3], until it was shown that its anti-hermitian part made the equation applicable to positronium [4]. Finally, it was shown how to extract $V(r)$ and $H_{hf,D}$ directly from the relevant T-matrix of perturbative QED, $S = 1 + iT$ [5], thereby avoiding nonrelativistic approximations and distance transformations. The resulting α_Z^6-terms have been checked against the NRQED results [6], but a number of radiative corrections are still missing, among these the anomalous magnetic moments.

To understand that the number of components for the isolated two-fermion system is eight, not sixteen, recall the use of time translation invariance in the derivation of (1): ψ_{as} satisfies two separate equations, $\boldsymbol{p}^2\psi_{as} = (E_1^2 - m_1^2)\psi_{as}$ and $\boldsymbol{p}^2\psi_{as} = (E_2^2 - m_2^2)\psi_{as}$, where the E_i are eigenvalues of the individual time shifters $i\partial/\partial t_i$. But as the interaction commutes only with a common time shift $i\partial/\partial t = i\partial/\partial t_1 + i\partial/\partial t_2$, the two equations must be combined into a single one that contains only $E_1 + E_2 = E$, which is an eigenvalue of $i\partial/\partial t$. The form (1) follows uniquely. In analogy, after the two free single-particle Dirac equations have been correctly combined into one 16-component equation which contains only $E_1 + E_2$,

$$(E_1 + E_2 - \gamma_1^5 \boldsymbol{p}_1 \boldsymbol{\sigma}_1 - \gamma_2^5 \boldsymbol{p}_2 \boldsymbol{\sigma}_2 - m_1\beta_1 - m_2\beta_2)\psi_{as}^{(16)} = 0, \tag{5}$$

it remains to eliminate those components which arise from single-particle parity transformations, $\psi_{as}^{P_i}(\boldsymbol{r}_i, \boldsymbol{r}_j) = \beta_i \psi_{as}(-\boldsymbol{r}_i, \boldsymbol{r}_j)$. In other words, as the interaction commutes only with the total parity transformation, the desired function ψ must only allow for that transformation,

$$\psi^P(\boldsymbol{r}_i, \boldsymbol{r}_j) = \beta\psi(-\boldsymbol{r}_i, -\boldsymbol{r}_j), \quad \beta \equiv \beta_1\beta_2. \tag{6}$$

Although one may avoid $\psi^{(16)}$ altogether, we briefly indicate the elimination method [7]. $\psi^{(16)}$ is split into octets of eigenstates of the total chirality operator $\gamma_1^5\gamma_2^5$ which commutes with β:

$$\psi^{(16)} = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad \gamma_1^5\gamma_2^5\psi = \psi, \quad \gamma_1^5\gamma_2^5\chi = -\chi. \tag{7}$$

The phases between the components of ψ and χ are conveniently chosen such that

$$\beta_2\psi = \chi, \quad \beta_2\chi = \psi. \tag{8}$$

Finally, ψ and χ may be subdivided into large and small components which are eigenstates of β. The asymptotic components of ψ and χ will be called v and w, respectively. According to (5), they satisfy coupled equations, one of which is used for the elimination of w:

$$w = m_+^{-1}[E - \gamma^5(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\boldsymbol{p}]v, \quad m_\pm = m_2 \pm \beta m_1. \tag{9}$$

The distinction between γ_1^5 and γ_2^5 is now unnecessary, as $\gamma_2^5 v = \gamma_1^5 v$. In the following, we generally set $\gamma_1^5 \equiv \gamma^5$. The only obstacle in the application of this method has been the m_+^{-1}, which must be removed because it is singular for $m_2 = m_1$ and $\beta = -1$. This is achieved by the transformation $v = C_1 v_1$,

$$C_1 = (m_+ m_-)^{-1/2}(m_2 + m_1\beta P_{\text{spin}}), \quad P_{\text{spin}} = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2), \tag{10}$$

where P_{spin} is the spin permutation operator. It also has the property of removing the $\boldsymbol{\sigma}_2$ of (5), which explains the absence of a term $\boldsymbol{\sigma}_2\boldsymbol{p}$ in (3). A similar transformation C_2 exists which removes instead the $\boldsymbol{\sigma}_2$ [7].

It is instructive to compare this formalism with the Bethe-Salpeter equation and with NRQED, which both use the 16-component formalism. For this purpose, we multiply (3) by $2E$, obtaining the simpler "basic" form

$$[E^2 - 2EV - m_+^2 - 2E\gamma^5\boldsymbol{\sigma}_1\boldsymbol{p} - 2i\gamma^5 V\boldsymbol{\sigma}^\times\boldsymbol{p} - 2EH_{an}]\psi_1 = 0. \tag{11}$$

The reduction of the 16-component equation to 8 components produces the operator $(E - V)^2$ instead of the $E^2 - 2EV$ of (11) (plus a more complicated structure of operators linear in V). The V^2-contribution to the energy levels is then cancelled by interactions derived from two-photon exchange [8]. The 16-component formalism is not incorrect, only less efficient.

The nonrelativistic reduction of (3) produces the known tensor and Fermi contact potentials between two Dirac particles, with $V' = dV/dr \approx \alpha_Z/r^2$,

$$V_{t,D} = \sigma_t V'/4m_1m_2r, \quad \sigma_t = 3\sigma_{1r}\sigma_{2r} - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2, \tag{12}$$

$$V_{\text{con,D}} = \tfrac{2}{3}\pi\alpha_Z\delta(r)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2/m_1m_2, \tag{13}$$

plus three spin-orbit potentials,

$$V_{s1l,D} = \boldsymbol{\sigma}_1\boldsymbol{l}V'/4\mu^2r, \quad V_{s2l,D} = \boldsymbol{\sigma}_2\boldsymbol{l}V'/2m_1m_2r, \tag{14}$$

$$V_{s\times l,D} = i\boldsymbol{\sigma}^{\times}\boldsymbol{l}V'/4m_1m_2r. \tag{15}$$

$V_{s1l,D}$ is the spin-orbit potential of a Dirac particle of mass μ, $V_{s2l,D}$ is the "other-spin"-orbit potential emerging from $H_{hf,D}$, and $V_{s\times l,D}$ comes from the antihermitian part of $H_{hf,D}$. The spin-orbit potentials differ drastically from the two spin-orbit potentials of the 16-component reduction, which were first calculated by Chraplyvy and by Barker and Glover [9]:

$$V^{\rm CBG}_{s1l,D} = \boldsymbol{\sigma}_1\boldsymbol{l}V'(m_1^{-1} + 2m_2^{-1})/4m_1r, \quad V^{\rm CBG}_{s2l,D} = \boldsymbol{\sigma}_2\boldsymbol{l}V'(m_2^{-1} + 2m_1^{-1})/4m_2r. \tag{16}$$

The results to order α_Z^4 are nevertheless identical: The sum of their expectation values (with $\langle\boldsymbol{\sigma}_1\rangle = \langle\boldsymbol{\sigma}_2\rangle$ and $\langle V_{s12l,D}\rangle = 0$) are the same, and the inclusion of $V_{s12l,D}$ provides the known mixing between singlet and triplet spin states in degenerate states. It is this latter mixing which is modified by the inclusion of κ_1 and κ_2.

2 The Matrices T, M, M^C

The one-photon exchange of the QED T-matrix is, with $\boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_1' = \boldsymbol{k}_2' - \boldsymbol{k}_2$,

$$T^{(16\times16)} = 4\pi\alpha_Z J^{\mu}_{11'}g_{\mu\nu}J^{\nu}_{22'}/t, \quad t = q^{02} - \boldsymbol{q}^2. \tag{17}$$

For $\boldsymbol{q}^2 \ll m_1^2$, one may approximate the two fermion form factors by $F_1 = 1$, $F_2 = \kappa$,

$$J^{\mu}_{11'} = \overline{u}_1'(\gamma_1^{\mu} + \sigma_1^{\mu\nu}q_\nu\kappa_1/2m_1)u_1. \tag{18}$$

The free spinors u_i and $u_i'^{\dagger}$ will now be omitted, and the remaining oprators will be distinguished by hats. In the cms one has $q^0 = 0$, and with $\sigma^{0j} = \gamma^5\sigma^j$, $\sigma^{jk} = i\sigma_l$ (jkl cyclic),

$$\hat{J}^0_{11'} = 1 + \gamma_1^5\beta_1\boldsymbol{q}\boldsymbol{\sigma}_1\kappa_1/2m_1, \quad \hat{\boldsymbol{J}}_{11'} = \gamma_1^5\boldsymbol{\sigma}_1 + i\beta_1\boldsymbol{q}\times\boldsymbol{\sigma}_1\kappa_1/2m_1, \tag{19}$$

The corresponding $\hat{J}^{\nu}_{22'}$ has $\boldsymbol{q}_2 = -\boldsymbol{q}$:

$$\hat{J}^0_{22'} = 1 - \gamma_2^5\beta_2\boldsymbol{q}\boldsymbol{\sigma}_2\kappa_2/2m_2, \quad \hat{\boldsymbol{J}}_{22'} = \gamma_2^5\boldsymbol{\sigma}_2 - i\beta_2\boldsymbol{q}\times\boldsymbol{\sigma}_2\kappa_2/2m_2. \tag{20}$$

Insertion of these expressions into $\hat{T}$ gives several terms, which may be classified by their respective powers of anomalous magnetic moments:

$$\hat{T} = \hat{T}_{\rm D} + \kappa_1\hat{T}_{\kappa1} + \kappa_2\hat{T}_{\kappa2} + \kappa_1\kappa_2\hat{T}_{\kappa1,\kappa2}. \tag{21}$$

$\hat{T}_{\rm D}$ denotes the pure "Dirac" matrix element:

$$\hat{T}_{\rm D} = -4\pi\alpha_Z(1 - \gamma_1^5\gamma_2^5\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)/\boldsymbol{q}^2. \tag{22}$$

The remaining three matrices are

$$\hat{T}_{\kappa1} = -4\pi\alpha_Z\boldsymbol{q}(\gamma_1^5\boldsymbol{\sigma}_1 + i\gamma_2^5\boldsymbol{\sigma}^{\times})\beta_1/2m_1\boldsymbol{q}^2, \tag{23}$$

$$\hat{T}_{\kappa2} = -4\pi\alpha_Z\boldsymbol{q}(-\gamma_2^5\boldsymbol{\sigma}_2 + i\gamma_1^5\boldsymbol{\sigma}^{\times})\beta_2/2m_2\boldsymbol{q}^2, \tag{24}$$

$$\hat{T}_{\kappa 1,\kappa 2} = -4\pi\alpha_Z[-\gamma_1^5\gamma_2^5(\boldsymbol{q}\boldsymbol{\sigma}_1)(\boldsymbol{q}\boldsymbol{\sigma}_2) + (\boldsymbol{q}\times\boldsymbol{\sigma}_1)(\boldsymbol{q}\times\boldsymbol{\sigma}_2)]\beta_1\beta_2/4m_1m_2\boldsymbol{q}^2. \quad (25)$$

Next, the 16 components of u_1u_2 are split into two octets, v and w, which are eigenstates of $\gamma_1^5\gamma_2^5$, with eigenvalues $+1$ and -1, respectively. We rewrite (17) as

$$T^{(16\times 16)} = v'^\dagger \hat{T}_{vv} v + w'^\dagger \hat{T}_{ww} w + v'^\dagger \hat{T}_{vw} w + w'^\dagger \hat{T}_{wv} v. \quad (26)$$

By using (9) as well as the free-particle equations (with $\boldsymbol{k}_1 = -\boldsymbol{k}_2 \equiv \boldsymbol{k}$),

$$w = m_+^{-1}(E - \gamma^5\Delta\boldsymbol{\sigma}\boldsymbol{k})v, \quad v'^\dagger = w'^\dagger(E - \gamma^5\boldsymbol{\sigma}\boldsymbol{k}')m_+^{-1}, \quad (27)$$

$$\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2, \quad \gamma^5 = \gamma_1^5, \quad (28)$$

all four submatrices of (26) may be put into the form of the last one,

$$T^{(16\times 16)} = w'^\dagger m_+^{-1} M v = w_1'^\dagger m_+^{-1} M^C v_1, \quad M^C \equiv C_1^{-1} M C_1. \quad (29)$$

(The form of T between Lorentz-irreducible spinors is necessarily asymmetric and non-hermitian, anologous to the $u_l'^\dagger M^{(2\times 2)} u_r$ between the right- and left-handed Weyl components of a single Dirac particle.) The matrices $\hat{T}_{vw}$ and $\hat{T}_{wv}$ of $\hat{T}_{\mathrm{D}}$ and $\hat{T}_{\kappa 1,\kappa 2}$ vanish; the other two give

$$M = m_+[(E - \gamma^5\boldsymbol{\sigma}\boldsymbol{k}')\hat{T}_{vv}/m_+ + \hat{T}_{ww}/m_+(E - \gamma^5\Delta\boldsymbol{\sigma}\boldsymbol{k})]. \quad (30)$$

As $(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)/4$ is the projector on the spin-singlet state, $\boldsymbol{\sigma}\boldsymbol{k}'(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$ vanishes. The m_+ cancels, and $-(1 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\Delta\boldsymbol{\sigma} = 2i\boldsymbol{\sigma}^\times$ leads to

$$M_D = -4\pi\alpha_Z[2E + 2i\gamma^5\boldsymbol{k}\boldsymbol{\sigma}^\times]/\boldsymbol{q}^2 = M_D^C. \quad (31)$$

The Fourier transform of M_D^C gives the Dirac interaction of (5); the Dirac potential in (3) is the Fourier transforms of $M_D^C/2E$.

As $\beta_1\beta_2 \equiv \beta$ commutes with $\gamma_1^5\gamma_2^5$, it does not exchange v with w:

$$\hat{T}_{\kappa 1,\kappa 2}{}^{vv}_{ww} = -4\pi\alpha_Z[\mp(\boldsymbol{q}\boldsymbol{\sigma}_1)(\boldsymbol{q}\boldsymbol{\sigma}_2) + (\boldsymbol{q}\times\boldsymbol{\sigma}_1)(\boldsymbol{q}\times\boldsymbol{\sigma}_2)]\beta/4m_1m_2\boldsymbol{q}^2. \quad (32)$$

Whereas m_+ has cancelled out in M_D, it does not cancel completely in $M_{\kappa 1,\kappa 2}$ calculated from (32), because of $m_+\gamma^5 = \gamma^5 m_-$. C_1 has the following properties:

$$C_1^{-1}\gamma^5 = \gamma^5 C_1, \quad C_1\boldsymbol{\sigma}C_1 = \boldsymbol{\sigma}m_+/m_-, \quad (33)$$

$$C_1\Delta\boldsymbol{\sigma}C_1 = \Delta\boldsymbol{\sigma}, \quad C_1\boldsymbol{\sigma}^\times C_1 = \boldsymbol{\sigma}^\times. \quad (34)$$

For (32), the C_1-transformation commutes with $\hat{T}_{vv}$ and $\hat{T}_{ww}$ which are symmetric under the exchange of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$:

$$C_1^{-1}(\sigma_{1i}\sigma_{2j} + \sigma_{1j}\sigma_{2i})C_1 = \sigma_{1i}\sigma_{2j} + \sigma_{1j}\sigma_{2i}, \quad (35)$$

$$C_1^{-1}M_{\kappa 1\kappa 2}C_1 = [(E - \gamma^5\boldsymbol{\sigma}\boldsymbol{k}')\hat{T}_{vv} + \hat{T}_{ww}(E - \gamma^5\Delta\boldsymbol{\sigma}\boldsymbol{k})]. \quad (36)$$

The leading terms $E(\boldsymbol{q}\boldsymbol{\sigma}_1)(\boldsymbol{q}\boldsymbol{\sigma}_2)$ in (32) vanish. In the nonrelativistic limit, one may also set $\beta = 1$:

$$(M_{\kappa 1\kappa 2}^C/2E)_{nr} = 4\pi\alpha_Z[\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{q}^2 - (\boldsymbol{\sigma}_1\boldsymbol{q})(\boldsymbol{\sigma}_2\boldsymbol{q})]/4m_1m_2\boldsymbol{q}^2. \quad (37)$$

The other two matrices (23) and (24) contain a single factor β_i, which implies $\hat{T}_{vv} = \hat{T}_{ww} = 0$. We first treat (24), using $\beta_2 w = v,\ v'^\dagger \beta_2 = w'^\dagger$,

$$T_{wv} = T_{vw} = -4\pi\alpha_Z \gamma^5 \boldsymbol{q}(\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}^\times)/2m_2\boldsymbol{q}^2, \tag{38}$$

$$M_{\kappa 2} = -4\pi\alpha_Z \gamma^5 m_- \boldsymbol{q}(\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}^\times)/m_2\boldsymbol{q}^2. \tag{39}$$

The C-transformation replaces $m_-\boldsymbol{\sigma}_2$ by

$$m_- \tfrac{1}{2}(\boldsymbol{\sigma} m_+/m_- - \Delta\boldsymbol{\sigma}) = \tfrac{1}{2}(\boldsymbol{\sigma} m_+ - \Delta\boldsymbol{\sigma} m_-) = \boldsymbol{\sigma}_2 m_2 + \beta\boldsymbol{\sigma}_1 m_1, \tag{40}$$

$$M_{\kappa 2}^C = -4\pi\alpha_Z \gamma^5 \boldsymbol{q}[\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}^\times + \beta(\boldsymbol{\sigma}_1 - i\boldsymbol{\sigma}^\times)m_1/m_2]/\boldsymbol{q}^2. \tag{41}$$

The calculation of $T_{\kappa 1}$ and $M_{\kappa 1}^C$ is formally different, but the result follows from $M_{\kappa 2}^C$ simply by exchange of m_1 and m_2. The β_2 is replaced by $\beta_1 = \beta\beta_2$; the C_1-transformation replaces $m_-\boldsymbol{\sigma}_1\beta$ by

$$m_- \tfrac{1}{2}(\boldsymbol{\sigma} m_+/m_- + \Delta\boldsymbol{\sigma})\beta = (\boldsymbol{\sigma}_1 m_2 + \beta\boldsymbol{\sigma}_2 m_1)\beta = \boldsymbol{\sigma}_2 m_1 + \beta\boldsymbol{\sigma}_1 m_2, \tag{42}$$

which follows from (40) by the exchange of m_1 and m_2. As a result,

$$M_{\kappa 1}^C = -4\pi\alpha_Z \gamma^5 \boldsymbol{q}[\boldsymbol{\sigma}_2 + i\boldsymbol{\sigma}^\times + \beta(\boldsymbol{\sigma}_1 - i\boldsymbol{\sigma}^\times)m_2/m_1]/\boldsymbol{q}^2. \tag{43}$$

3 The Nonrelativistic Potentials and Hyperfine Mixing

The interaction in the basic form (11) is the Fourier transform of the Born term,

$$I(\boldsymbol{r}) = 2EH_I(\boldsymbol{r}) = (2\pi)^{-3}\int d^3q e^{-i\boldsymbol{q}\boldsymbol{r}} M^C. \tag{44}$$

$I_D = 2EV + 2i\gamma^5 V\boldsymbol{\sigma}^\times \boldsymbol{p}$ results from M_D^C. Using

$$(2\pi)^{-3}\int d^3q e^{-i\boldsymbol{q}\boldsymbol{r}} \boldsymbol{q}(-4\pi\alpha_Z/q^2) = i\nabla V = i\boldsymbol{r}V'/r, \tag{45}$$

$$I_{\kappa 2} = V'/r\gamma^5 \boldsymbol{r}[i\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}^\times + \beta(i\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}^\times)m_1/m_2]. \tag{46}$$

$I_{\kappa 1}$ has m_1/m_2 replaced by m_2/m_2.

Next, we construct the nonrelativistic hyperfine operators, by eliminating the small components ψ_f of the Dirac equation (3). We immediately restrict ourselves to the terms linear in V',

$$V_{\kappa 2} = \{\boldsymbol{\sigma}_1\boldsymbol{p}, (V'/r)\boldsymbol{r}(i\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}^\times)\}/4\mu E + [\boldsymbol{\sigma}_1\boldsymbol{p}, V'/r\boldsymbol{r}(i\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}^\times)m_1/m_2]/4\mu E. \tag{47}$$

The commutator arises from the extra factor β, because of $\beta\psi_g = \psi_g$, $\beta\psi_f = -\psi_f$.

The operator containing $\boldsymbol{\sigma}^\times \boldsymbol{r}V'/r$ in the anticommutator is the hermitian part of the Dirac hyperfine operator, and we simply adopt the "Dirac" results (12)- (15):

$$V_{s2l,\kappa 2} = V_{s2l} = V'\boldsymbol{\sigma}_2\boldsymbol{l}/2E\mu, \quad V_{t,\kappa 2} = V_t = \tfrac{1}{2}\sigma_t V'/2E\mu. \tag{48}$$

In the commutator, the part containing $i\boldsymbol{\sigma}_1$ is also hermitian, giving

$$V_{s1l,\kappa 2} = V' \boldsymbol{\sigma}_1 \boldsymbol{l}/2m_2^2 r. \tag{49}$$

For comparison, the CBG reduction finds only one potential [10],

$$V_{\kappa 2}^{\mathrm{CBG}} = \boldsymbol{\sigma}_2 \boldsymbol{l} V'/2\mu m_2 r. \tag{50}$$

Again, there is agreement in the expectation values,

$$\langle V_{s2l,\kappa 2} + V_{s1l,\kappa 2}\rangle = \langle V_{s2l,\kappa 2}^{\mathrm{CBG}}\rangle. \tag{51}$$

The part containing $\boldsymbol{\sigma}^\times$ is antihermitian and leads to

$$V_{s12l,\kappa 2} = -iV' \boldsymbol{\sigma}^\times \boldsymbol{l}/4m_2^2 r. \tag{52}$$

The operator $i\boldsymbol{\sigma}_2$ in the anticommutator is relatively complicated. With $i\boldsymbol{\sigma}_1 \boldsymbol{p} = \sigma_{1r}(\partial_r - \boldsymbol{\sigma}_1 \boldsymbol{l}/r) = (\partial_r + \boldsymbol{\sigma}_1 \boldsymbol{l}/r)\sigma_{1r}$, one obtains, neglecting an operator $\sim \Delta V$,

$$V_{ah,\kappa 2} = -[\sigma_{1r}\sigma_{2r}, \boldsymbol{\sigma}_1 \boldsymbol{l}]V'/4E\mu r, \tag{53}$$

where the index $_{ah}$ stands for "antihermitian".

As a result, the total operator linear in κ_1 and κ_2 is

$$V_\kappa = [(\boldsymbol{\sigma}_1 \boldsymbol{l} - \tfrac{1}{2} i\boldsymbol{\sigma}^\times \boldsymbol{l})\hat{\kappa} + (\boldsymbol{\sigma}_2 \boldsymbol{l} + \tfrac{1}{2}\sigma_t - \tfrac{1}{2}[\sigma_{1r}\sigma_{2r}, \boldsymbol{\sigma}_1 \boldsymbol{l}])\kappa]V'/2E\mu r, \tag{54}$$

$$\hat{\kappa} \equiv \kappa_1 m_2/m_1 + \kappa_2 m_1/m_2, \quad \kappa \equiv \kappa_1 + \kappa_2. \tag{55}$$

4 Results and Discussion

Finally, we construct the matrix elements and energy shifts of the new operators. Let $\boldsymbol{f} = \boldsymbol{l} + \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$ denote the total angular momentum operator, and $f(f+1) \equiv F^2$ an eigenvalue of its square. In the following, we only consider states with $l = f > 0$, where spin-orbit and hyperfine operators mix already at the order α_Z^4, such that the antihermitian operators are essential. The triplet-singlet spin basis will be used; the relevant spin states are denoted by $\chi_{f,1}$ and $\chi_{f,0}$, respectively:

$$\chi = \begin{pmatrix} \chi_{f,1} \\ \chi_{f,0} \end{pmatrix}, \quad \boldsymbol{\sigma}_1 \boldsymbol{l} = \begin{pmatrix} -1 & F \\ F & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 \boldsymbol{l} = \begin{pmatrix} -1 & -F \\ -F & 0 \end{pmatrix}. \tag{56}$$

In this two-component space, the tensor spin operator arises as follows:

$$\sigma_{1r}\sigma_{2r} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad \sigma_t = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{57}$$

The following antihermitian matrices are also needed:

$$i\boldsymbol{\sigma}^\times \boldsymbol{l} = 2\begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix} = -[\sigma_{1r}\sigma_{2r}, \boldsymbol{\sigma}_1 \boldsymbol{l}]. \tag{58}$$

Although all spin potentials except $V_{s1l,D}$ contain $2E\mu = 2m_1 m_2$ in their denominators, it is convenient to separate a factor $4\mu^2$, because the expectation value of $V'/4\mu^2 r$ is simple for the hydrogenic Schrödinger equation for a particle of mass μ:

$$\langle V'/4\mu^2 r\rangle = \langle \alpha_Z/4\mu^2 r^3\rangle = \alpha_Z^4 \mu/[4n^3(f+1/2)F^2]. \tag{59}$$

Collecting the matrices, one obtains

$$V_\kappa = \begin{pmatrix} -\hat{\kappa} & 2F(\hat{\kappa}-\kappa) \\ 0 & 0 \end{pmatrix} \frac{V'}{2E\mu r}, \quad \hat{\kappa}-\kappa = (\kappa_1 m_2 - \kappa_2 m_1)\frac{m_2 - m_1}{m_1 m_2}. \tag{60}$$

To this one has to add the tensor interaction $\kappa_1\kappa_2 I_{\kappa 1,\kappa 2}$ from (37) and the Dirac spin potentials (12)-(15),

$$V_{s,tot} = \begin{pmatrix} -1-2y & Fx \\ F & 0 \end{pmatrix} \frac{V'}{4\mu^2 r}, \quad y = (\hat{\kappa} - \kappa_1\kappa_2)\mu/E, \tag{61}$$

$$x = 1 - 4(1-\hat{\kappa}+\kappa)\mu/E = (m_2 - m_1)(m_2 - m_1 + 4\kappa_1 m_2 - 4\kappa_2 m_1)/E^2. \tag{62}$$

(In [3], both σ_t and a minus sign were forgotten in eq (5.72).) The eigenvalues of $V_{s,tot}$ have been called $-2a_l^{(\pm)}F^2$ [4] , with $F^2 = f(f+1) = l(l+1)$. They are now found as

$$-2a_l^{(\pm)}F^2 = -\tfrac{1}{2} - y \pm \sqrt{(\tfrac{1}{2}+y)^2 + F^2 x}. \tag{63}$$

They may be compared with those of the CBG spin matrix. As the off-diagonal elements of a real 2×2 hermitian matrix are equal, the corresponding $x^{\rm CBG}$ must be a square:

$$x^{\rm CBG} = (m_2 - m_1 + 2\kappa_1 m_2 - 2\kappa_2 m_1)^2/E^2. \tag{64}$$

One thus finds

$$x^{\rm CBG} = x + 4(\kappa_1 m_2 - \kappa_2 m_1)^2/E^2. \tag{65}$$

The difference vanishes both for positronium and for protonium. For muonium, $\kappa_1 = \kappa_2 = \alpha/2\pi$ makes $x^{\rm CBG} - x = E^{-2}(m_1 - m_2)^2\alpha^2/\pi^2$. A contribution of this order could be cancelled by two-loop vertex graphs, analogous to the V^2-cancellation observed in the Bethe-Salpeter equation. For hydrogen, the difference is strongly suppressed relative to the main term $x \approx 1$. Antiprotonic atoms such as $\overline{p}\,^3$He ($g_1 = 5.58$, $\kappa_1 = 1.79$ $g_2 = -3.185$, $\kappa_2 = -2.592$ [11]) have $x^{\rm CBG} - x > 1$. As the baryon magnetic moments involve the strong interaction, the results could in principle differ in different QED formalisms. The author is not aware of any relevant experiments. There exist however atomic states for which the square in (65) is verified beyond doubt, namely the large-l Rydberg states of helium [12]. The inner electron makes the core appear like a fermion of mass $m_2 = m(^4{\rm He})$ and g-factor $g_2 = -2m_2/m_1 \equiv -2(1+\kappa_2)$. The resulting "hyperfine structure" is in fact much larger than the fine structure and is correctly reproduced by $x^{\rm CBG}$. Clearly, when the square in (65) is much larger than x, the 8-component method breaks down, at least in its present form.

Even without the square, one may wonder how a perturbation theory with non-hermitian operators works [7]. In the case at hand, the orthogonality relations are derived from an unperturbed operator which is hermitian. In more general cases, there is in principle also the possibility of expressing a non-hermitian operator H in terms of an equivalent hermitian operator H_h by a similarity transformation, $H = V_{sim} H_h V_{sim}^{-1}$. The V_{sim} of the relativistic equation (3) has not yet been found; for the nonrelativistic reduction (61) it is a 2×2 matrix. The result is elegant for $\kappa_1 = \kappa_2 = 0$,

$$V_{sim}(\kappa_i = 0) = \begin{pmatrix} \delta\hat{m}^{1/2} & 0 \\ 0 & \delta\hat{m}^{-1/2} \end{pmatrix}, \quad V_{s,h}(\kappa_i = 0) = \begin{pmatrix} -1 & F\delta\hat{m} \\ F\delta\hat{m} & 0 \end{pmatrix}, \tag{66}$$

with $\delta\hat{m} = (m_2 - m_1)/E$. A particular advantage of the hermitian form is presently not evident.

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